International Journal of Pure and Applied Mathematics
Volume 70 No. 1 2011, 15-23

A NOVEL FUZZY CLUSTERING APPROACH INDUCED BY COVERING

Zhanhong Shi
College of Science
Gansu Agricultural University
Lanzhou, 730070, P.R. CHINA

Abstract: According to the similarity of objects and the corresponding criteria, we can classify the objects which studied, this mathematical method is called clustering analysis. However, there don't exist clear border between objects usually. In fact, the relationship between them is the fuzzy relationship, we can classify these objects using fuzzy mathematics method, this mathematical method is called fuzzy clustering analysis. In the covering rough set theory, classify the objects of universe of discourse according to the different standard, different coverings of universe can be obtained. In this paper, the problem of how to classify the objects of universe according to these different coverings is discussed. Firstly, the significant degree of covering and the covering degree between the objects in the universe are defined, and by viewing the significant degree of covering as its weight, the corresponding fuzzy matrix is constructed. Secondly, the fuzzy clustering approach induced by covering is built. Finally, a concrete example is employed to illustrate the effectivity of approach which presented in this paper.

AMS Subject Classification: 62H30
Key Words: fuzzy clustering, covering, significant degree

1. Introduction

The theory of rough sets, first proposed by Pawlak [1], is a new mathematical tool to handle uncertain knowledge, and has been successfully applied in pattern recognition, data mining, machine learning and so on. In Pawlak' rough set theory, the equivalence classes are the building blocks for the construction of

Received: January 5, 2011

the lower and the upper approximations. However, the equivalence relation or partition in Pawlak rough set theory can only deal with complete information systems, which limits the application scope of the theory to a certain extent. To address this issue, some authors have extended equivalence relation to tolerance relations [4], similarity relations [5], ordinary binary relations [7]. Meanwhile, some authors have adopted the approach of relaxing the partition of the universe to a covering and obtain the covering rough sets [2,3,6,8,10,].

Clustering analysis is used for clustering a data set into groups of similar individuals. It is an unsupervised learning approach, and is one of the major techniques in pattern recognition.

In covering-based rough set theory, if classifying the objects in universe according to the different standards, we can obtain different coverings of universe. Naturally, we should consider how to classify these objects again according to these different coverings. Certainly, the covering significant degree must be considered. In this paper, according to the covering significant degree, a novel fuzzy clustering approach is proposed.

2. Basic Concepts

Definition 2.1. (see [10]) Let U be a universe of discourse, $C = \{X | X \subseteq U\}$ a family of subsets of U. If no element of C is empty, and $\cup_{X \in C} X = U$, then C is called a covering of U. We call the ordered pair (U, C) a covering approximation space.

Definition 2.2. (see [10]) Let (U, C) be a covering approximation space, $x \in U$, then the set family

$$Md(x) = \{K \in C | x \in K \wedge (\forall S \in C \wedge x \in S \wedge S \subseteq K \Rightarrow K = S)\}$$

is called the minimal description of x.

Definition 2.3. (see [6]) Let (U, C) be a covering approximation space, U be a nonempty and finite set called universe of discourse, C be a covering of U. Then for any $X \subseteq U$, the lower and the upper approximations of X with respect to approximation space (U, C) are defined as follows:

$$\underline{C}(X) = \{x \in U | (\cap Md(x)) \subseteq X\}, \qquad \overline{C}(X) = \{x \in U | (\cap Md(x)) \cap X \neq \phi\}.$$

Definition 2.4. (see [9]) Let $(X, \leq)$ be a partial set, a real function D: $X \times X \to [0, 1]$ is called an inclusion degree on X if for every $x, y \in X$, there always exists a real number $D(y/x)$ satisfies:

(1) $0 \leq D(y/x) \leq 1$,

(2) $x \leq y \Rightarrow D(y/x) = 1$,

(3) $x \leq y \leq z \Rightarrow D(x/z) \leq D(x/y)$.

Obviously, if $P(U)$ denotes the powerset of U, and for $E, F \in P(U)$, let $D(F/E) = \frac{|E \cap F|}{|E|}$, then D is an inclusion degree on $P(U)$. In the following, we suppose that $D(F/E) = \frac{|E \cap F|}{|E|}$.

3. Covering Significant Degree and Covering Degree

Definition 3.1. (see [2]) Let U be a universe of discourse, $C = \{C_1, C_2, \cdots, C_n\}$ be a cover of U , $Cov(C) = \{\cap Md(x) | x \in U\}$ is then also a cover of U , we call it the induced cover of C.

