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CONVERGENCE THEOREMS BY HYBRID METHODS FOR MONOTONE MAPPINGS AND A COUNTABLE FAMILY OF NONEXPANSIVE MAPPINGS AND ITS APPLICATIONS

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1. Introduction

Let C be a closed convex subset of a real Hilbert space H and let P_C be the metric projection of H onto C. A mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \le \|x - y\|, \tag{1.1}$$

for all $x, y \in C$. We denote by $\mathrm{F}(S)$ the set of fixed points of S. A mapping A of C into H is called *monotone* if

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$$\langle Au - Av, u - v\rangle \geq 0, \tag{1.2}$$

for all $u, v \in C$. A is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2, \tag{1.3}$$

for all $u, v \in C$. It is obvious that any α -inverse-strongly-monotone mapping A is monotone and Lipschitz continuous.

The classical *variational inequality problem* is to find $u \in C$ such that $\langle v-u, Au\rangle \geq 0$ for all $v \in C$. We denoted by $VI(C, A)$ the set of solutions of this variational inequality problem. The variational inequality has been extensively studied in the literature. See, e.g. [29, 31] and the references therein.

Construction of fixed points of nonexpansive mapping is an important subject in the theory of nonexpansive mappings. However, the sequence $\{S^n x\}_{n=0}^{\infty}$ of iterates of the mapping S at a point $x \in C$ may not converge even in weak topology. More precisely, a Mann's iterated procedure is a sequence $\{x_n\}$ which is generation in the following recursive way:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sx_n, \quad n \geq 0, \tag{1.4}$$

where the initial guess $x_0 \in C$ is chosen arbitrary. However, we note that Mann's iterations have only weak convergence even in a Hilbert space [12].

Attempts to modify the Mann iteration method (1.4) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [14] proposed the following modification of Mann iteration method (1.4):

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)Sx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots. \end{cases} \tag{1.5}$$

They proved that if the sequence $\{\alpha_n\}$ is bounded above by 1, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(S)}x_0$.

In the case where the space is the Euclidean space $\mathbb{R}^N$, for finding a zero point of an inverse-strongly-monotone operator, Gol'shtein and Tret'yakov [9] introduced the following scheme: $x_1 = x \in \mathbb{R}^N$ and

$$x_{n+1} = x_n - \lambda_n Ax_n \tag{1.6}$$

for every $n = 1, 2, ...$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ generated by (1.6) converges to some element of $A^{-1}0$, where $A^{-1}0 = \{u \in \mathbb{R}^N : Au = 0\}$.

When the space is a Hilbert space H, one method of solving a point in $VI(C, A)$ is the projection algorithm which starts with any $x_1 = x \in C$ and updates iteratively x_{n+1} according to the formula

$$x_{n+1} = P_C(x_n - \lambda_n A x_n) \tag{1.7}$$

for every $n = 1, 2, ...$, where A is a monotone operator of C into H, P_C is the metric projection of H onto C and $\{\lambda_n\}$ is a sequence of positive numbers. In the case where A is inverse-strongly-monotone, Iiduka, Takahashi and Toyoda [11] proved that the sequence $\{x_n\}$ generated by (1.7) converges weakly to some element of $VI(C, A)$.

In 2004, Iiduka, Takahashi and Toyoda [11] introduced the following iterative scheme via the hybrid method in mathematical programming; see [14]: $x_1 = x \in C$ and

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases} \tag{1.8}$$

for every $n \in \mathbb{N}$, where $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. They proved that the sequence $\{x_n\}$ generated by (1.8) converges strongly to $P_{VI(C,A)}x$, where $P_{VI(C,A)}$ is the metric projection from C onto $VI(C, A)$.

Concerning a family of nonexpansive mappings it has been considered by many authors. The well-know convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, for example, [2]. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance (see [30]). For finding an element of $\bigcap_{n=1}^{\infty} F(S_n) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, the mappings $\{S_n\}$ of C into itself is a family of nonexpansive and a mapping A of C into H is α -inverse-strongly-monotone, Kumam and Plubtieng [21] extended Takahashi and Toyoda [27] and introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - \lambda_n A x_n) \tag{1.9}$$

for every $n = 0, 1, 2, ...$, where $x_0 = x \in C, \{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Suppose that

$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty$ for any bounded subset B of C. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. They shown that, if $F(S) \cap VI(C, A) \neq \emptyset$, then such a sequence $\{x_n\}$ converges weakly to some $z \in P_{F(S)\cap VI(C,A)}x$.

Recently, Takahashi, Takeuchi and Kubota [28] proved the following strong convergence theorem by using the hybrid method in mathematical programming.

Theorem 1.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{1.10}$$

where $0 \le \alpha_n < \alpha < 1$ for all $n \in \mathbb{N}$. Let S be a mapping of C into itself such that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Suppose that for each bounded sequence $\{z_n\} \subset C$, $\lim_{n\to\infty} \|z_{n+1} - S_n z_n\| = 0$ implies that $\lim_{n\to\infty} \|z_n - S_m z_n\| = 0$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathrm{F}(S)}x_0$.

Very recently, Nakajo et al.[16] introduced the more general condition so-called the *the NST*-condition*, $\{S_n\}$ is said to satisfy the NST*-condition if for each bounded sequence $\{z_n\}$ in C,

$$\lim_{n\to\infty} \|z_{n+1} - z_n\| = \lim_{n\to\infty} \|z_n - S_n z_n\| = 0 \quad \text{implies} \quad \omega_w(z_n) \subset \cap_{n=1}^{\infty} F(S_n),$$

where $\omega_w(z_n)$ denotes the set of all weak subsequential limits of a bounded sequence z_n in C. It follows directly the definitions above that if $\{S_n\}$ satisfies the NST*-condition and $S_n z \to z \in C$, then $z \in \bigcap_{n=1}^{\infty} F(S_n)$.

