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SOME QUANTUM MECHANICAL EVIDENCE FOR THE AMPLIFICATION OF THERMAL NOISE IN AN ELECTROSTATIC FIELD

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Abstract: Recently the phenomenon of thermal noise amplification in a conductor exposed to an electrostatic field has been discovered in a laboratory experiment. The challenge behind the present contribution is a qualitative explanation of this phenomenon based on a quantum mechanical model. It turns out that the amplification of thermal noise in an conductor complies with computations concerning a tight binding model of an electron confined to a lattice which is interpreted as the quantum mechanical position space; the dependence of the thermal noise level on the strength of an external electrostatic field is explored in a computer experiment providing theoretical evidence for the physical phenomenon.

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1. Introduction

In [3] an experimental confirmation of the phenomenon of thermal noise amplification in a conductor is reported; the conductor is exposed to an electrostatic field which is realized by imposing a high dc voltage on a parallel-plate capacitor. The thermal noise level is measured as the dispersion of the voltage signal

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between appropriate points of the conductor. This phenomenon is noteworthy insofar as it can be realized by applying only negligible electric power.

In the present contribution we introduce a simplified quantum mechanical model of a conductor exposed to an electrostatic field. We consider an electron which is confined to a discrete position space and compute the corresponding Schrödinger operator (Section 2). This kind of quantum mechanical models (tight binding models, cf. [1], p. 168) is subject of intensive explorations (cf. [1], [2] and literature cited therein) and offers an attractive field for computer experimental studies. In Section 3 we introduce the quantum mechanical Gibbs state describing the electron at thermal equilibrium; an example illustrates the physical plausibility of this simplified description. In Section 4 the quantum mechanical dispersion of the velocity operator is defined; it can be viewed as an indicator of the thermal noise level inherent in a conductor. We report on the outcome of a computer experiment confirming thermal noise amplification by an electrostatic field based on the introduced quantum mechanical model.

2. A Schödinger Operator for an Electron in a Discrete Position Space

Let us consider a finite lattice

$$L_a := \{na | n = 1, \dots, N\}$$

of N points modeling a discrete position space; parameter $a > 0$ is called lattice constant. L_a serves as a model of an 1-dimensional conductor. A quantum mechanical state of an electron confined to the conductor is described by a function $\varphi : L_a \to \mathbb{C}$ satisfying the condition

$$\sum_{n=1}^{N} |\varphi(na)|^2 = 1.$$

In this context $|\varphi(na)|^2$ is interpreted as the probability of spatial association of the electron with lattice point $na \in L_a$. By a standard identification, the set of all electronic states can be viewed as the unit sphere in $\mathbb{C}^N$.

The quantum mechanical momentum operator $\widehat{p} : \mathbb{C}^N \to \mathbb{C}^N$ is defined by

$$(\widehat{p}\varphi)(na) = -i\hbar \cdot \frac{\varphi((n+1)a) - \varphi((n-1)a)}{2a} \qquad (n = 1, \dots, N) \tag{2.1}$$

where $\hbar$ denotes Planck's constant; in (2.1) the convention

$$\varphi(na) = 0 \quad \text{for} \quad n < 1 \quad \text{and for} \quad n > N$$

is applied and can be interpreted as Dirichlet boundary condition (cf. [2], p. 28ff). $\widehat{p}$ is self-adjoint and serves as a discrete central difference approximation of the 1-dimensional momentum operator

$$-i\hbar \cdot \frac{d}{dx}$$

for the position space modeled by the real line. Accordingly, the kinetic energy of the electron is expressed by the operator $\widehat{p}^2/2m$ where m denotes the electronic mass.

Remark 2.1. All entries in the matrix representing operator $\widehat{p}$ are purely imaginary; consequently, all entries in the matrix representing $\widehat{p}^2/2m$ are real numbers.

In our simplified discrete model the electric field induced by the ions positioned at the lattice points is averaged over the whole conductor yielding the potential V which is constant; without loss of generality we can choose the gauge $V = 0$ of this potential.

Let us now suppose that the conductor is exposed to a homogeneous electrostatic field of strength E. If the electron is spatially associated with site $na \in L_a$, then its potential energy w.r.t. the field is given by $-E \cdot e \cdot na$; therefore the operator $U_E : \mathbb{C}^N \to \mathbb{C}^N$ modeling the potential energy of the electron is given by

$$(U_E\varphi)(na) = -E \cdot e \cdot na \cdot \varphi(na) \qquad (n = 1, \ldots, N).$$

Finally, the announced Schrödinger operator $H_E : \mathbb{C}^N \to \mathbb{C}^N$ describing the electron confined to lattice L_a and exposed to a homogeneous electrostatic field of strength E is defined by

$$H_E = \frac{\widehat{p}^2}{2m} + U_E. \tag{2.2}$$

Remark 2.2. All entries in the matrix describing operator H_E are real numbers. The extension of H_E to an appropriate infinitely dimensional Hilbert space is called Stark-Jacobi operator, cf. [4].