For $\forall x \in U$, $\cap Md(x)$ is the minimal set including x in $Cov(C)$, $Cov(C) = C$ if and only if C is a partition of U. At the same time, $Cov(C)$ is also the minimal cover of U, that is to say, for every $K \in Cov(C)$, K isn't a union of some sets in $Cov(C) - K$.

Definition 3.2. (see [2]) Let $\Delta = \{C_1, C_2, \cdots, C_m\}$ be a family of covers of U. For every $x \in U$, let $\Delta_x = \cap\{\cap Md(x) | \cap Md(x) \in Cov(C_i)\}$, then $Cov(\Delta) = \{\Delta_x | x \in U\}$ is also a cover of U, we call it the induced cover of Δ.

Definition 3.3. (see [2]) Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universe, $\Delta = \{C_1, C_2, \cdots, C_m\}$ be a family of covering on U. D is an inclusion degree on $(P(U), \subseteq)$. For every $C \in \Delta$, we denote
$Cov(\Delta) = \{\Delta_{x_1}, \Delta_{x_2}, \cdots, \Delta_{x_n}\}$, $Cov(\Delta - \{C\}) = \{\Delta'_{x_1}, \Delta'_{x_2}, \cdots, \Delta'_{x_n}\}$,
$\mathfrak{D}(Cov(\Delta)/Cov(\Delta - \{C\})) = \bigwedge_{i=1}^{n} D(\Delta_{x_i}/\Delta'_{x_i})$.
Then the significance degree of C in Δ is defined by

$$sig(C) = 1 - \mathfrak{D}(Cov(\Delta)/Cov(\Delta - \{C\})).$$

Example 3.1. Let $U = \{x_1, x_2, \cdots, x_8\}$, $\Delta = \{C_1, C_2, C_3\}$, where
$C_1 = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_6, x_7, x_8\}\}$,
$C_2 = \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}, \{x_6, x_7, x_8\}\}$,
$C_3 = \{\{x_1, x_2\}, \{x_3, x_4, x_7\}, \{x_4, x_5, x_6, x_8\}\}$.
By Definition 3.2, we have
$\Delta_{x_1} = \{x_1, x_2\}$, $\Delta_{x_2} = \{x_2\}$, $\Delta_{x_3} = \{x_3\}$, $\Delta_{x_4} = \{x_3, x_4\}$, $\Delta_{x_5} = \{x_5, x_6\}$, $\Delta_{x_6} = \{x_6\}$, $\Delta_{x_7} = \{x_7\}$, $\Delta_{x_8} = \{x_6, x_8\}$.

Therefore:

$Cov(\Delta) = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_6\}, \{x_7\}, \{x_6, x_8\}\}$.

On the other hand:

$$Cov(\Delta - \{C_1\}) = \{\{x_1, x_2\}, \{x_1, x_2\}, \{x_3, x_4\}, \{x_4\}, \{x_5, x_6\}, \{x_6\}, \{x_7\}, \{x_6, x_8\}\};$$

$$Cov(\Delta - \{C_2\}) = \{\{x_1, x_2\}, \{x_2\}, \{x_3\}, \{x_3, x_4\}, \{x_5, x_6\}, \{x_6\}, \{x_7\}, \{x_6, x_8\}\};$$

$$Cov(\Delta - \{C_3\}) = \{\{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_6\}, \{x_6, x_7\}, \{x_6, x_7, x_8\}\}.$$

Thus, we obtain the significant degree of C_1, C_2, C_3 as follows, respectively:

$$sig(C_1) = 1 - \mathfrak{D}(Cov(\Delta)/Cov(\Delta - \{C_1\})) = 1 - \frac{1}{2} = \frac{1}{2};$$
$$sig(C_2) = 1 - \mathfrak{D}(Cov(\Delta)/Cov(\Delta - \{C_2\})) = 1 - 1 = 0;$$
$$sig(C_3) = 1 - \mathfrak{D}(Cov(\Delta)/Cov(\Delta - \{C_3\})) = 1 - \frac{1}{3} = \frac{2}{3}.$$

Definition 3.4. Let $A = (U, C)$ be a covering approximation space, for any $x_i, x_j \in U$, the covering degree between x_i and x_j is defined by

$$D_C(x_i, x_j) = \frac{|\sim (\cup Md(x_i)) \cup (\cup Md(x_j))|}{|U|}.$$

Obviously, $\frac{1}{|U|} \leq D_C(x_i, x_j) \leq 1$.

