Pfaff fields and sectional genus

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Abstract: Here we give an extension of two results of M. Corrêa and M. Jardin on Pfaff fields on projective schemes.

AMS Subject Classification: 37F75, 14J60, 58A17
Key Words: Pfaff field, sectional genus

1. Introduction

Esteves and Kleiman introduced the general set-up of Pfaff systems and Pfaff fields with the right amount of generality (see [4]). Let $Y$ be an equidimensional reduced projective variety defined over an algebraically closed field $\mathbb{K}$. Set $m := \dim(Y)$. Fix an integer $s \in \{1, \ldots, m\}$. A rank $s$ Pfaff field on $Y$ is a map $\Omega^s_Y \to L$, where $L \in \text{Pic}(Y)$. For any ample line $H$, the $H$-sectional genus $g(X, H)$ of $H$ is the unique integer such that $2g(X, H) - 2 = (\omega_X + (m - 1)H) \cdot H^{m-1}$.

We first consider an extension of [2], Theorem 2.

**Theorem 1.** Assume the existence of a rank $m$ Pfaff field $\eta : \Omega^m_Y \to L$ not vanishing identically on any irreducible component of $Y$. Let $\mathcal{O}_Y(1)$ be any ample line bundle on $Y$. Let $T \subset Y$ denote the sum of all codimension 1 components of the singular set of $\eta$ and of the singular scheme $\Sigma_Y$ if $Y$ with the multiplicities coming from the scheme-structure described in [3], Subsection 4.1. Then $2g(Y, \mathcal{O}_Y(1)) \leq L \cdot \mathcal{O}_Y(1)^{m-1} + T \cdot \mathcal{O}_Y(1)^{m-1}$ (intersection numbers).

Received: February 20, 2011 c⃝ 2011 Academic Publications, Ltd.
If $Y \subset \mathbb{P}^n$ and $\eta$ comes from a rank $m$ Pfaff field $\eta_1$ of $\mathbb{P}^n$ (i.e. $Y$ is a solution of $\eta_1 = 0$ not contained in the singular locus of $\eta_1$) then $\eta$ is singular on a codimension 1 subvariety of $Y$ (see [4], Proposition 3.4, for much more). Hence $T \neq \emptyset$ in this case. Notice that if $Y$ is smooth, then the obvious isomorphism $\Omega^m_Y \to \omega_Y$ is a rank $m$ Pfaff field with $T = \emptyset$.

From now on in the introduction we assume $\text{char}(\mathbb{K}) = 0$. For any $R \in \text{Pic}(X)$ let $q(R, H)$ denote the infimum of all rational numbers $u$ such that there is an injective map $R \hookrightarrow M$ with $M \in \text{Pic}(X)$ and $M$ numerically equivalent to the $\mathbb{Q}$-divisor $uH$. Obviously $q(R, H) \in \mathbb{R}$. We use the integer $q(R, H)$ to extend [2], Theorem 1, to the case of Pfaff fields not associated to multiples of $H$, i.e. we prove the following result.

**Theorem 2.** Let $X$ be a smooth and connected $m$-dimensional projective variety whose tangent bundle $\Theta_X$ is $\mu$-semistable with respect to at least one polarization $H$. Fix an integer $k$ such that $1 \leq k \leq m$ and assume the existence of a $k$-Pfaff field, i.e. the existence of a non-zero map $\Omega^k_X \to R$ with $R \in \text{Pic}(X)$. Then $mR \cdot H^{m-1} \geq -k \cdot \omega_X \cdot H^{m-1}$ and $mq(R, H)H^n \geq -k \cdot \omega_X \cdot H^{m-1}$.

Since $2g(X, H) - 2 = (\omega_X + (m - 1)H) \cdot H^{m-1}$, may use the equality $\omega_X \cdot H^{m-1} = 2g(X, H) - 2 - (m - 1)H^m$ to rephrase the second inequality of Theorem 2 in terms of the sectional genus $g(X, H)$.

For several examples of smooth and connected $m$-dimensional projective variety with stable tangent bundle, see several papers quoted in [2]. For varieties with semistable tangent bundles we may add the Abelian varieties and varieties with an Abelian variety as an étale covering.