In this paper, motivated from above, we consider and analyze the following new hybrid method for an inverse-strongly-monotone mapping and a countable family of nonexpansive mappings defined by (3.11) and (3.13) below. We will prove strong convergence theorems by the hybrid methods for an inverse-strongly-monotone mapping and a countable family of nonexpansive mappings. Moreover, we apply our main theorems to the W-mapping and the class of strictly pseudocontractive mappings. Finally, we also apply our results to the problem for finding a common element of the set of equilibrium problems and the set solutions of the variational inequality problems for a monotone mapping. The results obtained in this paper improve and extend the corresponding result of [11],[14], [15], [17], [18],[28] and many others.

2. Preliminaries

Let H be a real Hilbert space. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$. It is also known that H satisfies

1. the *Opial's condition* [20], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality
$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$
holds for every $y \in H$ with $y \neq x$.

2. the *Kadec-Klee property* [8], that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together imply $\|x_n - x\| \to 0$.

Let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad \text{for all } y \in C.$$

P_C is called the *metric projection* of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.3}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.4}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.5}$$

for all $x \in H, y \in C$.

In the context of the variational inequality problem, this implies that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \text{ for all } \lambda > 0. \tag{2.6}$$

Let C be a subset of a Banach space E and let $\{T_n\}$ be a family of mappings from C into H with $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$. $\{S_n\}$ is said to satisfy:

(a) The AKTT-condition [1] if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup\{\|S_{n+1}z - S_n z\| : z \in B\} < \infty.$$

(b) The NST-condition [15] if for each bounded sequence $\{z_n\}$ of C,

$$\lim_{n\to\infty} \|z_n - S_n z_n\| = 0 \quad \text{implies} \quad \omega_w(z_n) \subset \cap_{n=1}^{\infty} F(S_n).$$

For more information, see Aoyama et al. [1], Nakajo et al [15] and Takahashi et al. [25].

The following lemma will be useful for proving the convergence result of this paper.

An operator A of C into E^* is said to be *hemicontinuous* if for all $x, y \in C$, the mapping F of $[0,1]$ into E^* defined by $F(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We define the *normal cone for C at a point $v \in C$*, $N_C(v)$ by

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^* \rangle \geq 0, \ \forall y \in C\}.$$

Lemma 2.1. *[22] Let C be a closed convex subset of a Banach space E, and let A be a monotone, hemicontinuous operator of C into E^*. Let $T_e \subset E \times E^*$ be an operator defined as follows:*

$$T_e v = \begin{cases} Av + N_C v, & v \in C; \\ \emptyset, & \textit{otherwise}. \end{cases}$$

Then T_e is maximal monotone and $T_e^{-1}0 = VI(A, C)$.

Throughout the paper, we will use the notations:
1. $\rightarrow$ for strong convergence.
2. $\omega_w(x_n) = \{x : \exists x_{n_r} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

3. Main Theorems

In this section, we prove strong convergence theorems for finding a common element of the set of solutions of the variational inequality of inverse-strongly monotone mappings and the set of common fixed points for a countable family of nonexpansive mappings in Hilbert spaces by using the hybrid method in mathematical programming.

3.1. The Hybrid Method

We consider the iterative scheme computing by the hybrid method (some authors call the CQ-method).

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{cases} \tag{3.1}$$

where $0 < \alpha_n \leq \alpha < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the NST-condition. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.*

Proof. For all $x, y \in C$ and $\lambda_n \in [0, 2\alpha]$, we note that

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x-y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda_n \langle x - y, Ax - Ay \rangle \\ &\quad + \lambda_n^2 \|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 \\ &\quad + \lambda_n(\lambda_n - 2\alpha)\|Ax - Ay\|^2, \end{aligned} \tag{3.2}$$

which implies that $I - \lambda_n A$ is nonexpansive.

We now prove that C_n and Q_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. Since $\|y_n - z\| \leq \|x_n - z\|$ is equivalent to

$$\|y_n - x_n\|^2 + 2\langle y_n - x_n, x_n - z \rangle \leq 0, \quad (\text{ by } (2.1))$$

it follows that C_n is convex. Next, we show that

$$\mathcal{F} \subset C_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Let $p \in \mathcal{F}$ and $n \in \mathbb{N} \cup \{0\}$. Thus, we have $p = P_C(p - \lambda_n A p)$

$$\|y_n - p\| = \|\alpha_n x_n + (1 - \alpha_n) S_n P_C(x_n - \lambda_n A x_n) - p\|$$

$$\begin{aligned}&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|S_nP_C(x_n - \lambda_n Ax_n) - p\|\\&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|P_C(I - \lambda_n A)x_n - p\|\\&\le \alpha_n\|x_n - p\| + (1-\alpha_n)\|x_n - p\|\\&= \|x_n - p\|.\end{aligned}$$

Hence, we have $p \in C_n$. Therefore we obtain (3.18). Next, we show that

$$\mathcal{F} \subset Q_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

We prove this by induction. For $n = 0$, we have $\mathcal{F} \subset C = Q_0$. Suppose that $\mathcal{F} \subset Q_n$. Then $\mathcal{F} \subset C_n \cap Q_n$ and there exists unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n} x_0$. Then

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0$$

for each $z \in C_n \cap Q_n$. In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1}\rangle \ge 0$$

for each $p \in \mathcal{F}$. It follows that $\mathcal{F} \subset Q_{n+1}$ and hence (3.4) holds. Therefore

$$\mathcal{F} \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

This implies that $\{x_n\}$ is well-defined. It follows from the definition of Q_n that $x_n = P_{Q_n} x_0$. Therefore

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } z \in Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}.$$

Let $z \in \mathcal{F} \subset Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Therefor $\{\|x_n - x_0\|\}$ is nondecreasing and bounded. So $\lim_{n\to\infty} \|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. Since $x_{n+1} = P_{C_n \cap Q_n} x_0 \in Q_n$, we have $\langle x_n - x_{n+1}, x_0 - x_n\rangle \ge 0$ and hence