Furthermore, the covering degree that C including x_i is defined by

$$D_C(x_i) = \frac{1}{|U|} \sum_{j=1}^{|U|} D_C(x_i, x_j).$$

Definition 3.5. Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universe, $\Delta = \{C_1, C_2, \cdots, C_m\}$ be a family of covering on U. The significant degree of $C_1, C_2, \cdots, C_m$ denoted as $\omega_1, \omega_2, \cdots, \omega_m$, respectively. For x_i, the covering degree under $C_k (k = 1, 2, \cdots, m)$ denoted as $D_{C_k}(x_i)$, then the total covering degree of x_i is defined by

$$D(x_i) = \sum_{k=1}^{m} \frac{\omega_k}{\omega_1 + \omega_2 + \omega_m} D_{C_k}(x_i).$$

4. Fuzzy Clustering Approach Induced by Covering

Let $U = \{x_1, x_2, \cdots, x_n\}$ be the universe, $\Delta = \{C_1, C_2, \cdots, C_m\}$ be a family of covering on U. The significant degree of $C_1, C_2, \cdots, C_m$ denoted as $\omega_1, \omega_2, \cdots, \omega_m$, respectively.

The fuzzy clustering approach induced by covering as follows:

Step 1. According to the covering C_k, compute the covering degree between x_i and x_j ($x_i, x_j \in U$), denoted as $D_{C_k}(x_i, x_j)$. Then, the fuzzy matrix can be obtained, denoted as M_k.

$$(k = 1, 2, \cdots, m; i = 1, 2, \cdots, n; j = 1, 2, \cdots, n.)$$

Step 2. Construct whole fuzzy matrix M:

$$M = \sum_{k=1}^{m} \frac{\omega_k}{\omega_1 + \omega_2 + \cdots + \omega_m} M_k.$$

Step 3. If writing the fuzzy matrix in step 2 as $M = (x_{ij})_{n\times n}$, we notice that M isn't a symmetrical matrix, i.e., $x_{ij} \neq x_{ji}$. Let $d_{ij} = \frac{1}{2}(x_{ij} + x_{ji})$, then $R = (d_{ij})_{n\times n}$ becomes a fuzzy similarity matrix, i.e., R is reflexive and symmetrical.

Step 4. Calculate the transitive closure of R, denoted as $t(R)$, as follows:

$$R \to R^2 \to R^4 \to \cdots \to R^{2^i} \to \cdots.$$

When there exists $R^k \circ R^k = R^k$ the first time, then the transitive closure $t(R) = R^k$. That is, $t(R)$ is a fuzzy equivalence matrix.

Step 5. Given a threshold λ, compute $\lambda-$cut matrix with respect to λ, and then classify the objects in universe according to this $\lambda-$cut matrix.

Example 4.1. Let $U = \{x_1, x_2, \cdots, x_8\}$ be a universe of discourse, C_1, C_2, C_3 are three different coverings of U, and

$C_1 = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_5, x_6\}, \{x_6, x_7, x_8\}\}$,

$C_2 = \{\{x_1, x_2\}, \{x_3, x_4, x_7\}, \{x_4, x_5, x_6, x_8\}\}$,

$C_3 = \{\{x_1, x_2, x_3, x_4\}, \{x_5, x_6, x_7\}, \{x_6, x_7, x_8\}\}$.

By Example 3.1, we can obtain that the significant degree of C_1, C_2, C_3 is $w_1 = \frac{1}{2}, w_2 = \frac{2}{3}, w_3 = 0$, respectively.

Step 1. According to the covering C_1, we have

$$Md(x_1) = \{\{x_1, x_2, x_3\}\}, \quad Md(x_2) = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}\},$$

$$Md(x_3) = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}\}, \quad Md(x_4) = \{\{x_2, x_3, x_4\}\},$$

$$Md(x_5) = \{\{x_5, x_6\}\}, \quad Md(x_6) = \{\{x_5, x_6\}, \{x_6, x_7, x_8\}\},$$

$$Md(x_7) = \{\{x_6, x_7, x_8\}\}, \quad Md(x_8) = \{\{x_6, x_7, x_8\}\}.$$

Thus, according to C_1, we can obtain the fuzzy matrix M_1:

