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THE INTERSECTION PROPERTY OF QUASI-IDEALS IN RINGS OF STRICTLY UPPER TRIANGULAR MATRICES OVER $\mathbb{Z}_m$

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Abstract: The notion of quasi-ideals for rings was first introduced by O. Steinfeld. It is known that the intersection of a left ideal and a right ideal of a ring R is a quasi-ideal of R. However, a quasi-ideal of R need not be obtained in this way. A quasi-ideal Q of R is said to *have the intersection property* if Q is the intersection of a left ideal and a right ideal of R. If every quasi-ideal of R has the intersection property, R is said to *have the intersection property of quasi-ideals.* Let $SU_n(R)$ denote the ring of all strictly upper triangular $n \times n$ matrices over a ring R. In this paper, we characterize when the ring $SU_n(\mathbb{Z}_m)$ has the intersection property of quasi-ideals.

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1. Introduction and Preliminaries

The notion of quasi-ideals for rings was first introduced by O. Steinfeld in [5]. Quasi-ideals generalize left ideals and right ideals. For nonempty subsets A, B of a ring R, let $\mathbb{Z}A$ and AB denote respectively the set of all finite sums of the form $\sum k_i a_i$ where $k_i \in \mathbb{Z}$ and $a_i \in A$ and the set of all finite sums of the form $\sum a_i b_i$ where $a_i \in A$ and $b_i \in B$. A subring Q of R is called a *quasi-ideal* of R if $RQ \cap QR \subseteq Q$. For a nonempty subset X of a ring R, let $(X)_q$ be the quasi-ideal of R generated by X which is the intersection of all quasi-ideals of R containing X ([6], page 11). H. J. Weinert [7] has given the following fact:

Proposition 1.1. (see [7]) *For a nonempty subset X of a ring R*

$$(X)_q = \mathbb{Z}X + (RX \cap XR).$$

It is known that the intersection of a left ideal and a right ideal of a ring R is a quasi-ideal of R. However, a quasi-ideal of R need not be obtained in this way, we can see some examples in [1], [2], [3], [4], [6], page 8 and [8]. We say that a quasi-ideal Q of R *has the intersection property* if Q is the intersection of a left ideal and a right ideal of R. Every left ideal and every right ideal of R is clearly seen to have the intersection property. If every quasi-ideal of R has the intersection property, R is said to *have the intersection property of quasi-ideals.* Every commutative ring is clearly seen to have this property. Moreover, a ring having a one-sided identity and a regular ring have the intersection property of quasi-ideals ([5], page 9 and page 73, respectively). In [8], M. C. Zhang, Y. Q. Chen and Y. H. Li have characterized the following result:

Proposition 1.2. (see [8]) *A ring R has the intersection property of quasi-ideals if and only if for any finite nonempty subset X of R*

$$RX \cap (\mathbb{Z}X + XR) \subseteq \mathbb{Z}X + (RX \cap XR).$$

Next, let $SU_n(R)$ be the ring of all strictly upper triangular $n \times n$ matrices over a ring R. In [4], Y. Kemprasit and P. Juntarakajorn have characterized when $SU_n(F)$ has the intersection property of quasi-ideals where F is a field.

Proposition 1.3. (see [4]) *Let F be a field. The ring $SU_n(F)$ has the intersection property of quasi-ideals if and only if $n \leq 3$.*

Next, R. Chinram and C. Rungsaripipat [2] have characterized when $SU_n(m\mathbb{Z})$ where $\mathbb{Z}$ is the set of all integers, has the intersection property of quasi-ideals by using some technique in [4] but it is more complicated.

Proposition 1.4. (see [2]) *The ring $SU_n(m\mathbb{Z})$ has the intersection property of quasi-ideals if and only if $m = 0$ or $n \leq 3$.*

Let $\mathbb{Z}_m$ denote the set of all integers modulo m. The purpose of this paper is to give necessary and sufficient conditions for n and m that the ring $SU_n(\mathbb{Z}_m)$ has the intersection property of quasi-ideals by using technique in [2].

2. Main Result

To prove the main theorem, the following three lemmas are provided.

Lemma 2.1. *Let $a, b, c \in \mathbb{Z}$. The following statements are true.*

(i) $a\mathbb{Z}_m + b\mathbb{Z}_m = gcd(a,b)\mathbb{Z}_m$.

(ii) $a\mathbb{Z}_m \cap b\mathbb{Z}_m = lcm(a,b)\mathbb{Z}_m$.

(iii) $gcd(a, lcm(b,c)) | lcm(gcd(a,b), c)$.

