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A POINTWISE NEGATIVE BINOMIAL APPROXIMATION BY *w*-FUNCTIONS

P. Malingam[1], K. Teerapabolarn[2 §]

Department of Mathematics
Faculty of Science
Burapha University
Chonburi 20131, THAILAND
[2]e-mail: kanint@buu.ac.th

Abstract: We use Stein's method and w-functions to determine a result in approximating the distribution of a non-negative integer-valued random variable X by negative binomial distribution with parameters $r \in [1, \infty)$ and $p = 1 - q \in (0, 1)$ in terms of the point metric between two such distributions together with its non uniform upper bound. In addition, when $r = 1$, we also give a non-uniform upper bound on pointwise geometric approximation to the distribution of X. For applications, we use these results to approximate some discrete distributions such as Pólya, negative Pólya, hypergeometric and negative hypergeometric distributions.

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1. Introduction

The negative binomial distribution with parameters $r > 0$ and $p \in (0, 1)$ is a discrete distribution with a long history as same as the binomial distribution. It has many applications in fields such as automobile insurance, inventory analysis, telecommunications networks analysis and population genetics. For $r \in \mathbb{N}$, it is called the Pascal distribution, which can be thought of as the distribution of

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§Correspondence author

the number of failures before the number of successes reaches a fixed integer r in a sequence of independent Bernoulli trials, where success occurs on each trial with a probability of p and failure occurs on each trial with a probability of $q = 1-p$. A special case, $r = 1$, it is referred to as the geometric distribution with parameter p, which models the number of failures before the first success. Let Y be the negative binomial random variable with parameters r and p, then its probability function can be written as

$$p_Y(k) = \frac{\Gamma(r+k)}{\Gamma(r)k!}q^k p^r, \; k = 0, 1, ..., \tag{1.1}$$

and its mean and variance are $\mathbb{E}(Y) = \frac{rq}{p}$ and variance $\mathbb{V}ar(Y) = \frac{rq}{p^2}$, respectively. It is noted that Y is a non-negative integer-valued random variable.

Let X be a non-negative integer-valued random variable with probability function $p_X(x) > 0$ for every x in the support of X, $\mathcal{S}(x)$. It is well known that the distribution of X can be approximated by some discrete distributions, such as Poisson, binomial and compound Poisson distributions, provided that certain conditions on their parameters are satisfied. Similarly, if parameters of the distribution of X and the negative binomial distribution are given under appropriate conditions, then the negative binomial distribution can be used as an approximation of the distribution of X. Furthermore, if we expect the distribution of X to be closer to the negative binomial distribution than other distributions, then it is reasonable to approximate the distribution of X by the negative binomial distribution.

The first work of the negative binomial approximation was introduced and applied to random indicators by Brown and Phillips [2]. They used Stein's method to give a uniform upper bound for approximating the distribution of a sum of m random indicators, $X = \sum_{i=1}^{m} X_i$, by a negative binomial distribution, and they applied this result to approximate the Pólya distribution. Vellaisamy and Upadhye [12] used Kerstan's method to give a uniform upper bound on negative binomial approximation to the distribution of a sum of n independent geometric random variables, $X = \sum_{i=1}^{n} X_i$. The recent article of negative binomial approximation, Teerapabolarn and Boondirek [11] used Stein's method and Stein's identity to give a uniform upper bound on this approximation to the distribution of a non-negative integer-valued random variable X as the following result.

$$\begin{aligned}|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \le \frac{(1-p^r)p}{rq}\mathbb{E}\left|(r+X)q/p - \sigma^2 w(X)\right| \\ + (1-p_X(0))\left|rq/p - \mu\right|,\end{aligned} \tag{1.2}$$

where $A \subseteq \mathbb{N} \cup \{0\}$, μ and $\sigma^2 \in (0,\infty)$ are mean and variance of X and $w(x) = \frac{1}{p_X(x)\sigma^2}\sum_{k=0}^{x}(\mu - k)p_X(k),\ x \in \mathcal{S}(x)$.

