ABSTRACT: By making use of the linear differential operator $D_{\sigma, \rho}^{\alpha, \mu}$ defined recently by the authors, a class of analytic functions is introduced. The sharp upper bound for the nonlinear functional $|a_2a_4 - a_3^2|$ is obtained.

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1. Introduction and Preliminaries

Let $S$ denote the class of normalized analytic univalent functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$  \hspace{1cm} (1)

which are analytic in the unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

In a recent paper [1], we introduced a generalized linear differential operator

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as follows:

\[ D_{\alpha,\mu}^{\sigma,\rho} f(z) = z + \sum_{n=2}^{\infty} \left[ 1 + (\alpha\mu n + \alpha - \mu)(n - 1) \right]^{\sigma} G(\rho, n) a_n z^n, \]

where \( G(\rho, n) = \left( \frac{n + \rho - 1}{\rho} \right) \) and \( \rho \in \mathbb{N}_0, 0 \leq \mu \leq \alpha \leq 1, \sigma \in \mathbb{N}_0. \)

**Definition 1.1.** For the function \( f \) given by (1.1) and \( q \in \mathbb{N} = \{1, 2, 3, \ldots\} \), the \( q^{th} \) Hankel determinant of \( f \) is defined by

\[ H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \]

This determinant has also been considered by several authors. For examples, Noor [9] determined the rate of growth of \( H_q(n) \) as \( n \to \infty \) with bounded boundary, Ehrenborg [4] studied the Hankel determinant of exponential polynomials, and some of its properties were discussed by Layman in [6]. Recently, Al-Refai and Darus [2] studied the Hankel determinant for \( q = 2 \) and \( n = 2 \) defined by a fractional operator.

It is well-known that the Fekete-Szegö functional is \( H_2(1) \). It is also known that Fekete and Szegö gave sharp estimates of \( |a_3 - \mu a_2^2| \) for \( \mu \) real and \( f \in S \). Then many authors studied the estimation of \( |a_3 - \mu a_2^2| \) for various subclasses (see for example [11]-[15]).

We now introduce the following class of functions.

**Definition 1.2.** Let \( f \) be given by (1.1). Then \( f \in R_{\alpha,\mu}(\sigma, \rho) \) if it satisfies the inequality

\[ \Re\{(D_{\alpha,\mu}^{\sigma,\rho} f(z))'\} > 0, \ z \in U. \quad (2) \]

The subclass \( R_{\alpha,\mu}(0,0) \) was studied by MacGregor [10] and Janteng et al. [5]. Recently, many authors considered different classes for Hankel determinant, see for example [16].

In the present paper, we consider the Hankel determinant for \( q = 2 \) and \( n = 2 \) that lead to find the sharp upper bound for the functional \( |a_2a_4 - a_3^2| \) for functions \( f \) belong to the class \( f \in R_{\alpha,\mu}(\sigma, \rho) \). This paper is motivated by Janteng et al.[5]. The techniques here follow the same with the one in [5].

We first state some preliminary lemmas which shall be used in our proof.
2. Preliminary Results

Let \( \wp \) be the family of all functions \( p \) analytic in \( U \) for which \( \Re\{p(z)\} > 0 \)

\[
p(z) = 1 + c_1 z + c_2 z^2 + \ldots \quad (3)
\]

for \( z \in U \).

**Lemma 2.1.** (see [3]) If \( p \in \wp \), then \( |c_k| \leq 2 \) for each \( k \in N \) and inequality is sharp.

**Lemma 2.2.** (see [8], [7]) Let the function \( p \in \wp \) and be given by the power series (2.1). Then

\[
2c_2 = c_1^2 + x(4 - c_1^2) \quad (4)
\]

for some \( x, |x| \leq 1 \), and

\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z \quad (5)
\]

for some \( z, |z| \leq 1 \).

3. Main Result

**Theorem 3.1.** Let \( f \in R_{\alpha, \mu}(\sigma, \rho) \). Then

\[
|a_2 a_4 - a_3^2| \leq \frac{16}{9(\rho + 1)^2(\rho + 2)^2[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}}.
\]

The result obtained is sharp.

**Proof:** Since \( f \in R_{\alpha, \mu}(\sigma, \rho) \), it follows from (1.2) that

\[
(D_{\alpha, \mu}^\sigma f(z))' = p(z) \quad (6)
\]

Comparing the coefficients, we get

\[
\begin{cases}
2(\rho + 1)[1 + \alpha - \mu + 2\alpha\mu]^{\sigma}a_2 = c_1, \\
3(\rho + 1)(\rho + 2)[1 + 2\alpha - 2\mu + 6\alpha\mu]^{\sigma}a_3 = 2c_2, \\
2(\rho + 1)(\rho + 2)(\rho + 3)[1 + 3\alpha - 3\mu + 12\alpha\mu]^{\sigma}a_4 = 3c_3.
\end{cases} \quad (7)
\]