(iv) $(a\mathbb{Z}_m + b\mathbb{Z}_m) \cap c\mathbb{Z}_m \subseteq a\mathbb{Z}_m + (b\mathbb{Z}_m \cap c\mathbb{Z}_m)$.

Proof. The proofs of (i) and (ii) are straight forward.

(iii) We have that

$$\begin{aligned}
\operatorname{lcm}(\gcd(a,b),c) &= \operatorname{lcm}\Big(\frac{ab}{\operatorname{lcm}(a,b)}, c\Big) \\
&= \frac{\frac{abc}{\operatorname{lcm}(a,b)}}{\gcd(\frac{ab}{\operatorname{lcm}(a,b)}, c)} \\
&= \frac{\frac{abc}{\gcd(b,c)}}{\operatorname{lcm}(\frac{bc}{\gcd(b,c)}, a)} \frac{\gcd(b,c)}{\gcd(\frac{ab}{\operatorname{lcm}(a,b)}, c)} \frac{\operatorname{lcm}(a, \frac{bc}{\gcd(b,c)})}{\operatorname{lcm}(a,b)} \\
&= \gcd\Big(a, \frac{bc}{\gcd(b,c)}\Big) \frac{\gcd(b,c)}{\gcd(\frac{ab}{\operatorname{lcm}(a,b)}, c)} \frac{\operatorname{lcm}(a, \frac{bc}{\gcd(b,c)})}{\operatorname{lcm}(a,b)} \\
&= \gcd(a, \operatorname{lcm}(b,c)) \frac{\gcd(b,c)}{\gcd(\gcd(a,b),c)} \frac{\operatorname{lcm}(a, \operatorname{lcm}(b,c))}{\operatorname{lcm}(a,b)}.
\end{aligned}$$

Since $\frac{\gcd(b,c)}{\gcd(\gcd(a,b),c)}$ and $\frac{\operatorname{lcm}(a,\operatorname{lcm}(b,c))}{\operatorname{lcm}(a,b)}$ are integers, $\gcd(a, \operatorname{lcm}(b,c)) | \operatorname{lcm}(\gcd(a,b),c)$.

(iv) By (i) and (ii), we have that

$$(a\mathbb{Z}_m + b\mathbb{Z}_m) \cap c\mathbb{Z}_m = \gcd(a,b)\mathbb{Z}_m \cap c\mathbb{Z}_m = \operatorname{lcm}(\gcd(a,b),c)\mathbb{Z}_m$$

and

$$a\mathbb{Z}_m + (b\mathbb{Z}_m \cap c\mathbb{Z}_m) = a\mathbb{Z}_m + \operatorname{lcm}(b,c)\mathbb{Z}_m = \gcd(a, \operatorname{lcm}(b,c))\mathbb{Z}_m.$$

By (iii), we have $\gcd(a, \operatorname{lcm}(b,c))|\operatorname{lcm}(\gcd(a,b),c)$, this implies that

$$\operatorname{lcm}(\gcd(a,b),c)\mathbb{Z}_m \subseteq \gcd(a, \operatorname{lcm}(b,c))\mathbb{Z}_m.$$

Hence $(a\mathbb{Z}_m + b\mathbb{Z}_m) \cap c\mathbb{Z}_m \subseteq a\mathbb{Z}_m + (b\mathbb{Z}_m \cap c\mathbb{Z}_m)$, as required. □

Lemma 2.2. *The ring $SU_3(\mathbb{Z}_m)$ has the intersection property of quasi-ideals.*

Proof. Let X be any finite nonempty subset of $SU_3(\mathbb{Z}_m)$. Then

$$XSU_3(\mathbb{Z}_m) \subseteq \left\{ \begin{bmatrix} \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x \in \mathbb{Z} \right\}$$

and

$$SU_3(\mathbb{Z}_m)X \subseteq \left\{ \begin{bmatrix} \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x \in \mathbb{Z} \right\}.$$

To show $SU_3(\mathbb{Z}_m)X \cap (\mathbb{Z}X + XSU_3(\mathbb{Z}_m)) \subseteq \mathbb{Z}X + (SU_3(\mathbb{Z}_m)X \cap XSU_3(\mathbb{Z}_m))$, let $M \in SU_3(\mathbb{Z}_m)X \cap (\mathbb{Z}X + XSU_3(\mathbb{Z}_m))$. This implies that $M \in SU_3(\mathbb{Z}_m)X$ and $M = Q + N$ for some $Q \in \mathbb{Z}X$ and $N \in XSU_3(\mathbb{Z}_m)$. Then

$$Q = M - N \in \left\{ \begin{bmatrix} \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x \in \mathbb{Z} \right\} \cap \mathbb{Z}X.$$