For geometric approximation, $r = 1$, the most results are obtained by Stein's method and the first work in this direction was presented by Barbour and Grübel [1]. They gave a uniform upper bound on this approximation for the problem of finding the first sum of a random positive integer sequence with given divider. Peköz [6] gave two uniform upper bounds for measuring the error in the geometric approximation of a random variable counting the number of failures before the first success in a sequence of dependent Bernoulli trials. He applied the results to Markov hitting time and sequence pattern applications. Brown and Phillips [2] considered this approximation in connection with a sum of indicator random variables, and they gave a uniform upper bound on the rate of convergence of the Pólya distribution. Phillips and Weinberg [7] gave a non-uniform upper bound for approximating the distribution of a sum of indicator random variables by improving the bound in Brown and Phillips [2], Teerapabolarn [9] gave a better uniform upper bound on the rate of convergence of the Pólya distribution by a different manner and Teerapabolarn [10] used this method and w-functions to give uniform and non-uniform upper bounds on this approximation to the distribution and the distribution function of a non-negative integer-valued random variable X as the following results.

$$|\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)| \le \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \frac{|(k+1)q - p\sigma^2 w(k)|p(k)}{k} + |q/p - \mu|\,[1 - p_X(0)q], \tag{1.3}$$

for $A = \{0, ..., x_0\}$ and $x_0 \in \mathbb{N} \cup \{0\}$,

$$|\mathbb{P}(X \le x_0) - \mathbb{P}(Y \le x_0)| \le \sum_{k\in\mathcal{S}(x)} \left| q - \frac{\sigma^2 w(k)p}{k+1}\right| p(k) + |q/p - \mu| \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \frac{p(k)}{k} \tag{1.4}$$

and, for $0 < q \le 1/2$,

$$|\mathbb{P}(X \le x_0) - \mathbb{P}(Y \le x_0)| \le \frac{1}{x_0+1} \sum_{k\in\mathcal{S}(x)} \left|(k+1)q - \sigma^2 w(k)p\right| p(k) + |q/p - \mu|\,\frac{1 - p_X(0)}{x_0+1}. \tag{1.5}$$

Let us consider the results in (1.2) and (1.3), for $A = \{x_0\}$ as $x_0 \in \mathcal{S}(x)$, then the two point metrics and corresponding upper bounds satisfied this condition are as the following.

$$|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| \leq \frac{(1-p^r)p}{rq}\mathbb{E}\left|(r+X)q/p - \sigma^2 w(X)\right| + (1-p_X(0))\,|rq/p - \mu| \tag{1.6}$$

and

$$|p_X(x_0) - \mathrm{G}_p(x_0)| \leq \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \frac{|(k+1)q - p\sigma^2 w(k)|p(k)}{k} + |q/p - \mu|\,[1 - p_X(0)q], \tag{1.7}$$

where $\mathrm{NB}_{r,p}(x_0) = \frac{\Gamma(r+x_0)}{\Gamma(r)x_0!}q^{x_0}p^r$ and $\mathrm{G}_p(x_0) = pq^{x_0}$ at x_0. It is observed that the upper bounds in (1.6) and (1.7) can be seen to be uniform with respect to x_0, or they do not depend on x_0. Hence, the uniform upper bounds in (1.6) and (1.7) may not be sufficiently good for measuring the accuracy of these approximations. In this case, a non-uniform upper bound for each result, which depends on x_0, is required. In this study, we determine two non-uniform upper bounds for the point metrics $|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)|$ and $|p_X(x_0) - \mathrm{G}_p(x_0)|$ as $x_0 \in \mathcal{S}(x)$. The tools for giving our results consist of the so-called w-functions and Steins method for the negative binomial distribution, which are in Section 2. In Section 3, we use Stein's method and w-functions to give each result in terms of the point metric together with its non-uniform upper bound. For applications, we use the obtained results to approximate some discrete distributions such as Pólya, negative Pólya, hypergeometric and negative hypergeometric distributions, these are presented in the last section.

2. Method

In order to give main results of negative binomial and geometric approximations, we use the same methodology as in Teerapabolarn [10], which consists of Stein's method and w-functions as the following.