$$M_1 = \begin{pmatrix} 1 & 1 & 1 & \frac{7}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} \\ \frac{7}{8} & 1 & 1 & \frac{7}{8} & \frac{4}{8} & \frac{4}{8} & \frac{4}{8} & \frac{4}{8} \\ \frac{7}{8} & 1 & 1 & \frac{7}{8} & \frac{4}{8} & \frac{4}{8} & \frac{4}{8} & \frac{4}{8} \\ \frac{7}{8} & 1 & 1 & 1 & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} \\ \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & 1 & 1 & \frac{7}{8} & \frac{7}{8} \\ \frac{4}{8} & \frac{4}{8} & \frac{4}{8} & \frac{4}{8} & \frac{6}{8} & 1 & \frac{7}{8} & \frac{7}{8} \\ \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{6}{8} & 1 & 1 & 1 \\ \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{5}{8} & \frac{6}{8} & 1 & 1 & 1 \end{pmatrix}.$$

Similarly, according to the covering C_2, we have

$$Md(x_1) = \{\{x_1, x_2\}\}, \quad Md(x_2) = \{\{x_1, x_2\}\},$$

$$Md(x_3) = \{\{x_3, x_4, x_7\}\}, \quad Md(x_4) = \{\{x_3, x_4, x_7\}, \{x_4, x_5, x_6, x_8\}\},$$

$$Md(x_5) = \{\{x_4, x_5, x_6, x_8\}\}, \quad Md(x_6) = \{\{x_4, x_5, x_6, x_8\}\},$$

$$Md(x_7) = \{\{x_3, x_4, x_7\}\}, \quad Md(x_8) = \{\{x_4, x_5, x_6, x_8\}\}.$$

Thus, according to C_2, we can obtain the fuzzy matrix M_2:

$$M_2 = \begin{pmatrix} 1 & 1 & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} \\ 1 & 1 & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} & \frac{6}{8} \\ \frac{5}{8} & \frac{5}{8} & 1 & 1 & \frac{6}{8} & \frac{6}{8} & 1 & \frac{6}{8} \\ \frac{2}{8} & \frac{2}{8} & \frac{5}{8} & 1 & \frac{6}{8} & \frac{6}{8} & \frac{5}{8} & \frac{6}{8} \\ \frac{4}{8} & \frac{4}{8} & \frac{5}{8} & 1 & 1 & 1 & \frac{5}{8} & 1 \\ \frac{4}{8} & \frac{4}{8} & \frac{5}{8} & 1 & 1 & 1 & \frac{5}{8} & 1 \\ \frac{5}{8} & \frac{5}{8} & 1 & 1 & \frac{6}{8} & \frac{6}{8} & 1 & \frac{6}{8} \\ \frac{4}{8} & \frac{4}{8} & \frac{5}{8} & 1 & 1 & 1 & \frac{5}{8} & 1 \end{pmatrix}.$$

Step 2. Construct the whole fuzzy matrix, we write as M:

$$M = \begin{pmatrix} 1.0000 & 1.0000 & 0.8333 & 0.7917 & 0.7083 & 0.7083 & 0.7083 & 0.7083 \\ 0.9583 & 1.0000 & 0.8333 & 0.7917 & 0.6667 & 0.6667 & 0.6667 & 0.6667 \\ 0.7083 & 0.7500 & 1.0000 & 0.9583 & 0.6667 & 0.6667 & 0.8333 & 0.6667 \\ 0.4583 & 0.5000 & 0.7500 & 1.0000 & 0.7083 & 0.7083 & 0.6250 & 0.7083 \\ 0.5833 & 0.5833 & 0.6667 & 0.9167 & 1.0000 & 1.0000 & 0.7083 & 0.9583 \\ 0.5000 & 0.5000 & 0.5833 & 0.8333 & 0.9167 & 1.0000 & 0.7083 & 0.9583 \\ 0.6250 & 0.6250 & 0.8750 & 0.8750 & 0.7500 & 0.8333 & 1.0000 & 0.8333 \\ 0.5417 & 0.5417 & 0.6250 & 0.8750 & 0.9167 & 1.0000 & 0.7500 & 1.0000 \end{pmatrix}.$$