Define $\varphi : (\left\{ \begin{bmatrix} \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x \in \mathbb{Z} \right\}, +) \to (\mathbb{Z}_m, +)$ by

$$\varphi(\begin{bmatrix} \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{bmatrix}) = x \text{ for all } x \in \mathbb{Z}.$$

It is clearly seen that φ is an isomorphism. Since φ is an isomorphism and $XSU_3(\mathbb{Z}_m), SU_3(\mathbb{Z}_m)X$ and $\left\{\begin{bmatrix}\overline{0} & \overline{0} & \overline{x}\\ \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \mid x \in \mathbb{Z}\right\} \cap \mathbb{Z}X$ are subgroups of $\left\{\begin{bmatrix}\overline{0} & \overline{0} & \overline{x}\\ \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \mid x \in \mathbb{Z}\right\}$, we have $\varphi(XSU_3(\mathbb{Z}_m))$, $\varphi(SU_3(\mathbb{Z}_m)X)$ and $\varphi(\left\{\begin{bmatrix}\overline{0} & \overline{0} & \overline{x}\\ \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \mid x \in \mathbb{Z}\right\} \cap \mathbb{Z}X)$ are subgroups of $\mathbb{Z}_m$. Therefore

$$\varphi(XSU_3(\mathbb{Z}_m)) = a\mathbb{Z}_m, \varphi(SU_3(\mathbb{Z}_m)X) = b\mathbb{Z}_m \text{ and}$$
$$\varphi(\left\{\begin{bmatrix}\overline{0} & \overline{0} & \overline{x}\\ \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \mid x \in \mathbb{Z}\right\} \cap \mathbb{Z}X) = c\mathbb{Z}_m$$

for some $a, b, c \in \mathbb{Z}$. Since $M = Q + N$, $\varphi(M) = \varphi(Q) + \varphi(N) \in c\mathbb{Z}_m \cap (a\mathbb{Z}_m + b\mathbb{Z}_m) \subseteq c\mathbb{Z}_m + (a\mathbb{Z}_m \cap b\mathbb{Z}_m)$ by Lemma 2.1(iv). Therefore $\varphi(M) = y + z$ for some $y \in c\mathbb{Z}_m$ and $z \in a\mathbb{Z}_m \cap b\mathbb{Z}_m$. Since φ is an isomorphism, there exist $A \in \left\{\begin{bmatrix}\overline{0} & \overline{0} & \overline{x}\\ \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \mid x \in \mathbb{Z}\right\} \cap \mathbb{Z}X$ and $B \in XSU_3(\mathbb{Z}_m) \cap SU_3(\mathbb{Z}_m)X$ such that $y = \varphi(A)$ and $z = \varphi(B)$. Since φ is one-to-one, $M = A + B \in \mathbb{Z}X + XSU_3(\mathbb{Z}_m) \cap SU_3(\mathbb{Z}_m)X$. Therefore $SU_3(\mathbb{Z}_m)X \cap (\mathbb{Z}X + XSU_3(\mathbb{Z}_m)) \subseteq \mathbb{Z}X + (SU_3(\mathbb{Z}_m)X \cap XSU_3(\mathbb{Z}_m))$. By Proposition 1.2, $SU_3(\mathbb{Z}_m)$ has the intersection property of quasi-ideals.

Therefore the lemma is proved. □

Lemma 2.3. *If $n \geq 4$, then the ring $SU_n(\mathbb{Z}_m)$ does not have the intersection property of quasi-ideals.*

Proof. Let $A, B \in SU_n(\mathbb{Z}_m)$ be defined by

$$A = \begin{bmatrix}\overline{0} & \dots & \overline{0} & \overline{1} & \overline{0} & \overline{0}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{1}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\end{bmatrix} \text{ and } B = \begin{bmatrix}\overline{0} & \dots & \overline{0} & \overline{0} & \overline{1} & \overline{0}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{1}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\\ \dots & \dots & \dots & \dots & \dots & \dots\\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0}\end{bmatrix}.$$

Thus

$$SU_n(\mathbb{Z}_m)\{A,B\} = \left\{ \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{x} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{y} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x,y \in \mathbb{Z} \right\}$$

$$\{A,B\}SU_n(\mathbb{Z}_m) = \left\{ \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{0} & \overline{x} & \overline{y} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid x,y \in \mathbb{Z} \right\}$$

and

$$\mathbb{Z}\{A,B\} = \left\{ \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{k} & \overline{l} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{l} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{k} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid k,l \in \mathbb{Z} \right\}.$$