For w-functions, Cacoullos and Papathanasiou [3] defined a function w associated with non-negative integer-valued random variable X in the relation

$$w(x)p_X(x) = \frac{1}{\sigma^2}\sum_{i=0}^{x}(\mu - i)p_X(i), \; x \in \mathcal{S}(x) \tag{2.1}$$

and, afterwards, Majsnerowska [5] expressed the relation (2.1) as the form

$$w(0) = \frac{\mu}{\sigma^2},\ w(x) = \frac{1}{\sigma^2}\left\{\mu + \frac{\sigma^2 w(x-1)p_X(x-1)}{p_X(x)} - x\right\},\ x \in \mathcal{S}(x)\setminus\{0\} \tag{2.2}$$

and

$$w(x) \geq 0,\ x \in \mathcal{S}(x), \tag{2.3}$$

where $p_X(x) > 0$ for every $x \in \mathcal{S}(x)$. The following relation is an important property for proving the result, which was stated by Cacoullos and Papathanasiou [3].

If a non-negative integer-valued random variable X is defined as in Section 1, then

$$\mathbb{E}[(X-\mu)f(X)] = \sigma^2\mathbb{E}[w(X)\Delta f(X)], \tag{2.4}$$

for any function $f : \mathbb{N}\cup\{0\} \to \mathbb{R}$ for which $\mathbb{E}|w(X)\Delta f(X)| < \infty$, where $\Delta f(x) = f(x+1) - f(x)$. For $f(x) = x$, we have that $\mathbb{E}[w(X)] = 1$.

For Stein's method, Stein [8] introduced a powerful and general method for bounding the error in the normal approximation. This method was first developed and applied in the setting of the Poisson approximation by Chen [4]. Brown and Phillips [2] applied Stein's method to the negative binomial case. Stein's equation for the negative binomial distribution with parameters $r > 0$ and $p = (1-q) \in (0,1)$ is, for given h, of the form

$$h(x) - \mathcal{NB}_{r,p}(h) = q(r+x)f(x+1) - xf(x), \tag{2.5}$$

where $\mathcal{NB}_{r,p}(h) = \sum_{k=0}^{\infty} h(k)\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k$ and f and h are bounded real-valued functions defined on $\mathbb{N}\cup\{0\}$.

For $A \subseteq \mathbb{N}\cup\{0\}$, let $h_A : \mathbb{N}\cup\{0\} \to \mathbb{R}$ be defined by

$$h_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{2.6}$$

By following Brown and Phillips [2] and Teerapabolarn and Boondirek [11], for $A = \{x_0\}$, $x_0 \in \mathbb{N}\cup\{0\}$, the solution $f_{x_0} = f_{\{x_0\}}$ of (2.5) can be written as

$$f_{x_0}(x) = \begin{cases} \frac{-\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(h_{C_{x-1}})}{x\mathcal{NB}_{r,p}(h_x)} & \text{if } x \leq x_0, \\ \frac{\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(1-h_{C_{x-1}})}{x\mathcal{NB}_{r,p}(h_x)} & \text{if } x > x_0, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.7}$$

where $C_x = \{0, ..., x\}$ and $h_{x_0} = h_{\{x_0\}}$.

The following lemma presents some properties of function f_{x_0} in (2.7), which are also need for proving the related results.

Lemma 2.1. *For $x_0 \in \mathbb{N} \cup \{0\}$ and $x \in \mathbb{N}$, let $\Delta f_{x_0}(x) = f_{x_0}(x+1) - f_{x_0}(x)$, then we have the following.*

1. If $r \geq 1$, then

$$|f_{x_0}(x)| \leq \begin{cases} \frac{1-p^r}{rq} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} & \text{if } x_0 > 0 \end{cases} \tag{2.8}$$

and

$$|\Delta f_{x_0}(x)| \leq \begin{cases} \frac{1-p^r}{rq} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} & \text{if } x_0 > 0. \end{cases} \tag{2.9}$$

2. If $r = 1$, then

$$|f_{x_0}(x)| \leq \frac{1}{x} \text{ and } |\Delta f_{x_0}(x)| \leq \frac{1}{x}. \tag{2.10}$$