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2\\&= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle\end{aligned}$$

$$\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Since $x_{n+1} \in C_n$, it follow that

$$\begin{aligned}
\|x_n - S_n P_C(x_n - \lambda_n A x_n)\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\
&\leq \frac{1}{1-\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\|\right) \\
&\leq \frac{2}{1-\alpha_n}\|x_n - x_{n+1}\|
\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. From (3.5) and $\alpha_n \leq \alpha < 1$, we obtain

$$\lim_{n\to\infty} \|x_n - S_n P_C(x_n - \lambda_n A x_n)\| = 0. \tag{3.6}$$

Next, we put $z_n = P_C(x_n - \lambda_n A x_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Let $u \in \mathcal{F}$. Since $I - \lambda_n A$ is nonexpansive and $u = P_C(u - \lambda_n A u)$, we have

$$\begin{aligned}
\|z_n - u\| &= \|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\| \\
&\leq \|(x_n - \lambda_n A x_n) - (u - \lambda_n A u)\| \\
&\leq \|(I - \lambda_n A)x_n - (I - \lambda_n A)u\| \\
&\leq \|x_n - u\|
\end{aligned}$$

for every $n = 0, 1, 2, ...$ Hence $\{z_n\}$ is bounded.
Since $x_{n+1} \in C$, we have $\|y_n - x_{n+1}\| \leq \|x_n + x_{n+1}\|$ and hence

$$\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \leq 2\|x_n - x_{n+1}\|, \ \ \forall n \geq 0.$$

From $\|x_{n+1} - x_n\| \to 0$, we obtain

$$\|x_n - y_n\| \to 0. \tag{3.7}$$

On the other hand, we note that

$$\begin{aligned}
\|y_n - u\|^2 &\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n z_n - u\|^2 \\
&\leq \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\
&= \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|P_C(x_n - \lambda_n A x_n) - P_C(u - \lambda_n A u)\|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|(I-\lambda_nA)x_n-(I-\lambda_nA)u\|^2\\
&\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)(\|x_n-u\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax_n-Au\|^2)\\
&\qquad\qquad \text{by}(3.2)\\
&\leq \alpha_n\|x_n-u\|^2+\|x_n-u\|^2+(1-\alpha_n)a(b-2\alpha)\|Ax_n-Au\|^2.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
-(1-\alpha_n)a(b-2\alpha)\|Ax_n-Au\|^2 &\leq \alpha_n\|x_n-u\|^2+\|x_n-u\|^2-\|y_n-u\|^2\\
&\leq \alpha_n\|x_n-u\|^2\\
&\quad +(\|x_n-u\|+\|y_n-u\|)\|x_n-y_n\|.
\end{aligned}
$$

Since $\alpha_n \to 0, (a,b) \subset (0,2\alpha)$ and $\|x_n-y_n\| \to 0$, it follow that $\|Ax_n-Au\| \to 0$. From (2.3), we have

$$
\begin{aligned}
\|z_n-u\|^2 &= \|P_C(x_n-\lambda_nAx_n)-P_C(u-\lambda_nAu)\|^2\\
&\leq \langle (x_n-\lambda_nAx_n)-(u-\lambda_nAu), z_n-u\rangle\\
&= (1/2)\{\|(x_n-\lambda_nAx_n)-(u-\lambda_nAu)\|^2+\|z_n-u\|^2\\
&\quad -\|[(x_n-\lambda_nAx_n)-(u-\lambda_nAu)]-(z_n-u)\|^2\}\\
&\leq (1/2)\{\|x_n-u\|^2+\|z_n-u\|^2-\|(x_n-z_n)-\lambda_n(Ax_n-Au)\|^2\}\\
&= (1/2)\{\|x_n-u\|^2+\|z_n-u\|^2-\|(x_n-z_n)\|^2\\
&\quad +2\lambda_n\langle x_n-z_n, Ax_n-Au\rangle-\lambda_n^2\|Ax_n-Au\|^2\}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\|z_n-u\|^2 &\leq \|x_n-u\|^2-\|x_n-z_n\|^2\\
&\quad +2\lambda_n\langle x_n-z_n, Ax_n-Au\rangle-\lambda_n^2\|Ax_n-Au\|^2.
\end{aligned}
$$

From this, we note that

$$
\begin{aligned}
\|y_n-u\|^2 &= \|\alpha_nx_n+(1-\alpha_n)S_nz_n-u\|^2\\
&\leq \alpha_n\|x_n-u\|^2+(1-\alpha_n)\|z_n-u\|^2\\
&\leq \alpha_n\|x_n-u\|^2+\|x_n-u\|^2-\|x_n-z_n\|^2\\
&\quad +2\lambda_n\langle x_n-z_n, Ax_n-Au\rangle-\lambda_n^2\|Ax_n-Au\|^2
\end{aligned}
$$

and hence

$$
\begin{aligned}
\|x_n-z_n\|^2 &\leq \alpha_n\|x_n-u\|^2+(\|x_n-u\|^2-\|y_n-u\|^2)\\
&\quad +2\lambda_n\langle x_n-z_n, Ax_n-Au\rangle-\lambda_n^2\|Ax_n-Au\|^2
\end{aligned}
$$

$$\leq \alpha_n\|x_n - u\|^2 + (\|x_n - u\| + \|y_n - u\|)\|x_n - y_n\| \\ +2\lambda_n\langle x_n - z_n, Ax_n - Au\rangle - \lambda_n^2\|Ax_n - Au\|^2.$$

Since $\alpha_n \to 0$, $\|x_n - y_n\| \to 0$ and $\|Ax_n - Au\| \to 0$, we obtain

$$\|x_n - z_n\| \to 0. \tag{3.8}$$

It follows from $\alpha_n \leq \alpha < 1$ that

$$\begin{aligned} \|x_n - S_n x_n\| &\leq \|x_n - S_n z_n\| + \|S_n z_n - S_n x_n\| \\ &= \frac{1}{1-\alpha_n}\|y_n - x_n\| + \|z_n - x_n\| \\ &\leq \frac{1}{1-\alpha}\|y_n - x_n\| + \|z_n - x_n\|. \end{aligned} \tag{3.9}$$

This together with (3.7) and (3.8), we obtain

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0.$$

From (3.5) and $\{S_n\}$ satisfies the NST*-condition, we have $z_0 \in \cap_{n=1}^{\infty} F(S_n)$.