The set of all states of an electron can be embedded into the set of positive semi-definite operators with trace 1 by associating any unit vector $\varphi \in \mathbb{C}^N$ with the orthogonal projection onto the 1-dimensional space spanned by φ. Therefore we call any positive semi-definite operator $Z : \mathbb{C}^N \to \mathbb{C}^N$ with trace 1 (generalized) quantum mechanical state of the electron confined to lattice L_a.

3. The Gibbs State of the Electron

Let us consider Schrödinger operator H_E introduced in Section 2. Let $T > 0$ denote the temperature of the conductor. The operator $G_{T,E} : \mathbb{C}^N \to \mathbb{C}^N$ modeling the Gibbs state of the electron is given by

$$G_{T,E} = \frac{1}{Z(T,E)} \cdot \exp\left(-\frac{1}{k_B \cdot T} \cdot H_E\right) \tag{3.1}$$

where

$$Z(T,E) := \operatorname{trace}\left(\exp\left(-\frac{1}{k_B \cdot T} \cdot H_{T,E}\right)\right) \tag{3.2}$$

denotes the partition function and k_B the Boltzmann constant. $G_{T,E}$ is a positive semidefinite operator whose trace is equal to 1. Operator $G_{T,E}$ is motivated by the entropy principle (cf. [5], p. 384) and describes the thermal equilibrium state of the electron confined to lattice L_a with the interpretation of the diagonal entry $G_{T,E}(n,n)$ as the probability of spatial association of the electron with lattice point na.

Remark 3.1. All entries in the matrix describing operator $G_{T,E}$ are real numbers; cf. Remark 2.2.

Example 3.2. Put $N = 200, a = 10^{-10}$m, $E = 5 \cdot 10^6$V/m and $T = 300$K. Note that 10^{-10}m$= 1$Å corresponds to the typical order of magnitude for the distance between adjacent ions in a conductor.

In Figure 1 the horizontal axis corresponds to lattice L_a. The graph shows the probabilities $G_{T,E}(n,n)$ of finding the electron at lattice points $na, n = 1, \ldots, N$. The diagram suggests that the electron prefers sites with low potential energy w.r.t. the external electrostatic field, which underlines the physical plausibility of the presented simplified quantum mechanical description.

4. The Dispersion of the Electronic Velocity Operator

The operator $\widehat{v} : \mathbb{C}^N \to \mathbb{C}^N$,

$$\widehat{v} := \frac{\widehat{p}}{m},$$

describes the velocity of the electron confined to lattice L_a where $\widehat{p}$ is the momentum operator introduced in Section 2.

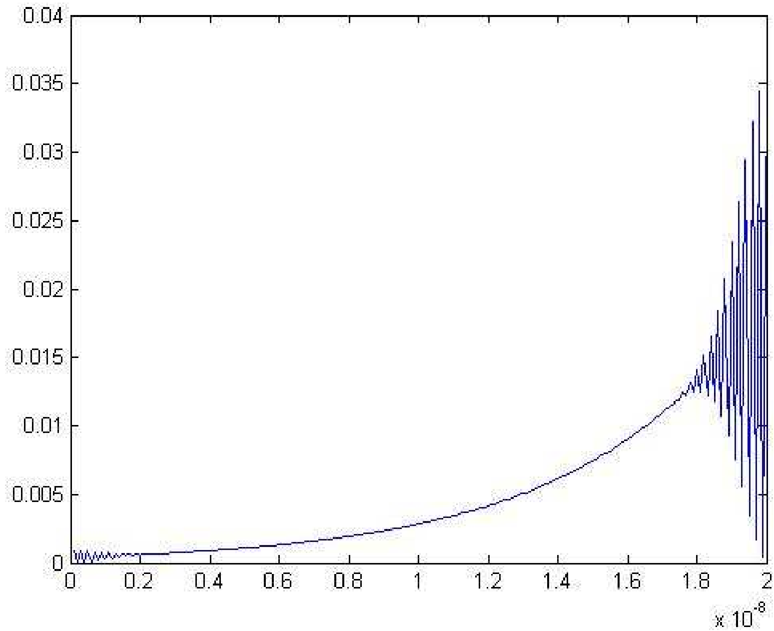

Figure 1: Gibbsian probabilities

The quantum mechanical expectation $\mathbb{E}_{\text{qm}}(\widehat{v})$ of the velocity of the electron whose state is described by $G_{T,E}$, is given by

$$\mathbb{E}_{\text{qm}}(\widehat{v}) = \text{trace}(G_{T,E}\widehat{v}). \tag{4.1}$$