Proof. 1. For $x_0 = 0$, it follows from Brown and Phillips [2] that f_0 is positive and decreasing in $x \in \mathbb{N}$. Therefore, by (2.7), $|\Delta f_0(x)| \leq |f_0(x)| \leq f_0(1) = \frac{1-p^r}{rq}$ for every $x \in \mathbb{N}$. For $x_0 > 0$, by following Brown and Phillips [2], f_{x_0} is positive and decreasing in $x \in \{x_0+1, x_0+2, ...\}$ and is negative and decreasing in $x \in \{1, ..., x_0\}$. Thus, we have $|f_{x_0}(x)| \leq \Delta f_{x_0}(x_0)$ and $|\Delta f_{x_0}(x)| \leq \Delta f_{x_0}(x_0)$, and we then have to show that $\Delta f_{x_0}(x_0) \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\}$. Since, by (2.7),

$$\begin{aligned} \Delta f_{x_0}(x_0) &= \frac{\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(1-h_{C_{x_0}})}{(x_0+1)\mathcal{NB}_{r,p}(h_{x_0+1})} + \frac{\mathcal{NB}_{r,p}(h_{x_0})\mathcal{NB}_{r,p}(h_{C_{x_0-1}})}{x_0\mathcal{NB}_{r,p}(h_{x_0})} \\ &= \frac{1}{(r+x_0)q}\sum_{k=x_0+1}^{\infty}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k + \frac{1}{x_0}\sum_{k=0}^{x_0-1}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k \qquad (2.11) \\ &= \frac{1}{x_0}\left\{\sum_{k=x_0+1}^{\infty}\frac{x_0(r+k-1)}{k(r+x_0)}\frac{\Gamma(r+k-1)}{\Gamma(r)(k-1)!}p^r q^{k-1} \right. \\ &\quad \left. + \sum_{k=0}^{x_0-1}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k\right\} \end{aligned}$$

$$\le \frac{1}{x_0}\left\{\sum_{k=x_0}^{\infty}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k + \sum_{k=0}^{x_0-1}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k\right\}$$

$$= \frac{1}{x_0}$$

and, by (2.11),

$$\Delta f_{x_0}(x_0) \le \frac{1}{(r+x_0-1)q}\left\{\sum_{k=x_0+1}^{\infty}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k\right.$$

$$\left. + \sum_{k=0}^{x_0-1}\frac{(r+x_0-1)(k+1)}{(r+k)x_0}\frac{\Gamma(r+k+1)}{\Gamma(r)(k+1)!}p^r q^{k+1}\right\}$$

$$\le \frac{1}{(r+x_0-1)q}\left\{\sum_{k=x_0+1}^{\infty}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k + \sum_{k=1}^{x_0}\frac{\Gamma(r+k)}{\Gamma(r)k!}p^r q^k\right\}$$

$$= \frac{1-p^r}{(r+x_0-1)q},$$

we obtain $|f_{x_0}(x)| \le \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\}$ and $|\Delta f_{x_0}(x)| \le \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\}$, which implies the inequalities (2.8) and (2.9).

2. It follows the fact that $|f_{x_0}(x)| \le \Delta f_x(x)$ and $|\Delta f_{x_0}(x)| \le \Delta f_x(x)$ and, by the proof of 1, $\Delta f_x(x) \le \frac{1}{x}$. $\square$

3. Results

The following theorem shows a result of negative binomial approximation to the distribution of a non-negative integer-valued random variable X in terms of the point metric together with its non-uniform upper bound, which is determined by Stein's method and w-functions.

Theorem 3.1. *Let a non-negative integer-valued random variable X together with corresponding w-function $w(X)$ be defined as above. Then, for $r \ge 1$, the following inequalities hold:*

1. For $x_0 = 0$,

$$|p_X(0) - \mathrm{NB}_{r,p}(0)| \le \frac{(1-p^r)p}{rq}\left\{\mathbb{E}\left|(r+X)q/p - \sigma^2 w(X)\right|\right.$$

$$+(1-p_X(0))\left|rq/p-\mu\right|\} \quad (3.1)$$

and, if $rq/p=\mu$, *then*

$$|p_X(0)-\mathrm{NB}_{r,p}(0)|\leq\frac{(1-p^r)p}{rq}\mathbb{E}\left|(r+X)q/p-\sigma^2w(X)\right|. \quad (3.2)$$