Define $T_e \subset E \times E^*$ be as in Lemma 2.1. By using the same argument as in the proof of Theorem [10, Theorem 3.1, pp. 346-347], we can show that $z_0 \in VI(C, A)$. Hence $z_0 \in \mathcal{F}$.

Finally, we show that $x_n \to w_0$, where $w_0 = P_{\mathcal{F}} x_0$. Since $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ and $x_{n_k} \rightharpoonup z_0$. Since $x_{n_k} \subset C$ and C is closed and convex, we obtain $z_0 \in C$.

Since $x_n = P_{Q_n} x_0$ and $w_0 \in \mathcal{F} \subset Q_n$, we have

$$\|x_n - x_0\| \leq \|w_0 - x_0\|.$$

It follows from $w_0 = P_{\mathcal{F}} x_0$ and the lower semicontinuity of the norm that

$$\|w_0 - x_0\| \leq \|z_0 - x_0\| \leq \liminf_{k\to\infty} \|x_{n_k} - x_0\| \leq \limsup_{k\to\infty} \|x_{n_k} - x_0\| \leq \|w_0 - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|z_0 - x_0\| = \|w_0 - x_0\|$. Using the Kadec-Klee property of H, we obtain that

$$\lim_{k\to\infty} x_{n_k} = z_0 = w_0.$$

Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{F(T)\cap VI(C,A)} x_0$. □

By the same argument as in the proof of Theorem 3.1, we obtain the following theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let S be nonexpansive mappings from C into itself such that operatornameF$(S) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{cases} \tag{3.10}$$

where $0 \leq \alpha_n < \alpha < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $P_{\mathcal{F}} x_0$.

Proof. Defined $S_n = S$ for all $n \in \mathbb{N}$. So, we obtain the desired result by using Theorem 3.1. □

Using Theorem 3.1, we obtain the following results.

Corollary 3.3. (see [17, Theorem 4.1]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $F := \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n) S_n x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{cases} \tag{3.11}$$

where $0 < \alpha_n \leq \alpha < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the NST-condition. Then $\{x_n\}$ converges strongly to $P_F x_0$.*

Proof. Putting $C = H$, then $P_C \equiv I$. By Theorem 3.1, we obtain the desired result easily. □

Corollary 3.4. (see [18, Corollary 11] and [28, Theorem 3.4]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$.*

Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined as follows:

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n x_n, \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \dots, \end{cases} \tag{3.12}$$

where $0 \le \alpha_n < \alpha < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the AKTT-condition or NST-condition. Let S be a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(S)} x_0$.

Proof. Putting $C = H$, then $P_C \equiv I$, and since the AKTT condition (see [18, Corollary 11]) or the NST-condition (see [28, Theorem 3.4]) implies the NST*-condition (see also Lemma 2.5 in [19]). So, by Theorem 3.1, we obtain the desired result easily. □

3.2. The Shrinking Projection Method

We next use the new hybrid method, so-called The shrinking projection method to obtain several strong convergence theorems.

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\mathcal{F} := \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A) \ne \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.13}$$

where $0 < \alpha_n \le c < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the NST-condition. Then $\{x_n\}$ converges strongly to $w = P_{\mathcal{F}} x_0$.*

Proof. We first show by induction that $\mathcal{F} \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\mathcal{F} \subset C = C_1$. Suppose that $\mathcal{F} \subset C_k$ for each $k \in \mathbb{N}$. Hence, for $u \in \mathcal{F} \subset C_k$ we have $u = P_C(u - \lambda_{k+1} A u)$ and

$$\|y_k - u\| = \|\alpha_k x_k + (1-\alpha_k) S_k P_C(x_k - \lambda_k A x_k) - u\|$$

$$
\begin{aligned}
&\leq \alpha_k\|x_k - u\| + (1-\alpha_k)\|S_kP_C(x_k - \lambda_k Ax_k) - u\| \\
&\leq \alpha_k\|x_k - u\| + (1-\alpha_k)\|P_C(x_k - \lambda_k Ax_k) - P_C(u - \lambda_k Au)\| \\
&\leq \alpha_k\|x_k - u\| + (1-\alpha_k)\|(I - \lambda_k A)x_k - (I - \lambda_k A)u\| \\
&\leq \alpha_k\|x_k - u\| + (1-\alpha_k)\|x_k - u\| \\
&= \|x_k - u\|.
\end{aligned}
$$

Hence $u \in C_{k+1}$. This implies that

$$
\mathcal{F} \subset C_n \quad \text{for all } n \in \mathbb{N}. \tag{3.14}
$$

Next, we prove that C_n is closed and convex for all $n \in \mathbb{N}$. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_k$, we know that $\|y_k - z\| \leq \|x_k - z\|$ is equivalent to

$$
\|y_k - x_k\|^2 + 2\langle y_k - x_k, x_k - z\rangle \geq 0. \quad (\text{ by } (2.1))
$$

So, C_{k+1} is closed and convex. Then, for any $n \in \mathbb{N}$, C_n is closed and convex. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n}x_0$, we have

$$
\langle x_0 - x_n, x_n - y\rangle \geq 0
$$

for each $y \in C_n$. Since $\mathcal{F} \subset C_n$, we also have

$$
\langle x_0 - x_n, x_n - u\rangle \geq 0 \text{ for each } u \in \mathcal{F} \text{ and } n \in \mathbb{N}.
$$

Hence, for $u \in \mathcal{F}$, we have

$$
\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - u\rangle \\
&= -\|x_0 - x_n\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}
$$

This implies that

$$
\|x_0 - x_n\| \leq \|x_0 - u\| \text{ for all } u \in \mathcal{F} \text{ and } n \in \mathbb{N}.
$$

