International Journal of Pure and Applied Mathematics
Volume 70 No. 1 2011, 11-14

REBUTTAL TO THE REVIEW OF MY PAPER "An implication of Gödel's incompleteness theorem" APPEARED IN ZENTRALBLATT FÜR MATHEMATIK

Hitoshi Kitada
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Meguro, Tokyo 153-8914, JAPAN
e-mail: kitada@ms.u-tokyo.ac.jp

Abstract: A rebuttal to the review of Zentralblatt für Mathematik on the author's paper [2] is given. The questions raised by the reviewer are answered.

AMS Subject Classification: 03F40, 03F15, 03B25, 03E99
Key Words: consistency, incompleteness, Gödel, transfinite extension

*

A reviewer of Zentralblatt für Mathematik wrote at http://www.zentralblatt-math.org/zmath/?q=an:05575645&type=pdf&format=complete as follows on the author's paper "An implication of Gödel's incompleteness theorem" [2].

> "This long paper gives a classic proof of Gödel's First Incompleteness Theorem, in the lines of Rosser's proof which cleverly avoids the assumption of ω-consistency (and uses only the consistency of the sufficiently strong theories). Almost 38 pages (511–549) are devoted to this task. Though the author mentions in the abstract of the paper that his proof is "new", it is not clear which part (or technique) of this proof is new; it is pretty similar to the proof of Gödel's theorem in textbooks. The author also mentions in the abstract that, by considering a transfinite extension of Gödel's theorem, he shows that assuming the set theory ZFC on the meta level

Received: November 1, 2010

> (like on the object level) a contradiction arises. The arguments of the paper in Sections 7 and 8 are roughly as below. For a sufficiently strong (arithmetical) theory $S = S_0$, by Gödel's First Incompleteness Theorem, there exists a sentence G_0 which is undecidable in S_0 (i.e., $S_0 \not\vdash G_0$ and $S_0 \not\vdash \neg G_0$). Now let $S_1 = S_0 \cup \{G_0\}$, and go on by induction, $S_{n+1} = S_n \cup \{G_n\}$ for an undecidable G_n, and finally put $S_\omega = \bigcup_n S_n$. Again, by applying the theorem for S_ω, we get an undecidable sentence G_ω, and then put $S_{\omega+1} = S_\omega \cup \{G_\omega\}$. Continue this on transfinite ordinals by taking $S_{\alpha+1}$ to be an extension of S_α by an S_α-undecidable sentence (like G_α), and for limit ordinal λ, let $S_\lambda = \bigcup_{\alpha<\lambda} S_\alpha$. Then the author claims to have proved (in Theorem 7.5) that for a countable limit ordinal β the system S_β has no undecidable proposition, and S_β is complete. Thus, any extension of S_β is inconsistent. Then, in Subsection 7.5, it is argued that this leads to a contradiction, since β is a countable limit ordinal and thus $\beta = \lim \alpha_n$ for a monotone increasing sequence of countable ordinals $\{\alpha_n\}$. But then S_β is the union of all S_α's, and then we can construct an S_β-undecidable sentence just like the way we got an undecidable sentence G_ω above. But this is a contradiction, since S_β is complete! Then the author draws some strange (and never heard before) philosophical conclusions. The point that the author misses, is that the reason that Gödel's theorem cannot find an undecidable sentence for S_β is not that it is complete! It is because S_β fails to be a recursively enumerable (r.e.) theory (which is required for sufficiently strong theories in Gödel's Incompleteness Theorems)."

In the rest of the review the reviewer gives an example of an incomplete but non-r.e. theory, for which Gödel's theorem cannot find an undecidable sentence. This will be clarified to be completely out of context and is omitted here.

The point which the reviewer assumes that his denial of the paper is right is exactly in the following part.

> "The point that the author misses, is that the reason that Gödel's theorem cannot find an undecidable sentence for S_β is not that it is complete! It is because S_β fails to be a recursively enumerable (r.e.) theory."

Obviously the reviewer overlooks the fundamental assumption made at the beginning of Subsection 7.1 on page 553 as follows.

> We now consider a formal set theory S equivalent to ZFC, and assume that we can use the same set theory ZFC also on the meta level.

which is reminded in the footnote 11 on page 556 in Subsection 7.3:

> However as we assume ZFC on the meta level, we can make this decision by the axioms of ZFC even if we cannot make this decision recursively.

If one assumes the axiom of choice with other axioms of set theory. it is well-known that one can construct every ordinal by a transfinite induction. The theory S_β has therefore an undecidable sentence under the axioms of ZFC, contradicting that S_β is complete.

After giving a long description of an example of an incomplete but non-r.e. theory, the reviewer wrote as follows.

> "In conclusion, the first six sections of the paper contain a proof of a famous theorem that can be found in several textbooks, and has nothing new to offer, and the last two sections are some wrong conclusions and mistaken propositions. Thus the paper may mislead the young researchers instead of being helpful for them, contrary to what is hoped at the end of Section 1. [Saeed Salehi (Tabriz)]"

The reviewer totally misreads the paper and overlooks the new point of the proof given in [2]. At the beginning of Section 5 on page 537, we wrote

> In the next section we will show that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ defined in Section 3 are numeralwise expressible in number theory S. To show this Gödel [4][1] proved the following theorem, and used the fact that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ are recursive.
>
> **Theorem 5.1.** For any recursive relation $\mathbf{R}(x_1,\dots,x_n)$ there exists a number-theoretic formula $r(u_1,\dots,u_n)$ with n free variables $u_1,\dots,u_n$ such that for any n-tuple of natural numbers $x_1,\dots,x_n$ the following i) and ii) hold.
>
> i) If $\mathbf{R}(x_1,\dots,x_n)$ is true, then $\vdash r(\lceil x_1\rceil,\dots,\lceil x_n\rceil)$ holds.
>
> ii) If $\mathbf{R}(x_1,\dots,x_n)$ is false, then $\vdash \neg r(\lceil x_1\rceil,\dots,\lceil x_n\rceil)$ holds.

[1] [1] in the present rebuttal.

> In this paper we do not prove this theorem. Instead we will prove directly that the predicates $\mathbf{G}(a,b)$, $\mathbf{H}(a,b)$ are numeralwise expressed by some formulae $g(a,b)$, $h(a,b)$, respectively.

Namely, Gödel assumes in his proof the recursive construction of the predicates so that his proof works only for r.e. theories. To avoid this restriction we preferred to directly translate the predicates into numeralwise expressions so that we avoided the restrictions associated with the recursive enumerability and we have the proof of Theorem 7.5 [2]. The meaning of the conclusion in the last Section 8 will be clear accordingly.

References

[1] K. Gödel, On formally undecidable propositions of Principia mathematica and related systems I, In: *Kurt Gödel Collected Works, Volume I, Publications 1929-1936*, Oxford University Press, New York, Clarendon Press, Oxford (1986), 144-195; Translated from: Über formal unentsceidebare Sätze der Principia mathematica und verwandter Systeme I, *Monatshefte für Mathematik und Physik*, **38** (1931), 173-198.

[2] H. Kitada, An implication of Gödel's incompleteness theorem, *Int. J. Pure Appl. Math.*, **52**, No. 4 (2009), 511-567.