2. For $x_0\in\mathcal{S}(x)\setminus\{0\}$,

$$|p_X(x_0)-\mathrm{NB}_{r,p}(x_0)|\leq\min\left\{\frac{1}{x_0},\frac{1-p^r}{(r+x_0-1)q}\right\}p\left\{\mathbb{E}\left|(r+X)q/p-\sigma^2w(X)\right|\right.$$
$$\left.+(1-p_X(0))\left|rq/p-\mu\right|\right\} \quad (3.3)$$

and, if $rq/p=\mu$, *then*

$$|p_X(x_0)-\mathrm{NB}_{r,p}(x_0)|$$
$$\leq\min\left\{\frac{1}{x_0},\frac{1-p^r}{(r+x_0-1)q}\right\}p\mathbb{E}\left|(r+X)q/p-\sigma^2w(X)\right|. \quad (3.4)$$

Proof. Substituting h by h_{x_0}, x by X and taking expectation in (2.5), it becomes

$$p_X(x_0)-\mathrm{NB}_{r,p}(x_0)=\mathbb{E}[q(r+X)f(X+1)-Xf(X)], \quad (3.5)$$

where $f=f_{x_0}$ is defined in (2.7) and

$$\begin{aligned}\mathbb{E}[q(r+X)f(X+1)-Xf(X)]&=\mathbb{E}[rqf(X+1)+qX\Delta f(X)-pXf(X)]\\&=rq\mathbb{E}[f(X+1)]+q\mathbb{E}[X\Delta f(X)]\\&\quad-p\mathbb{E}[Xf(X)]\\&=rq\mathbb{E}[f(X+1)]+q\mathbb{E}[X\Delta f(X)]\\&\quad-p\{\mathbb{E}[(X-\mu)f(X)]+\mu\mathbb{E}[f(X)]\}\\&=rq\mathbb{E}[\Delta f(X)]+q\mathbb{E}[X\Delta f(X)]\\&\quad+(rq-p\mu)\mathbb{E}[f(X)]-p\mathbb{E}[(X-\mu)f(X)].\end{aligned}$$

Since $\mathbb{E}[w(X)]=1$ and $\mathbb{E}|w(X)\Delta f(X)|=\mathbb{E}[w(X)|\Delta f(X)|]<\infty$. Thus, by (2.4), it follows that

$$\begin{aligned}\mathbb{E}[q(r+X)f(X+1)-Xf(X)]&=rq\mathbb{E}[\Delta f(X)]+q\mathbb{E}[X\Delta f(X)]\\&\quad+(rq-p\mu)\mathbb{E}[f(X)]-p\mathbb{E}[\sigma^2w(X)\Delta f(X)]\\&=\mathbb{E}\{[(r+X)q-\sigma^2w(X)p]\Delta f(X)\}\end{aligned}$$

$$+ (rq - p\mu)\mathbb{E}[f(X)],$$

which, by (3.5), gives

$$\begin{aligned}
|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| &= |\mathbb{E}\{[(r+X)q - \sigma^2 w(X)p]\Delta f(X)\} + (rq - p\mu)\mathbb{E}[f(X)]| \\
&\le \mathbb{E}\{|(r+X)q - \sigma^2 w(X)p||\Delta f(X)|\} + |rq - p\mu|\mathbb{E}|f(X)| \\
&\le \sup_{x\ge 1}|\Delta f(x)|\mathbb{E}|(r+X)q - \sigma^2 w(X)p| + |rq - p\mu|\mathbb{E}|f(X)|
\end{aligned}$$

and

$$\begin{aligned}
|rq - p\mu|\mathbb{E}|f(X)| &= |rq - p\mu| \sum_{x\in\mathcal{S}(x)\setminus\{0\}} |f(x)|p_X(x) \\
&\le |rq - p\mu| \sup_{k\ge 1}|f(k)| \sum_{x\in\mathcal{S}(x)\setminus\{0\}} p_X(x) \\
&= |rq - p\mu|(1 - p_X(0)) \sup_{k\ge 1}|f(k)|.
\end{aligned}$$

Hence, by using Lemma 2.1 (1), the theorem is proved. □

Immediately from the Theorem 3.1, we have the following corollary.