By the same as in the proof of [28, Theorem 3.3], we can show that $\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0$ and hence $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
On the other hand, from $x_{n+1} \in C_{n+1} \subset C$, we have

$$
\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\|. \tag{3.15}
$$

Further, we have

$$\begin{aligned}\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)S_n P_C(x_n - \lambda_n A x_n) - x_n\| \\ &= (1-\alpha_n)\|S_n P_C(x_n - \lambda_n A x_n) - x_n\|.\end{aligned}$$

From (3.15), we note that

$$\begin{aligned}\|x_n - S_n P_C(x_n - \lambda_n A x_n)\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\le \frac{1}{1-\alpha_n}\left(\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\|\right) \\ &\le \frac{2}{1-\alpha_n}\|x_n - x_{n+1}\|\end{aligned}$$

for all $n \in \mathbb{N}$. Putting $z_n = P_C(x_n - \lambda_n A x_n)$ for all $n \in \mathbb{N}$. Since $0 < \alpha_n \le c < 1$, it follow that

$$\lim_{n\to\infty} \|x_n - S_n z_n\| = 0. \tag{3.16}$$

For $u \in \mathcal{F}$, we note that

$$\begin{aligned}\|y_n - u\|^2 &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|S_n z_n - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|z_n - u\|^2 \\ &= \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|P_C(x_n - \lambda_n A x_n) - u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\|(I - \lambda_n A)x_n - (I - \lambda_n A)u\|^2 \\ &\le \alpha_n\|x_n - u\|^2 + (1-\alpha_n)\{\|x_n - u\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Au\|^2\} \\ &\le \|x_n - u\|^2 + (1-\alpha_n)a(b - 2\alpha)\|Ax_n - Au\|^2.\end{aligned}$$

Therefore, we have

$$\begin{aligned}-(1-\alpha_n)a(b-2\alpha)\|Ax_n - Au\|^2 &\le \|x_n - u\|^2 - \|y_n - u\|^2 \\ &\le \|y_n - x_n\|(\|x_n - u\| + \|y_n - u\|).\end{aligned}$$

Since $\alpha_n \to 0, (a,b) \subset (0, 2\alpha)$ and $\|y_n - x_n\| \to 0$, we obtain $\|Ax_n - Au\| \to 0$. As in the proof of Theorem 3.1, we note that

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0.$$

This together with $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and by $\{S_n\}$ satisfies the NST*-condition, we obtain $z_0 \in \cap_{n=1}^{\infty} F(S_n)$. By the same argument as in the proof of Theorem 3.1, we can show that $z_0 \in VI(C, A)$. Hence $z_0 \in \mathcal{F}$.

Finally, we show that $x_n \to w$, where $w = P_{\mathcal{F}}x_0$. Since $\{x_n\}$ is bounded. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. Since $\{x_{n_k}\} \subset C$ and C is closed and convex, we obtain $w' \in C$.

Since $x_n = P_{C_n}x_0$ and $w \in \mathcal{F} \subset C_n$, we have

$$\|x_n - x_0\| \leq \|w - x_0\|.$$

It follows from $w = P_{\mathcal{F}}x_0$ and the lower semicontinuity of the norm that

$$\|w - x_0\| \leq \|w' - x_0\| \leq \liminf_{k\to\infty} \|x_{n_k} - x_0\| \leq \limsup_{k\to\infty} \|x_{n_k} - x_0\| \leq \|w - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H, we obtain that $\lim_{k\to\infty} x_{n_k} = w' = w$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to w, where $w = P_{\mathcal{F}}x_0$. □

By using Theorem 3.5, we obtain the following results.

Theorem 3.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let S be a nonexpansive mappings from C into itself such that $\mathcal{F} := \bigcap_{n=1}^{\infty} F(S_n) \cap VI(C,A)$ is nonempty and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)SP_C(x_n - \lambda_n Ax_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.17}$$

where $0 \leq \alpha_n < a < 1$ and $0 < c < \lambda_n < d < 2\alpha$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathcal{F}}x_0$.

Proof. Defined $S_n \equiv S$ for all $n \in \mathbb{N}$. □

Corollary 3.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.18}$$

where $0 < \alpha_n \leq c < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the NST-condition. Then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(S_n)}x_0$.*

Corollary 3.8. (see [28, Theorem 3.3]) *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, define a sequence $\{x_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)S_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad n \in \mathbb{N}, \end{cases} \tag{3.19}$$

where $0 < \alpha_n \leq c < 1$ and $0 < a < \lambda_n < b < 2\alpha$ for all $n \in \mathbb{N}$. If $\{S_n\}$ satisfies the NST-condition and S is a mapping of C into itself defined by $Sz = \lim_{n\to\infty} S_n z$ for all $z \in C$ and suppose that $\mathrm{F}(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(S_n)$. Then $\{x_n\}$ converges strongly to $P_{\bigcap_{n=1}^{\infty} F(S_n)}x_0$.

Proof. Since the NST-condition implies the NST*-condition. □

4. W-Mappings

Using Theorem 3.1, we prove two theorems in Hilbert spaces. Let $T_1, T_2, \ldots$ be an infinite sequence of mappings of C into itself and let $\lambda_1, \lambda_2, \ldots$ be real numbers such that $0 \leq \lambda_i \leq 1$ for every $i \in \mathbb{N}$. Then for any $n \in \mathbb{N}$, Takahashi [24] (see also [23], [25]) defined a mapping W_n of C into itself as follows:

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1-\lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1-\lambda_{n-1})I, \\ \vdots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1-\lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1-\lambda_{k-1})I, \\ \vdots \\ U_{n,2} = \lambda_2 T_2 U_{n,3} + (1-\lambda_2)I, \\ W_n = U_{n,1} = \lambda_1 T_1 U_{n,2} + (1-\lambda_1)I. \end{cases} \tag{4.1}$$

Such a mapping W_n is called the W-mapping generated by $T_n, T_{n-1}, \ldots, T_1$ and $\lambda_n, \lambda_{n-1}, \ldots, \lambda_1$.

In the following Lemma, we can see the prove in Shimoji and Takahashi [23] and Chang et al.[6].