Theorems 1 and 2 are about the sectional genus. However, if $m \geq 2$, then they should imply bounds on the intermediate cohomology $\bigoplus_{t \in \mathbb{Z}} H^i(Y, \mathcal{O}_Y(t))$ and $\bigoplus_{t \in \mathbb{Z}} H^i(Y, L \otimes \mathcal{O}_Y(t))$, $1 \leq i \leq m - 1$ (see Proposition 1 for the smooth case).

## 2. The Proofs

**Lemma 1.** Let $F$ be a rank $k$ torsion free sheaf on $X$. Fix a rank $k$ vector bundle $E$ on $X$ and an ample line bundle $H$ on $X$. Then there is an integer $x$ such that for all integers $t \geq x$ there is an inclusion $F \hookrightarrow E \otimes H^\otimes t$. We call $k(F, E, H)$ the integer $x$ just introduced.

**Proof.** Let $y$ be the minimal integer $x$ such that for all integers $t \geq x$ the vector bundle $E \otimes H^\otimes t$ is spanned (it exists by the definition of ample line bundle). Fix any integer $t \geq y$. For a general $k$-dimensional vector space
V \subset H^0(X, E \otimes H^{\otimes t})$ the evaluation map $V \otimes O_X \to E \otimes H^{\otimes t}$. Thus we may take $y = k(O_X^{\otimes k}, E, H)$ (here we have = and not just $\geq$ by the minimality assumption of $y$).

Since $F$ is torsion free, the natural map $j_F : F \to F^{\vee \vee}$ is injective and for any vector bundle $G$ any injective map $F \to G$ factors through $j_F$. Thus $k(F, E, H)$ exists if and only if $k(F^{\vee \vee}, E, H)$ exists. Moreover, if any of them exists, then they are equal. Hence it is sufficient to prove the existence of the integer $k(F^{\vee \vee}, E, H)$. Let $w$ be the minimal integer such that $F^{\vee \vee} \otimes O_X$ is ample. Fix an integer $i$ such that $1 \leq i \leq m - 1$. Then $H^i(Y, O_Y(t)) = 0$ if $mR \cdot H^{m-1} \geq -k \cdot \omega_X \cdot H^{m-1}/m$. Hence the non-zero maps $\Omega^{\otimes k}_X \to M$ and $\Omega^{\otimes k}_X \to M$ give $mR \cdot H^{m-1} \geq -k \cdot \omega_X \cdot H^{m-1}$ and $(m_m - 1) qH^m \geq -\omega_X \cdot H^{m-1}$. Taking $0 < \eta \ll 1$ we conclude.

Proof of Theorem 1. For any coherent sheaf $\mathcal{F}$ on $Y$ let $T(\mathcal{F})$ denote its torsion subsheaf. There is an injective map $j_Y : \Omega^m_Y/T(\Omega^m_Y) \to I_{\Sigma_Y} \omega_Y$ (see [3], Subsection 3.1). Since $L$ is locally free, the map $\eta : \Omega^m_Y \to L$ induces a non-zero map $\eta' : \Omega^m_Y/T(\Omega^m_Y) \to L$. Set $\eta'' := \eta' \circ j : I_{\Sigma_Y} \omega_Y \to L$. Notice that $\eta''$ is an isomorphism on the Zariski open dense subset of $Y$. Take the intersection $m - 1$ times with $O_Y(1)$ and use the definition of $T$.

Kodaira’s vanishing gives the following result.

Proposition 1. Assume $\text{char}(K) = 0$ and $Y$ smooth. Let $x$ be the minimal integer such that $\omega_Y \otimes O_Y(x)$ is ample, $y$ the minimal integer such that $L \otimes O_Y(y)$ is ample and $z$ the minimal integer such that $O_Y(z) \otimes L^*$ is ample. Fix an integer $i$ such that $1 \leq i \leq m - 1$. Then $H^i(Y, O_Y(t)) = 0$ if
either $t \geq x + 1$ or $t < 0$ and $H^i(Y, L \otimes O_Y(t)) = 0$ if either $t \geq x + y$ or $t \leq -z$.

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References