Corollary 3.1. *For $r \ge 1$, if $(n+x)q/p - z(x) \ge / < 0$ for every $x \in \mathcal{S}(x)$, then*

1. For $x_0 = 0$,

$$|p_X(0) - \mathrm{NB}_{r,p}(0)| \le \frac{(1-p^r)p}{rq}\left\{\left|(r+\mu)q/p - \sigma^2\right| + (1 - p_X(0))\left|rq/p - \mu\right|\right\} \quad (3.6)$$

and, if $rq/p = \mu$, then

$$|p_X(0) - \mathrm{NB}_{r,p}(0)| \le \frac{(1-p^r)p}{rq}\left|(r+\mu)q/p - \sigma^2\right|. \quad (3.7)$$

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| \le \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} p\left\{\left|(r+\mu)q/p - \sigma^2\right| + (1 - p_X(0))\left|rq/p - \mu\right|\right\} \quad (3.8)$$

and, if $rq/p = \mu$, *then*

$$|p_X(x_0) - \text{NB}_{r,p}(x_0)| \leq \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} p\left|(r+\mu)q/p - \sigma^2\right|. \quad (3.9)$$

In case of $r = 1$, by using Lemma 2.1 (1 and 2) and the same arguments detailed as in the proof of Theorem 3.1, a result of geometric approximation for a non-negative integer-valued random variable X, in terms of the point metric together with its non-uniform upper bound, can be obtained as the following corollary.

Corollary 3.2. *For geometric approximation, we have the following.*
1. For $x_0 = 0$ (Theorem 2.2 in Teerapabolarn [10]),

$$|p_X(0) - \text{G}_p(0)| \leq \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \left|\frac{q}{k} - \frac{\sigma^2 w(k)p}{k(k+1)}\right| p(k) + |q - p\mu| \left\{p(0) + \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \frac{p(k)}{k}\right\} \quad (3.10)$$

and, if $q/p = \mu$, *then*

$$|p_X(0) - \text{G}_p(0)| \leq \sum_{k\in\mathcal{S}(x)\setminus\{0\}} \left|\frac{q}{k} - \frac{\sigma^2 w(k)p}{k(k+1)}\right| p(k). \quad (3.11)$$

2. For $x_0 \in \mathcal{S}(x) \setminus \{0\}$,

$$|p_X(x_0) - \text{G}_p(x_0)| \leq \sum_{x\in\mathcal{S}(x)\setminus\{0\}} \min\left\{\frac{1}{x}, \frac{1}{x_0}\right\} p\left\{\left|(1+x)q/p - \sigma^2 w(x)\right| + |q/p - \mu|\right\} p_X(x) + p|q/p - \mu| p_X(0) \quad (3.12)$$

and, if $q/p = \mu$, *then*

$$|p_X(x_0) - \text{G}_p(x_0)| \leq \sum_{x\in\mathcal{S}(x)\setminus\{0\}} \min\left\{\frac{1}{x}, \frac{1}{x_0}\right\} p\left|(1+x)q/p - \sigma^2 w(x)\right| p_X(x). \quad (3.13)$$

Remark. *By comparing the results in* (1.6) *and* (1.7) *(uniform upper bounds) with the the results in Theorem 3.1 and Corollary 3.2 (non-uniform upper bounds), it is found that the results in Theorem 3.1 and Corollary 3.2 are better than the results in* (1.6) *and* (1.7).

4. Applications

For applications of this study, we use the results in Theorem 3.1 and Corollary 3.2 to approximate some distributions such as Pólya, negative Pólya, hypergeometric and negative hypergeometric distributions.

4.1. An Approximation to Pólya Distribution

Suppose that a single urn contain r black balls and $N-r$ white balls, and draw a ball at random with extra replacement, that is, if a black ball is drawn replace it together with another black ball, and similarly for a white ball. Do this way for m draws and let X be the number of black balls drawn in the m drawings, then the distribution of X is a Pólya distribution with parameters N, m and r. The probability function of X is given by

$$p_X(x) = \frac{\binom{r+x-1}{x}\binom{N-r+m-x-1}{m-x}}{\binom{N+m-1}{m}}, \ x = 0, 1, ..., m$$

and the mean and variance of X are $\mu = \frac{rm}{N}$ and $\sigma^2 = \frac{rm(N+m)(N-r)}{N^2(N+1)}$, respectively.