Lemma 4.1. (see [23]) *Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, ...$ be nonexpansive mappings of C into itself such that $\bigcap_{n=1}^{\infty} F(T_n) \neq 0$ and let $\lambda_1, \lambda_2, ...$ be real numbers such that $0 < \lambda_n \leq b < 1$ for any $n \geq 1$. Then, for every $x \in C$ and $k \in \mathbb{N}$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

Remark 4.2. (see [5], Remark 3.2). It can be known from Lemma 4.1 that if D is a nonempty bounded subset of C, then for $\epsilon > 0$ there exists $n_0 \geq k$ such that for all $n > n_0$

$$\sup_{x\in D} \|U_{n,k}x - U_k x\| \leq \epsilon.$$

Remark 4.3. (see [5], Remark 3.3). Using Lemma 4.1, we define a mapping $W : C \to C$ as follows:

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for all $x \in C$. Indeed, observe that for each $x, y \in C$

$$\|Wx - Wy\| = \lim_{n\to\infty} \|W_n x - W_n y\| \leq \|x - y\|.$$

If $\{x_n\}$ is a bounded sequence in C, then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 4.2 that for an arbitrary $\epsilon > 0$, there exists $N_0 \geq 1$, such that for all $n > N_0$

$$\|W_n x_n - W x_n\| = \|U_{n,1}x_n - U_1 x_n\| \leq \sup_{x\in D} \|U_{n,1}x - U_1 x\| \leq \epsilon.$$

This implies that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0.$$

Setting $S_n \equiv W_n$ in Theorem 3.1 and using Lemma 4.1, we obtain the next theorems.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself such that $\Omega := \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A)$ is nonempty. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n) W_n P_C(x_n - \lambda_n A x_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.2}$$

where $0 \le \alpha_n < \alpha < 1$ *and* $0 < a < \lambda_n < b < 2\alpha$ *for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $P_\Omega x_0$.

Theorem 4.5. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\alpha > 0$ *and let* A *be an* α*-inverse-strongly monotone mapping of* C *into* H. *Let* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself such that* $\Omega := \bigcap_{n=1}^{\infty} \mathrm{F}(S_n) \cap VI(C, A) \neq \emptyset$ *and let* $x_0 \in H$. *For* $C_1 = C$ *and* $x_1 = P_{C_1} x_0$, *define a sequence* $\{x_n\}$ *of* C *as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) W_n P_C(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{4.3}$$

where $0 \le \alpha_n < c < 1$ *and* $0 < a < \lambda_n < b < 2\alpha$ *for all* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges strongly to* $w = P_\Omega x_0$.

5. Applications

5.1. Equilibrium Problems

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x, y) \ge 0 \quad \text{for all } y \in C. \tag{5.1}$$

The set of solutions of (5.1) is denoted by $\mathrm{EP}(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of (5.1). Some methods have been proposed to solve the equilibrium problem (see [3, 7, 13, 26]). In 2005, Combettes and Hirstoaga [4] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and they also proved a strong convergence theorem.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions (see [3]):

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} F(tz + (1-t)x, y) \le F(x, y);$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$.

By [3, Corollary 1] and [4, Lemma 2.12], we have the following lemma.

Lemma 5.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfies* (A1)-(A4) *and let $r > 0$ and $x \in H$. Then there exists unique $x^* \in C$ such that*

$$F(x^*, y) + \frac{1}{r}\langle y - x^*, x^* - x\rangle \geq 0 \quad \text{for all } y \in C.$$

Moreover, let T_r be a mapping of H into C defined by

$$T_r(x) = x^*$$

for all $x \in H$. Then, the following hold:

(i) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(ii) $F(T_r) = EP(F)$;

(iii) $EP(F)$ is closed and convex.

By [18, Theorem 16], we have the following lemma.

Lemma 5.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4)*. Let $\{r_n\}$ be a sequence of positive integers and T_{r_n} be mapping defined as in Lemma 5.1. If $\liminf_{n\to\infty} r_n > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then the following hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|T_{r_{n+1}} z - T_{r_n} z\| : z \in B\} < \infty$ for any bounded subset B of C,

(i) $F(T) = \bigcap_{n=1}^{\infty} F(T_{r_n})$ where T is a mapping defined by $Tx = \lim_{n\to\infty} T_{r_n} x$ for all $x \in C$.

Using Lemma 5.2 and Theorem 3.1, , we have the following theorem.

Theorem 5.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4)*. Let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \cap$*

$EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$ and

$$\begin{cases} w_n = P_C(y_n - \lambda_n A y_n), \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)u_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \ldots, \end{cases} \tag{5.2}$$

where $0 \leq \alpha_n < a < 1, \{\lambda_n\} \subset (c, d) \subset (0, 2\alpha)$, for all $n \in \mathbb{N}$ and $\{r_n\}$ is a sequence in $(0, \infty)$ satisfy $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $P_{VI(C,A)\cap \mathrm{EP}(F)} x_0$.

Proof. Putting $u_n = T_{r_n} w_n$ for all $n \in \mathbb{N}$. Hence by Theorem 3.1, $\{x_n\}$ converges strongly to $P_{VI(C,A)\cap \mathrm{EP}(F)} x_0$. □

Using Theorem 3.5 and Lemmas 5.2 we have the following theorem.

Theorem 5.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C, A) \cap EP(F) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define sequences $\{x_n\}$ as follows:*

$$\begin{cases} w_n = P_C(x_n - \lambda_n A x_n), \\ F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - w_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n < a < 1, \{\lambda_n\} \subset (c, d) \subset (0, 2\alpha)$, for all $n \in \mathbb{N}$ and $\{r_n\}$ is a sequence in $(0, \infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{VI}(C,A)\cap \mathrm{EP}(F)} x_0$.