From Linear Algebra it is known that the trace in (4.1) is a real number. Since the matrix describing operator $G_{T,E}$ is real and the Hermitean matrix corresponding to operator $\widehat{v}$ is purely imaginary, the trace in (4.1) is imaginary; we conclude that

$$\mathbb{E}_{\text{qm}}(\widehat{v}) = 0 \tag{4.2}$$

holds for arbitrary $T, E \in \mathbb{R}$. This means that there is no direct electronic current in the considered conductor and that the quantum mechanical variance $\mathbb{V}_{\text{qm}}(\widehat{v})$ of the velocity operator can be computed according to

$$\mathbb{V}_{\text{qm}}(\widehat{v}) = \text{trace}(G_{T,E}\widehat{v}^2). \tag{4.3}$$

The quantum mechanical dispersion of velocity operator $\widehat{v}$ is defined by

$$\mathbb{D}_{\text{qm}}(\widehat{v}) := \sqrt{\mathbb{V}_{\text{qm}}(\widehat{v})};$$

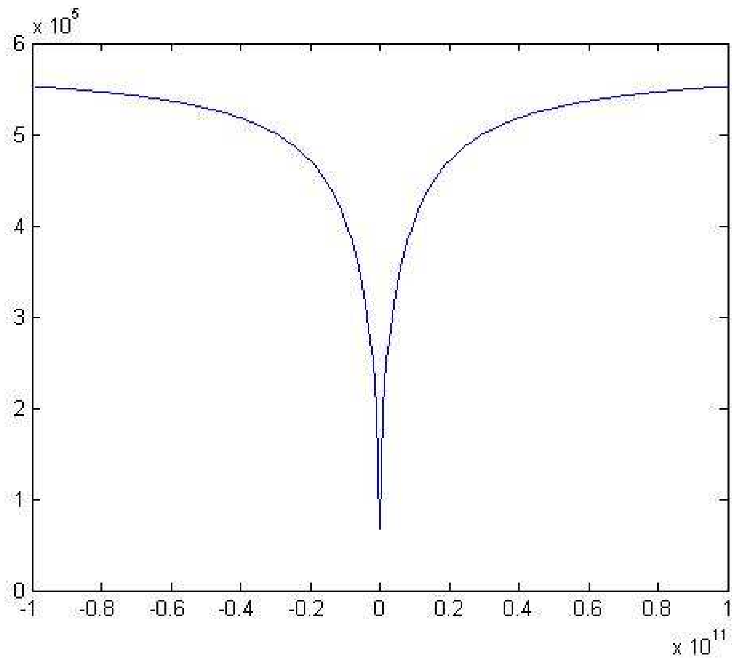

Figure 2: Dispersion of the velocity operator

this dispersion can be viewed as an indicator of the thermal noise level inherent in the conductor.

Example 4.1. Put $N = 500, a = 10^{-10}$m, $T = 300$K. In a computer experiment we let the strength E of the external electrostatic field vary and compute for each value of E the corresponding dispersion of velocity operator $\widehat{v}$.

In Figure 2 the horizontal axis corresponds to strength E of the external electrostatic field (the physical unit is V/m) and the vertical axis to the quantum mechanical dispersion of the velocity operator (the physical unit is m/s). The diagram shows that the dispersion of electronic velocity increases for increasing values of $|E|$. This complies qualitatively with the outcome of the laboratory experiment reported in [3]. Figure 2 suggests, moreover, that the thermal noise level is independent of the polarity of the external electrostatic field, which is in a good agreement with intuition.

References

[1] H.L. Cycon, R.G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators*, Springer, Berlin, Heidelberg, New York (2008).

[2] M. Disertori, W. Kirsch, A. Klein, F. Klopp, V. Rivasseau, *Random Schrödinger Operators*, Panoramas et Syntheses, Societe Mathematique de France, Paris (2008).

[3] E. Grycko, W. Kirsch, T. Mühlenbruch, Amplification of thermal noise by an electrostatic field, *Int. J. Pure Appl. Math.*, **61**, No. 2 (2010), 187-192.

[4] B. Nachtergaele, W.L. Spitzer, S. Starr, On the dynamics of interfaces in the ferromagnetic XXZ chain under weak perturbations, *Contemporary Mathematics*, **327** (2003), 251-270.

[5] W. Thirring, *Quantum Mathematical Physics*, Second edition, Springer, Berlin, Heidelberg, New York (2002).