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HANKEL DETERMINANT FOR A CLASS OF ANALYTIC FUNCTIONS INVOLVING A GENERALIZED LINEAR DIFFERENTIAL OPERATOR

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Abstract: By making use of the linear differential operator $D^{\sigma,\rho}_{\alpha,\mu}$ defined recently by the authors, a class of analytic functions is introduced. The sharp upper bound for the nonlinear functional $\left|a_2a_4 - a_3^2\right|$ is obtained.

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1. Introduction and Preliminaries

Let S denote the class of normalized analytic univalent functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

In a recent paper [1], we introduced a generalized linear differential operator

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as follows:

$$D^{\sigma;\rho}_{\alpha,\mu}f(z) = z + \sum_{n=2}^{\infty} [1 + (\alpha\mu n + \alpha - \mu)(n-1)]^{\sigma}\, G(\rho, n)\, a_n z^n,$$

where $G(\rho, n) = \begin{pmatrix} n+\rho-1 \\ \rho \end{pmatrix}$ and $\rho \in N_0, 0 \le \mu \le \alpha \le 1, \sigma \in N_0$.

Definition 1.1. For the function f given by (1.1) and $q \in N = \{1, 2, 3, ...\}$, the q^{th} Hankel determinant of f is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has also been considered by several authors. For examples, Noor [9] determined the rate of growth of $H_q(n)$ as $n \to \infty$ with bounded boundary, Ehrenborg [4] studied the Hankel determinant of exponential polynomials, and some of its properties were discussed by Layman in [6]. Recently, Al-Refai and Darus [2] studied the Hankel determinant for $q = 2$ and $n = 2$ defined by a fractional operator.

It is well-known that the Fekete-Szegö functional is $H_2(1)$. It is also known that Fekete and Szegö gave sharp estimates of $\left|a_3 - \mu a_2^2\right|$ for μ real and $f \in S$. Then many authors studied the estimation of $\left|a_3 - \mu a_2^2\right|$ for various subclasses (see for example [11]-[15]).

We now introduce the following class of functions.

Definition 1.2. Let f be given by (1.1). Then $f \in R_{\alpha,\mu}(\sigma, \rho)$ if it satisfies the inequality

$$\Re\{(D^{\sigma;\rho}_{\alpha,\mu}f(z))'\} > 0, \;\; z \in U. \tag{2}$$

The subclass $R_{\alpha,\mu}(0, 0)$ was studied by MacGregor [10] and Janteng et al. [5]. Recently, many authors considered different classes for Hankel determinant, see for example [16].

In the present paper, we consider the Hankel determinant for $q = 2$ and $n = 2$ that lead to find the sharp upper bound for the functional $\left|a_2a_4 - a_3^2\right|$ for functions f belong to the class $f \in R_{\alpha,\mu}(\sigma, \rho)$. This paper is motivated by Janteng et al.[5]. The techniques here follow the same with the one in [5].

We first state some preliminary lemmas which shall be used in our proof.

2. Preliminary Results

Let $\wp$ be the family of all functions p analytic in U for which $\Re\{p(z)\} > 0$

$$p(z) = 1 + c_1 z + c_2 z^2 + ... \tag{3}$$

for $z \in U$.

Lemma 2.1. (see [3]) *If* $p \in \wp$ *, then* $|c_k| \leq 2$ *for each* $k \in N$ *and inequality is sharp.*

Lemma 2.2. (see [8], [7]) *Let the function* $p \in \wp$ *and be given by the power series (2.1) . Then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{4}$$

for some $x, |x| \leq 1$*, and*

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \tag{5}$$

for some $z, |z| \leq 1$.

3. Main Result

Theorem 3.1. *Let* $f \in R_{\alpha,\mu}(\sigma, \rho)$*. Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{16}{9\,(\rho+1)^2(\rho+2)^2[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}.$$

The result obtained is sharp.

Proof: Since $f \in R_{\alpha,\mu}(\sigma, \rho)$, it follows from (1.2) that

$$(D_{\alpha,\mu}^{\sigma;\rho} f(z))' = p(z) \tag{6}$$

Comparing the coefficients, we get

$$\begin{cases} 2(\rho+1)[1+\alpha-\mu+2\alpha\mu]^{\sigma} a_2 = c_1, \\ 3(\rho+1)(\rho+2)[1+2\alpha-2\mu+6\alpha\mu]^{\sigma} a_3 = 2c_2, \\ 2(\rho+1)(\rho+2)(\rho+3)[1+3\alpha-3\mu+12\alpha\mu]^{\sigma} a_4 = 3\,c_3. \end{cases} \tag{7}$$