5.2. Accretive Operator

In this section, we consider the problem of finding a zero of an accretive operator. An operator $T \subset H \times H$ is said to be *accretive* if for each (x_1, y_1) and

$(x_2, y_2) \in T$, there exists $j \in J(x_1 - x_2)$ such that $\langle y_2 - y_1, j\rangle \geq 0$. A monotone operator T is said to be *maximal* if the graph of T is not properly contained in the graph of any other monotone operator. It is known that a monotone operator T is maximal if and only if $R(I + rT) = H$ for every $r > 0$, where $R(I + rT)$ is the rang of $I + rT$, An accretive operator T is said to satisfy the range condition if $\overline{D(T)} \subset R(I+rT)$ for all $r > 0$, where $D(T)$ is the domain of T, I is the identity mapping on $H, R(I+rT)$ is the rang of $I+rT$, and $\overline{D(T)}$ is the closure of $D(T)$. An accretive operator T is *m-accretive* if $R(I + rT) = H$ for each $r > 0$. If T is an accretive operator which satisfies the range condition, then we can defined, for each $r > 0$, a mapping $J_r : R(I + rT) \to D(T)$ by $J_r = (I + rT)^{-1}$, which is called the *resolvent* of T. We know that J_r is nonexpansive and $F(J_r) = T^{-1}0$ for all $r > 0$.

Lemma 5.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset H \times H$ be an accretive operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$, and $\{r_n\}$ be a sequence in $(0, \infty)$. If $inf\{r_n : n \in \mathbb{N}\} > 0$, and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then the followings hold:*

(i) $\sum_{n=1}^{\infty} \sup\{\|J_{r_{n+1}}z - J_{r_n}z\| : z \in B\} < \infty$ for any bounded subset B of C,

(ii) $F(S) = \bigcap_{n=1}^{\infty} \mathrm{F}(J_{r_n})$, where S is a mapping defined by $Sx = \lim_{n\to\infty} J_{r_n}x$ for all $x \in C$.

By Lemmas 5.5 and Theorem 3.1, we have the following theorem.

Theorem 5.6. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T \subset H \times H$ be an accretive operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H. Let $\{x_n\}$ be a sequence in C defined as follows:*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary,} \\ y_n = \alpha_n x_n + (1 - \alpha_n)J_{r_n}P_C(x_n - \lambda_n Ax_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{cases} \tag{5.3}$$

where $0 \leq \alpha_n < a < 1, \{\lambda_n\} \subset (c, d) \subset (0, 2\alpha)$, for all $n \in \mathbb{N}$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If $\inf\{r_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to z, where $z = P_{T^{-1}0 \cap VI(C,A)}x_0$.

Proof. Since H is Hilbert space $C = \overline{D(T)}$ is closed and convex. By Lemma

5.5, we have the following

$$F(S) = \bigcap_{n=1}^{\infty} F(J_{r_n}) = T^{-1}0 \neq \emptyset.$$

We note that $F(T) = VI(A, C)$. Therefore, by Theorem 3.1, we obtain $\{x_n\}$ converges strongly to $z = P_{T^{-1}0 \cap VI(C,A)} x_0$. ☐

Using Theorem 3.5 and Lemma 5.5, we have also the following theorem.

Theorem 5.7. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A \subset H \times H$ be an accretive operator such that $T^{-1}0 \neq \emptyset$ and $\overline{D(T)} \subset C \subset \bigcap_{r>0} R(I + rT)$. Let $\alpha > 0$ and let A be an α-inverse-strongly monotone mapping of C into H and $x_0 \in H$. For $C_1 = C$ and $x_1 = P_{C_1} x_0$, define sequences $\{x_n\}$ as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n) J_{r_n} P_C(x_n - \lambda_n A x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases}$$

where $0 \leq \alpha_n < a < 1, \{\lambda_n\} \subset (c, d) \subset (0, 2\alpha)$, for all $n \in \mathbb{N}$ and $\{r_n\}$ is a sequence in $(0, \infty)$. If $\inf\{r_n : n \in \mathbb{N}\} > 0$ and $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, then $\{x_n\}$ converges strongly to z, where $z = P_{T^{-1}0 \cap VI(C,A)} x_0$.

5.3. Strictly Pseudocontractive Mapping

A mapping $T : C \to C$ is called strictly pseudocontractive on C if there exists k with $0 \leq k < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x + (I - T)y\|^2, \text{ for all } x, y \in C.$$

If $k = 0$, then T is nonexpansive. Put $A = I - T$, where $T : C \to C$ is a strictly pseudocontractive mapping with k. We know that, A is $\frac{1-k}{2}-$ inverse strongly monotone and $A^{-1}0 = F(T)$ (see [10]).

Using Theorem 5.3, we have the following theorem.

Theorem 5.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings from C into itself. Let T be a strictly pseudocontractive mapping with constant k of C into itself which $\cap_{n=1}^{\infty} F(S_n) \cap F(T) \neq \emptyset$ and let $\{x_n\}$ be a sequence generated by $x_0 \in H$*

and

$$\begin{cases} x_0 \in C \text{ is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C((1-\lambda_n)x_n + \lambda_n T x_n), \\ C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2 \dots, \end{cases}$$

where $0 \le \alpha_n < a < 1$ *and* $\{\lambda_n\} \subset (c,d) \subset (0, 2\alpha)$, *for all* $n \in \mathbb{N}$ *.If* $\{S_n\}$ *satisfies the NST*-condition. Then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)} x_0$.

Proof. Put $A = I - T$. Then A is $\frac{1-k}{2}-$inverse-strongly monotone. We have that $F(T)$ is the solution set of $VI(A,C)$ i.e., $F(T) = VI(A,C)$ and

$$P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n.$$

Therefore, by Theorem 3.5, $\{x_n\}$ converges strongly to

$$z = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A)} x_0. \qquad \square$$

Theorem 5.9. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $\{S_n\}$ *be a sequence of nonexpansive mappings from* C *into itself and let* T *be a strictly pseudocontractive mapping with constant* k *of* C *into itself such that* $\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A) \neq \emptyset$ *and let* $x_0 \in H$. *For* $C_1 = C, x_1 = P_{C_1} x_0$, *define sequence* $\{x_n\}$ *as follows:*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n) S_n P_C((1-\lambda_n)x_n + \lambda_n T x_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \quad n \in \mathbb{N}, \end{cases} \tag{5.4}$$

where $0 \le \alpha_n < a < 1$ *and* $0 < c < \lambda_n < d < 2\alpha$ *for all* $n \in \mathbb{N}$. *If* $\{S_n\}$ *satisfies the NST*-condition. Then* $\{x_n\}$ *converges strongly to* z, *where* $z = P_{\cap_{n=1}^{\infty} F(S_n) \cap VI(C,A)} x_0$.