Therefore, (3.2) yields
\[ |a_2a_4 - a_3^2| = \frac{1}{(\rho + 1)^2(\rho + 2)} \left| \frac{3c_1c_3}{4(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma} \right| \]

\[ - \frac{4c_2^2}{9(\rho + 2)[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \cdot (8) \]

Since the function \( p(z) \) is a member of the class \( \wp \) simultaneously, we assume without loss of generality that \( c_1 > 0 \) and for convenience of notation, take \( c_1 = c \ (c \in [0, 2]) \). Then, using (2.2) along with (2.3), we get

\[ |a_2a_4 - a_3^2| = H(\rho, \sigma, \alpha, \mu) \left| \frac{(\rho + 2)}{144} \left\{ 27c^4 + 54c^2(4 - c^2)x - 27c^2(4 - c^2)x^2 \right\} \right| \]

\[ + \left( \frac{\rho + 3}{144} \left[ 1 + \alpha - \mu + 2\alpha\mu \right] [1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma \right) \]

\[ \left\{ -16c^4 - 32c^2(4 - c^2)x - 16(4 - c^2)^2x^2 \right\} \]

\[ + \frac{6c(4 - c^2)(1 - |x|^2)}{16}z \]

where

\[ H(\rho, \sigma, \alpha, \mu) = \frac{1}{(\rho + 1)^2(\rho + 2)^2(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma} \]

An application of triangle inequality and replacement of \( |x| \) by \( \nu \) give

\[ |a_2a_4 - a_3^2| \leq H(\rho, \sigma, \alpha, \mu) \]

\[ \left[ \frac{27(\rho + 2)}{144} - 16(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma \right] c^4 \]

\[ + \left\{ \frac{54(\rho + 2)}{144} - \frac{32(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{144[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right\} c^2(4 - c^2)\nu \]

\[ + \left\{ \frac{27c^2(\rho + 2)}{144} + \frac{16(4 - c^2)(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{144[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right\} (4 - c^2)\nu^2 \]

\[ + \frac{6c(4 - c^2)(1 - \nu^2)}{16} \]

\[ = H(\rho, \sigma, \alpha, \mu) \]

\[ \left[ \frac{27(\rho + 2)}{144} - 16(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma \right] c^4 \]
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\[
+ \left\{ \frac{54(\rho + 2)}{144} - \frac{32(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{144[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right\} c^2 (4 - c^2) \nu
\]

This concludes the proof of our theorem.

The choice \( \sigma = \rho = 0 \) yields what follows:

\[
\left\{ \frac{27c^2(\rho + 2)}{144} + \frac{16(\rho + 3)(4 - c^2)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{144[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right\}
\]

\[
= F(c, \nu) \quad (9)
\]

where \( 0 \leq c \leq 2 \) and \( 0 \leq \nu \leq 1 \).

We next maximize the function \( F(c, \nu) \) on the closed square \([0, 2] \times [0, 1]\).

Since

\[
\frac{\delta F}{\delta \nu} = H(\rho, \sigma, \alpha, \mu)
\]

\[
\left\{ \frac{54(\rho + 2)}{144} - \frac{32(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{144[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right\} c^2 (4 - c^2) + \frac{3c(c(\rho + 2) - 1)}{8} + \frac{16(4 - c^2)(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{72[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} (4 - c^2) \nu
\]

\[
< \frac{27}{16}, \quad \text{We have} \quad \frac{\delta F}{\delta \nu} > 0. \quad \text{Thus} \quad F(c, \nu) \quad \text{cannot have a maximum in the interior of the closed square} \quad [0, 2] \times [0, 1]. \quad \text{Moreover, for fixed} \quad c \in [0, 2],
\]

\[
\max_{0 \leq \nu \leq 1} F(c, \nu) = F(c, 1) = G(c)
\]

then

\[
G'(c) = H(\rho, \sigma, \alpha, \mu) \left[ \frac{3c(3 - c^2)(\rho + 2)}{2} \right.
\]

\[
- \frac{8c(4 - c^2)(\rho + 3)[1 + \alpha - \mu + 2\alpha\mu]^\sigma[1 + 3\alpha - 3\mu + 12\alpha\mu]^\sigma}{9[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}} \right],
\]

so that \( G'(c) < 0 \) for \( 0 < c < 2 \) and has real critical point at \( c = 0 \). Also observe that \( G(c) > G(2) \). Therefore, \( \max_{0 \leq c \leq 2} G(c) \) occurs at \( c = 0 \) and thus the upper bound of (3.4) corresponds to \( \nu = 1 \) and \( c = 0 \), in which case

\[
|a_2a_4 - a_3^2| \leq \frac{16}{9(\rho + 1)^2(\rho + 2)^2[1 + 2\alpha - 2\mu + 6\alpha\mu]^{2\sigma}}.
\]

(10)

This concludes the proof of our theorem.

The choice \( \sigma = \rho = 0 \) yields what follows:
Corollary 3.2. (see [5]) Let $f \in R_{\alpha,\mu}(0,0)$. Then

$$|a_2a_4 - a_3^2| \leq \frac{4}{9}.$$ 

The result obtained is sharp.

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References