By relation (2.2), we have $w(x) = \frac{(r+x)(m-x)}{N\sigma^2}$. Setting $p = \frac{N}{N+m}$ in Theorem 3.1, it follows that $\frac{(r+x)\mu}{r} - \sigma^2 w(x) = \frac{(r+x)x}{N} \geq 0$ for all $0 \leq x \leq m$ and, by Corollaries 3.1 and 3.2, we have the following results.

Corollary 4.1. *If $p = \frac{N}{N+m}$, then we have the following.*

$$|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| \leq \begin{cases} \frac{(1-p^r)(r+1)}{(N+1)p} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} \frac{r(r+1)m}{N(N+1)} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

Corollary 4.2. *For $r = 1$,*

$$|p_X(x_0) - \mathrm{G}_p(x_0)| \leq \begin{cases} \frac{m}{(N+m-1)(N+m)} & \text{if } x_0 = 0, \\ \frac{m}{N} \min\left\{\frac{2N+m-1}{(N+m-1)(N+m)}, \frac{2}{(N+1)x_0}\right\} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

It is noted that, the result in Corollary 4.1 gives a good approximation if N is large and m and r are small.

4.2. An Approximation to Negative Pólya Distribution

Let an urn contain n red and m black balls. Draw a ball at random, note the color, and return it into the urn together with another ball of the same color. Repeat this way until the number of red balls reaches a fixed number r. Let X be the number of black balls before r^{th} red ball, then the distribution of X is a negative Pólya distribution with parameters m, n and r. The probability function of X is given by

$$p_X(x) = \binom{r+x-1}{x} \frac{m(m+1)\cdots(m+x-1)n(n+1)\cdots(n+r-1)}{(m+n)(m+n+1)\cdots(m+n+r+x-1)},$$

where $x \in \mathbb{N} \cup \{0\}$ and $\mu = \frac{rm}{n-1}$ and $\sigma^2 = \frac{rm(r+n-1)(n+m-1)}{(n-2)(n-1)^2}$ are mean and variance of X, respectively.

Following the relation (2.2), we have $w(x) = \frac{(r+x)(m+x)}{(n-1)\sigma^2}$. Let $p = \frac{n-1}{n+m-1}$, then $\frac{(r+x)\mu}{r} - \sigma^2 w(x) = -\frac{(r+x)x}{n-1} \leq 0$ for all $n > 2$ and $x \geq 0$. Applying the Corollaries 3.1 and 3.2, we can get two required results.

Corollary 4.3. *For $n > 2$, if $p = \frac{n-1}{n+m-1}$, then*

$$|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| \leq \begin{cases} \frac{(1-p^r)(r+1)}{(n-2)p} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} \frac{r(r+1)m}{(n-2)(n-1)} & \text{if } x_0 \in \mathbb{N}. \end{cases}$$

Corollary 4.4. *For $r = 1$ and $n > 2$,*

$$|p_X(x_0) - \mathrm{G}_p(x_0)| \leq \begin{cases} \frac{m}{(n+m-1)(n+m)} & \text{if } x_0 = 0, \\ \frac{m}{n-1} \min\left\{\frac{2n+m-1}{(n+m-1)(n+m)}, \frac{2}{(n-2)x_0}\right\} & \text{if } x_0 \in \mathbb{N}. \end{cases}$$

Similarly, the result in Corollary 4.3 gives a good approximation if n is large and m and r are small.

4.3. An Approximation to Hypergeometric Distribution

Suppose a random sample of size m is taken from a finite population containing r elements of type $\mathcal{I}$ and $N-r$ (> 0) elements of type $\mathcal{II}$. Let X be the number of type $\mathcal{I}$ elements in the sample. Then X has the hypergeometric distribution with parameters N, m and r, and has probabilities

$$p_X(x) = \frac{\binom{r}{x}\binom{N-r}{m-x}}{\binom{N}{m}}, \ x = 0, 1, ..., \min\{m, r\},$$

and its mean and variance are $\mu = \frac{mr}{N}$ and $\sigma^2 = \frac{mr(N-m)(N-r)}{N^2(N-1)}$, respectively. It is well-known that a hypergeometric distribution can be approximated by Poisson and binomial distributions. However, in this application, we give a result of negative binomial approximation to hypergeometric distribution as follows.