Then

$$\begin{aligned} &\mathbb{Z}\{A,B\} + (SU_n(\mathbb{Z}_m)\{A,B\} \cap \{A,B\}SU_n(\mathbb{Z}_m)) \\ &= \left\{ \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{k} & \overline{l} & \overline{x} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{l} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{k} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix} \mid k,l,x \in \mathbb{Z} \right\}. \quad (1) \end{aligned}$$

Let $C, D \in SU_n(\mathbb{Z}_m)$ be defined by

$$C = \begin{bmatrix} \overline{0} & \overline{0} & \overline{1} & \overline{0} & \dots & \overline{0} \\ \overline{0} & \overline{0} & -\overline{1} & \overline{0} & \dots & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \dots & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \dots & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \overline{0} & \overline{0} & \overline{0} & \dots & \overline{0} \end{bmatrix} \text{ and } D = \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{1} & \overline{1} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}.$$

Then

$$CA = -B + AD = \begin{bmatrix} \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{1} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & -\overline{1} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \dots & \overline{0} & \overline{0} & \overline{0} & \overline{0} \end{bmatrix}.$$

By (1), we have

$$CA = -B + AD \notin \mathbb{Z}\{A, B\} + (SU_n(\mathbb{Z}_m)\{A, B\} \cap \{A, B\}SU_n(\mathbb{Z}_m)).$$

But

$$CA = -B + AD \in SU_n(\mathbb{Z}_m)\{A, B\} \cap (\mathbb{Z}\{A, B\} + \{A, B\}SU_n(\mathbb{Z}_m)),$$

so

$$SU_n(\mathbb{Z}_m)\{A, B\} \cap (\mathbb{Z}\{A, B\} + \{A, B\}SU_n(\mathbb{Z}_m)) \\ \nsubseteq \mathbb{Z}\{A, B\} + (SU_n(\mathbb{Z}_m)\{A, B\} \cap \{A, B\}SU_n(\mathbb{Z}_m)).$$

By Proposition 1.2, $SU_n(\mathbb{Z}_m)$ does not have the intersection property of quasi-ideals.

Therefore the lemma is proved. □

Now we are ready to prove our main result.

Theorem 2.4. *The ring $SU_n(\mathbb{Z}_m)$ has the intersection property of quasi-ideals if and only if $n \leq 3$.*

Proof. If $n \leq 2$, then we have that $SU_n(\mathbb{Z}_m)$ is a zero ring. Hence $SU_n(\mathbb{Z}_m)$ has the intersection property of quasi-ideals.

If $n = 3$, then by Lemma 2.2, $SU_3(\mathbb{Z}_m)$ has the intersection property of quasi-ideals.

For the converse, assume that $n \geq 4$, by Lemma 2.3, $SU_n(\mathbb{Z}_m)$ does not have the intersection property of quasi-ideals.

Hence the theorem is completely proved. □

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References

[1] R. Chinram, Y. Kemprasit, The intersection property of quasi-ideals in generalized rings of strictly upper triangular matrices, *East-West J. Math.*, **3** (2001), 201-210.

[2] R. Chinram, C. Rungsaripipat, The intersection property of quasi-ideals in rings of strictly upper triangular matrices over $m\mathbb{Z}$, *Adv. Algebra Anal.*, **1** (2006), 123-131.

[3] Y. Kemprasit, R. Chinram, Generalized rings of linear transformations having the intersection property of quasi-ideals, *Vietnam J. Math.*, **30** (2002), 283-288.

[4] Y. Kemprasit, P. Juntarakhajorn, Quasi-ideals of rings of strictly upper triangular matrices over a field, *Italian J. Pure Appl. Math.*, **12** (2002), 97-104.

[5] O. Steinfeld, On ideal-quotients and prime ideals, *Acta Math. Acad. Sci. Hung.*, **4** (1953), 289-298.

[6] O. Steinfeld, *Quasi-ideals in Rings and Semigroups*, Akadémiai Kiadó, Budapest (1978).

[7] H.J. Weinert, On quasi-ideals in rings, *Acta Math. Hung.*, **43** (1984), 85-99.

[8] M.C. Zhang, Y.Q. Chen, Y.H. Li, The intersection property of quasi-ideals in rings and semigroup rings, Southeast Asian Bull. Math., **20** (1996), 117-122.