A BOUNDEDNESS RESULT FOR PFAFF FIELDS

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Abstract: Fix positive integers $s, d, n$ with $n \geq s + 1$. Let $W \subseteq \mathbb{P}^n$ be an integral and Gorenstein projective variety of dimension $s + 1$ such that $\dim(\text{Sing}(W)) \leq s - 1$. Fix $M, H \in \text{Pic}(W)$ with $H$ ample. Here we prove the existence of an integer $x_0(H, d, M)$ with the following property. Fix any integer $x \geq x_0(H, d, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\text{Sing}(X)) \leq s - 2$; then there is no non-zero Pfaff field $\Omega^s_X \to \mathcal{O}_X(d)$. In particular $X$ is not a solution of a rank $s$ and degree $d$ Pfaff field on $\mathbb{P}^n$ whose singular locus does not contain $X$.

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Esteves and Kleiman introduced the general set-up of Pfaff systems and Pfaff fields with the right amount of generality ([5]). Let $Y$ be an equidimensional reduced projective variety defined over an algebraically closed field $\mathbb{K}$. Set $m := \dim(Y)$. Fix an integer $s \in \{1, \ldots, m\}$. A rank $s$ Pfaff field on $Y$ is a map $\Omega^s_Y \to L$, where $L \in \text{Pic}(Y)$. If $Y \subset \mathbb{P}^n$, and $\eta_1 : \Omega^s_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)$ is a non-zero rank $s$ Pfaff field, we say that $Y$ is a leaf of $\eta_1$ if $Y$ is not completely contained in singular locus of $\eta_1$ and $\eta_1$ factors through the natural map $\Omega^s_{\mathbb{P}^n} \to \Omega^s_Y$. A question of Poincaré aims to bounds $\deg(Y)$ in terms of $d$ for any leaf $Y$. As it stands the answer is negative and to give a positive answer we must both bounds the admissible singularities of $Y$ and the cohomology groups of $\mathcal{I}_Y$ ([2], [3], [4], [5] and references therein). In particular [2], Remark 21, gives an example

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with $Y$ a smooth curve. Here we bound $\deg(Y)$ allowing only codimension 2 singularities, but prescribing that the leaf varies in a fixed $(s + 1)$-dimensional variety. The case $s = n - 1$ is classical (see e.g. [6], [2], [3], [4], [5]).

**Theorem 1.** Fix an integer $s \geq 1$. Let $W$ be an integral and Gorenstein projective variety of dimension $s + 1$ such that $\dim(\text{Sing}(W)) \leq s - 1$. Fix $L, M, H \in \text{Pic}(W)$ with $H$ ample. Then there exists an integer $x_0(H, L, M)$ with the following property. Fix any integer $x \geq x_0(H, L, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\text{Sing}(X)) \leq s - 2$. Then there is no non-zero Pfaff field $\Omega^*_X \to L|X$.

Taking $W \subset \mathbb{P}^n$, $n \geq s + 1$ and $L := \mathcal{O}_W(d)$ for some integer $d$ we immediately get the following result.

**Corollary 1.** Fix positive integers $s, d, n$ with $n \geq s + 1$. Let $W \subset \mathbb{P}^n$ be an integral and Gorenstein projective variety of dimension $s + 1$ such that $\dim(\text{Sing}(W)) \leq s - 1$. Fix $M, H \in \text{Pic}(W)$ with $H$ ample. Then there exists an integer $x_0(H, d, M)$ with the following property. Fix any integer $x \geq x_0(H, d, M)$ and any integral $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\text{Sing}(X)) \leq s - 2$. Then there is no non-zero Pfaff field $\Omega^*_X \to \mathcal{O}_X(d)$. In particular $X$ is not a solution of a rank $s$ and degree $d$ Pfaff field on $\mathbb{P}^n$ whose singular locus does not contain $X$.

**Proof of Theorem 1.** Fix $x \in \mathbb{Z}$ such that there is $X \in |M \otimes H^{\otimes x}|$ such that $\dim(\text{Sing}(X)) \leq s - 2$ with $\dim(\text{Sing}(X)) \leq s - 2$ and $\eta : \Omega^*_X \to L|X$ such that $\eta \neq 0$. Hence $\text{Coker}(\eta)$ is supported by a closed subscheme of $X$ with dimension at most $s - 1$. For any coherent sheaf $F$ on $X$ let $T(F)$ denote the torsion subsheaf of $F$. Since $L$ has no torsion, $\eta$ induces a non-zero map $\eta' : \Omega^*_X/T(\Omega^*_X) \to L$ such that $\text{Im}(\eta') = \text{Im}(\eta)$, i.e. $\text{Coker}(\eta) = \text{Coker}(\eta')$. Thus the map $\eta' : H^s(X, \Omega^*_X/T(\Omega^*_X)) \to H^s(X, L)$ is surjective. In [1], subsection 3.1, the authors defined a scheme-structure on the algebraic set $\text{Sing}(X)$ using the natural map $\Omega^*_X \to \omega_X$, which gives an injective map $\Omega^*_X/T(\Omega^*_X) \to \omega_X$, because $\omega_X$ has no torsion ([1], 3.1). Call $\Sigma_X$ this scheme-structure on $\text{Sing}(X)$. Since $W$ is Gorenstein and $X$ is a Cartier divisor of $W$, $X$ is Gorenstein. Thus $\Omega^*_X/T(\Omega^*_X) \cong \mathcal{I}_{\Sigma_X} \omega_X$. Hence $H^s(X, \Omega^*_X/T(\Omega^*_X)) \cong H^s(X, \mathcal{I}_{\Sigma_X} \omega_X)$. Since $\dim(\Sigma(X)) \leq s - 2$, a standard exact sequence gives $H^s(X, \mathcal{I}_{\Sigma_X} \omega_X) \cong H^s(\omega_X)$. Duality gives that the latter vector space has dimension at most 1. Hence $h^s(X, L|X) \leq 1$. Thus $h^0(X, \omega_X \otimes (L|X)^*) \leq 1$ (duality). Since $X$ is a Cartier divisor of $W$, we have $\omega_X \cong \omega_W \otimes (M \otimes H^{\otimes x}|X)$. Look at the exact sequence of coherent sheaves on $W$:

$$0 \to L^* \otimes \omega_W \to L^* \to L^* \otimes \omega_W \otimes M \otimes H^{\otimes x} \to \omega_X \otimes (L^*|X) \to 0 \quad (1)$$
The integer $h^0(W, L \otimes \omega_W)$ does not depend from $x$. For $x \gg 0$ the integer $h^0(W, L^* \otimes \omega_W \otimes M \otimes H^{\otimes x})$ is arbitrarily large, because $H$ is ample. Thus for $x \gg 0$ we get $h^0(X, \omega_X \otimes (L|X)^*) \geq 2$, contradiction. 

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References