Using the relation (2.2), we then obtain $w(x) = \frac{(m-x)(r-x)}{N\sigma^2} = \frac{mr-x(m+r-x)}{N}$. Let $p = \frac{N}{N+m}$, then $\frac{(r+x)\mu}{r} - \sigma^2 w(x) = \frac{x(2m+r-x)}{N} \geq 0$ for all $0 \leq x \leq \min\{m, r\}$. By applying the Corollary 3.1, a result of this approximation can be expressed as the following.

Corollary 4.5. *If* $p = \frac{N}{N+m}$, *then, for* $x_0 \leq \min\{m, r\}$,

$$|p_X(x_0) - \mathrm{NB}_{r,p}(x_0)| < \begin{cases} \frac{(1-p^r)(2m+r-1)}{N-1} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} \frac{rm(2m+r-1)}{(N-1)(N+m)} & \text{if } 1 \leq x_0. \end{cases}$$

It is observed that the negative binomial distribution with parameters r and $p = \frac{N}{N+m}$ can be used as an approximation of the hypergeometric distribution with parameters N, m and r whenever $\frac{r}{N}$ and $\frac{m}{N}$ are sufficiently small.

4.4. An Approximation to Negative Hypergeometric Distribution

Suppose a finite population contain m elements of type $\mathcal{I}$ and $N - m$ elements of type $\mathcal{II}$. If elements in a random sample are drawn without replacement from this population until the number of types $\mathcal{II}$ elements reaches a fixed positive integer r. Let X be the number of types $\mathcal{I}$ elements in the sample. Then X has a negative hypergeometric distribution with parameters N, m and r, and its probability function can be expressed as

$$p_X(x) = \frac{\binom{r+x-1}{x}\binom{N-r-x}{m-x}}{\binom{N}{m}}, \; x = 0, 1, ..., m,$$

where $r \in \{1, ..., N - m\}$ and $\mu = \frac{rm}{N-m+1}$ and $\sigma^2 = \frac{rm(N-m-r+1)(N+1)}{(N-m+1)^2(N-m+2)}$ are the mean and variance of X, respectively.

By following the relation (2.2), we have $w(x) = \frac{(r+x)(m-x)}{(N-m+1)\sigma^2}$. Setting $p = \frac{N-m+1}{N+1}$ in Theorem 3.1, then $\frac{(r+x)\mu}{r} - \sigma^2 w(x) = \frac{(r+x)x}{N-m+1} \geq 0$ for all $0 \leq x \leq m$. By applying the Corollaries 3.1 and 3.2, the two results are obtained as follows:

Corollary 4.6. *If* $p = \frac{N-m+1}{N+1}$, *then we have the following.*

$$|p_X(x_0) - \text{NB}_{r,p}(x_0)| \leq \begin{cases} \frac{(1-p^r)(r+1)}{(N-m+2)p} & \text{if } x_0 = 0, \\ \min\left\{\frac{1}{x_0}, \frac{1-p^r}{(r+x_0-1)q}\right\} \frac{r(r+1)m}{(N-m+1)(N-m+2)} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

Corollary 4.7. *For* $r = 1$,

$$|p_X(x_0) - \text{G}_p(x_0)| \leq \begin{cases} \frac{m}{N(N+1)} & \text{if } x_0 = 0, \\ \frac{m}{N-m+1} \min\left\{\frac{2N-m+1}{N(N+1)}, \frac{2}{(N-m+2)x_0}\right\} & \text{if } 1 \leq x_0 \leq m. \end{cases}$$

For this application, a negative hypergeometric distribution can be approximated well by a negative binomial distribution if we set their parameters to be satisfied ($p = \frac{N-m+1}{N+1}$) under appropriate conditions (N is large and m and r are small).

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