Therefore, (3.2) yields

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&=\frac{1}{(\rho+1)^2(\rho+2)}\left|\frac{3c_1c_3}{4(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}\right.\\&\qquad\left.-\frac{4c_2^2}{9(\rho+2)[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\right|. \qquad (8)\end{aligned}$$

Since the function $p(z)$ is a member of the class $\wp$ simultaneously, we assume without loss of generality that $c_1 > 0$ and for convenience of notation, take $c_1 = c$ ($c \in [0, 2]$). Then, using (2.2) along with (2.3), we get

$$\begin{aligned}\left|a_2a_4-a_3^2\right| &= H(\rho,\sigma,\alpha,\mu)\left|\frac{(\rho+2)}{144}\left\{27c^4+54c^2(4-c^2)x-27c^2(4-c^2)x^2\right\}\right.\\&+\frac{(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\\&\quad\left\{-16c^4-32c^2(4-c^2)x-16(4-c^2)^2x^2\right\}\\&\left.+\frac{6c(4-c^2)(1-|x|^2)z}{16}\right|,\end{aligned}$$

where

$$\begin{aligned}&H(\rho,\sigma,\alpha,\mu)\\&=\frac{1}{(\rho+1)^2(\rho+2)^2(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}},\end{aligned}$$

An application of triangle inequality and replacement of $|x|$ by ν give

$$\begin{aligned}&\left|a_2a_4-a_3^2\right| \le H(\rho,\sigma,\alpha,\mu)\\&\left[\left\{\frac{27(\rho+2)}{144}-\frac{16(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\right\}c^4\right.\\&+\left\{\frac{54(\rho+2)}{144}-\frac{32(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\right\}c^2(4-c^2)\nu\\&+\left\{\frac{27c^2(\rho+2)}{144}+\frac{16(4-c^2)(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\right\}\\&\left.(4-c^2)\nu^2+\frac{6c(4-c^2)(1-\nu^2)}{16}\right] = H(\rho,\sigma,\alpha,\mu)\\&\left[\left\{\frac{27(\rho+2)}{144}-\frac{16(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\right\}c^4\right.\end{aligned}$$

$$+\{\frac{54(\rho+2)}{144}-\frac{32(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\}c^2(4-c^2)\nu$$
$$+\{\frac{27c^2(\rho+2)}{144}+\frac{16(\rho+3)(4-c^2)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}$$
$$-\frac{6c}{16}\}(4-c^2)\nu^2+\frac{6c(4-c^2)}{16}\Bigg]=F(c,\nu), \quad (9)$$

where $0\le c\le 2$ and $0\le \nu\le 1$.

We next maximize the function $F(c,\nu)$ on the closed square $[0,2]\times[0,1]$. Since

$$\frac{\delta F}{\delta \nu}=H(\rho,\sigma,\alpha,\mu)$$
$$\Bigg[\{\frac{54(\rho+2)}{144}-\frac{32(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{144\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\}c^2(4-c^2)+$$
$$\{\frac{3c(c(\rho+2)-1)}{8}+\frac{16(4-c^2)(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{72\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\}$$
$$(4-c^2)\nu\Big],$$

$\frac{(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{(\rho+2)\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}<\frac{27}{16}$, We have $\frac{\delta F}{\delta\nu}>0$. Thus $F(c,\nu)$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$. Moreover, for fixed $c\in[0,2]$.

$$\max_{0\le\nu\le1}F(c,\nu)=F(c,1)=G(c)$$

then

$$G'(c)=H(\rho,\sigma,\alpha,\mu)\Bigg[\frac{3c(3-c^2)(\rho+2)}{2}$$
$$-\frac{8c(4-c^2)(\rho+3)[1+\alpha-\mu+2\alpha\mu]^{\sigma}[1+3\alpha-3\mu+12\alpha\mu]^{\sigma}}{9\,[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}\Bigg],$$

so that $G'(c)<0$ for $0<c<2$ and has real critical point at $c=0$. Also observe that $G(c)>G(2)$. Therefore, $\max\limits_{0\le c\le2}G(c)$ occurs at $c=0$ and thus the upper bound of (3.4) corresponds to $\nu=1$ and $c=0$, in which case

$$\left|a_2a_4-a_3^2\right|\le\frac{16}{9\,(\rho+1)^2(\rho+2)^2[1+2\alpha-2\mu+6\alpha\mu]^{2\sigma}}. \quad (10)$$

This concludes the proof of our theorem.

The choice $\sigma=\rho=0$ yields what follows:

Corollary 3.2. (see [5]) *Let* $f \in R_{\alpha,\mu}(0,0)$. *Then*

$$\left|a_2 a_4 - a_3^2\right| \leq \frac{4}{9}.$$

The result obtained is sharp.

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