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References

[1] K. Aoyama, Y. Kimura, W. Takahashi, M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, *Nonlinear Anal.*, **67** (2007), 2350-2360.

[2] H.H. Bauschke, J.M. Borwein, On projection algorithms for solving convex feasibility problems, *SIAM Review*, **38**, No. 3 (1996), 367-426.

[3] E. Blum, W. Oettli, From optimization and variational inequalities to equilibrium problems, *Math. Student.*, **63** (1994), 123-145.

[4] P.L. Combettes, S.A. Hirstoaga, Equilibrium programming in Hilbert spaces, *J. Nonlinear Convex Anal.*, **6** (2005), 117-136.

[5] L.C. Ceng, J.C. Yao, A hybrid iterative scheme for mixed equilibrium problems and fixed point problems, *J. Comput. Appl. Math.*, **214** (2008), 186-201.

[6] S.S. Chang, H.W. Joseph Lee, C.K. Chan, A new method for solving equilibrium problem fixed point problem and variational inequality problem with application to optimization, *Nonlinear Anal.*, **70** (2009), 3307-3319.

[7] S.D. Flam, A.S. Antipin, Equilibrium progamming using proximal-link algolithms. *Math. Program.*, **78** (1997), 29-41.

[8] K. Goebel, W.A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge University Press, Cambridge (1990).

[9] E.G. Gol'shteĭn, N.V. Tret'yakov, Modified lograngians in convex programming and their generalizations, *Math. Program. Stu.*, **10** (1979), 86-97.

[10] H. Iiduka, W. Takahashi, Strong convergence theorems for nonexpansive mapping and inverse-strong monotone mappings, *Nonlinear Anal.*, **61** (2005), 341-350.

[11] H. Iiduka, W. Takahashi, M. Toyoda, Approximation of solutions of variational inequalities for monotone mappings, *Panamer. Math. J.*, **14** (2004), 49-61.

[12] W.R. Mann, Mean value methods in iteration, *Proc. Amer. Math. Soc.*, **4** (1953), 506-510.

[13] A. Moudafi, M. Thera, *Proximal and Dynamical Approaches to Equilibrium Problems*, Lecture note in Economics and Mathematical Systems, Springer-Verlag, New York **477** (1999), 187-201.

[14] K. Nakajo, W. Takahashi, Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups, *J. Math. Anal. Appl.*, **279** (2003), 372-379.

[15] K. Nakajo, K. Shimoji, W. Takahashi, Strong convergence theorems by the hybrid method for families of nonexpansive mappings in Hilbert spaces, *Taiwanese J. Math.*, **10** (2006), 339-360.

[16] K. Nakajo, K. Shimoji, W. Takahashi, On strong convergence by the hybrid method for families of map- pings in Hilbert spaces, *Nonlinear Anal.*, **71** (2008), 112-119.

[17] K. Nakprasit, W. Nilsrakoo, S. Saejung, Weak and strong convergence theorems of an implicit iteration process for a countable family of nonexpansive mappings, In: *Fixed Point Theory and Applications*, Volume 2008 (2008), Article ID 732193, 18 pages.

[18] W. Nilsrakoo, S. Saejung, Weak and strong convergence theorems for countable Lipschitzian mappings and its applications, *Nonlinear Anal.*, **69** (2008), 2695-2708

[19] W. Nilsrakoo, S. Saejung, Strong convergence theorems for a countable family of quasi-Lipschitzian mappings and its applications, *J. Math. Anal. Appl.*, **356** (2009), 154-167.

[20] Z. Opial, Weak convergence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.*, **73** (1967), 591-597.

[21] S. Plubtieng, P. Kumam, Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings, *J. Comput. Appl. Math.*, **224** (2009), 614-621.

[22] R.T. Rockafellar, On the maximality of sums of nonlinear monotone operators, *Trans. Amer. Math. Soc.*, **149**, No. 1 (1970), 75-88.

[23] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, *Taiwanese J. Math.*, **5** (2001), 387-404.

[24] W. Takahashi, Weak and strong convergence theorems for families of nonexpansive mappings and their applications, *Ann. Univ. Marjae Curie-Sklodowska Sect.*, **51** (1997), 277-292.

[25] W. Takahashi, K. Shimoji, Convergence theorems for nonexpansive mappings and feasibility problems, *Math. Comput. Model.*, **32** (2000), 1463-1471.

[26] S. Takahashi, W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, *J. Math. Anal. Appl.*, **331** (2007), 506-515.

[27] W. Takahashi, M. Toyoda, Weak convergence theorems for nonexpansive mappings and monotone mappings, *J. Optim. Theory Appl.*, **118** (2003), 417-428.

[28] W. Takahashi, Y. Takeuchi, R. Kubota, Strong Convergence Theorems by Hybrid Methods for Families of Nonexpansive Mappings in Hilbert Spaces, *J. Math. Anal. Appl.*, **341** (2008), 276-286.

[29] J.-C. Yao, O. Chadli, Pseudomonotone complementarity problems and variational inequalities, In: *Handbook of Generalized Convexity and Monotonicity* (Ed-s: J.P. Crouzeix, N. Haddjissas, S. Schaible) (2005), 501-558.

[30] D.C. Youla, *Mathematical theory of image restoration by the method of convex projections*, In: *Image Recovery: Theory and Applications* (Ed. H. Stark), Academic Press, Orlando, Florida, USA (1987), 29-77,

[31] L.C. Zeng, S. Schaible, J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, *J. Optim. Theory Appl.*, **124** (2005), 725-738.