Step 3. Denote the whole fuzzy matrix which is obtained in step 2 as $M = (x_{ij})_{8\times 8}$. Let $d_{ij} = \frac{1}{2}(x_{ij} + x_{ji})$, then $R = (d_{ij})_{8\times 8}$ is a fuzzy similarity matrix, and

$$R = \begin{pmatrix} 1.0000 & 0.9791 & 0.7916 & 0.7708 & 0.6875 & 0.6666 & 0.7708 & 0.6666 \\ 0.9791 & 1.0000 & 0.7916 & 0.7916 & 0.6875 & 0.6666 & 0.7916 & 0.6666 \\ 0.7916 & 0.7916 & 1.0000 & 0.8541 & 0.7500 & 0.7916 & 0.8541 & 0.7916 \\ 0.7708 & 0.7916 & 0.8541 & 1.0000 & 0.8125 & 0.8125 & 0.8125 & 0.8125 \\ 0.6875 & 0.6875 & 0.7500 & 0.8125 & 1.0000 & 0.9584 & 0.7916 & 0.9375 \\ 0.6666 & 0.6666 & 0.7916 & 0.8125 & 0.9584 & 1.0000 & 0.7916 & 0.9791 \\ 0.7708 & 0.7916 & 0.8541 & 0.8125 & 0.7916 & 0.7916 & 1.0000 & 0.7916 \\ 0.6666 & 0.6666 & 0.7916 & 0.8125 & 0.9375 & 0.9791 & 0.7916 & 1.0000 \end{pmatrix}.$$

Step 4. Calculate the transitive closure of R, denoted as $t(R)$, as follows:

$$t(R) = \begin{pmatrix} 1.0000 & 0.9791 & 0.7916 & 0.7916 & 0.7916 & 0.7916 & 0.7916 & 0.7916 \\ 0.9791 & 1.0000 & 0.7916 & 0.7916 & 0.7916 & 0.7916 & 0.7916 & 0.7916 \\ 0.7916 & 0.7916 & 1.0000 & 0.8541 & 0.8125 & 0.8125 & 0.8541 & 0.8125 \\ 0.7916 & 0.7916 & 0.8541 & 1.0000 & 0.8125 & 0.8125 & 0.8541 & 0.8125 \\ 0.7916 & 0.7916 & 0.8125 & 0.8125 & 1.0000 & 0.9584 & 0.8125 & 0.9584 \\ 0.7916 & 0.7916 & 0.8125 & 0.8125 & 0.9584 & 1.0000 & 0.8125 & 0.9791 \\ 0.7916 & 0.7916 & 0.8541 & 0.8541 & 0.8125 & 0.8125 & 1.0000 & 0.8125 \\ 0.7916 & 0.7916 & 0.8125 & 0.8125 & 0.9584 & 0.9791 & 0.8125 & 1.0000 \end{pmatrix}.$$

Step 5. Suppose $\lambda = 0.8$, then the $\lambda-$cut matrix of $t(R)$ can be obtained.

$$t(R)_\lambda = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Thus, we classify the objects in universe as $\{x_1, x_2\}, \{x_3, x_4, x_5, x_6, x_7, x_8\}$.

References

[1] Z. Pawlak, Rough sets, *International Journal of Computer and Information Sciences*, **11** (1982), 314-356.

[2] Z.H. Shi, Z.T. Gong, Knowledge reduction and knowledge significance measure based on covering rough sets, *International Journal of Pure and Applied Mathematics*, **48** (2008), 1-9.

[3] Z.H. Shi, Z.T. Gong, The further investigation of covering-based rough sets: uncertainty characterization, similarity measure and generalized models, *Information Sciences*, **180** (2010), 3745-3763.

[4] A. Skowron, J. Stepaniuk, Tolerance approximation spaces, *Fundamenta Informaticae*, **27** (1996), 245-253.

[5] R. Slowinski, D. Vanderpooten, A generalized definition of rough approximations based on similarity, *IEEE Trans. Data Knowledge Eng.*, **2** (2000), 331-336.

[6] W.H. Xu, W.X. Zhang, Measuring roughness of generalized rough sets induced by a covering, *Fuzzy sets and systems*, **158** (2007), 2443-2455.

[7] Y.Y. Yao, Relational interpretations of neighborhood operators and rough set approximation operators, *Information Sciences*, **101** (1998), 239-259.

[8] W. Zakowski, Approximations in the space(U, α), *Demonstratio Mathematica*, **16** (1983), 761-769.

[9] W.X. Zhang, W.Z. Wu, J.Y. Liang, D.Y. Li, *Theory and Method of Rough Sets*, Science Press, Beijing (2001).

[10] W. Zhu, F.Y. Wang, Reduction and axiomization of covering generalized rough sets, *Information Sciences*, **152** (2003), 217-230.