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CONSTRUCTION OF NON-MSF WAVELETS FROM TWO-INTERVAL AND THREE-INTERVAL WAVELET SETS

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Abstract: In this paper, we construct non-MSF wavelets in $L^2(\mathbb{R})$ from two-interval and three-interval wavelet sets.

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1. Introduction

Chui and Shi [3], proved that for a dilation a such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{N}$, the only wavelets that exist are MSF wavelets. However, the converse is not necessarily true. Bownik and Speegle [2], showed the existence of non-MSF wavelets for the dilation $a > 1$, for which there exists a $p \in \mathbb{Z} \setminus \{0\}$ such that $a^p\mathbb{Z} \cap \mathbb{Z} \neq \{0\}$, although their result dealt with higher dimensions.

In this paper, we construct non-MSF wavelets from two-interval and three-interval wavelet sets characterized by Ha, Kang, Lee and Seo [7], following a procedure whose germ lies in the paper of Bownik and Speegle [2]. In this process, the measure of the support of the Fourier transform of the newly constructed wavelet becomes larger than 2π, and also the modulus of its Fourier transform becomes different from a characteristic function, and thus the wavelet constructed happened to be a non-MSF wavelet. Furthermore, we obtain that non-MSF wavelets constructed from two-interval wavelet sets are all MRA wavelets

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while some of the non-MSF wavelets constructed from the three-interval wavelet sets are non-MRA. It is pertinent to mention that Vyas [11], has constructed non-MSF non-MRA wavelets for $L^2(\mathbb{R})$ through a similar technique using symmetric four-interval wavelet sets. Besides she constructed a family of non-MSF non-MRA H^2-wavelets which includes the one constructed by Behera [1].

2. Preliminaries

A *dyadic orthonormal wavelet* (or, simply a *wavelet*) in one dimension is a unit vector $\psi \in L^2(\mathbb{R})$, with the property that the set

$$\left\{2^{\frac{j}{2}}\psi\left(2^j x - k\right) \;:\; j, k \in \mathbb{Z}\right\}$$

of dilations of ψ by integral powers of 2 followed by all integral translates, forms an orthonormal basis for $L^2(\mathbb{R})$. The following result gives a characterization of wavelets.

Theorem 1. (see [6], Theorem 1) *A function* $\psi \in L^2(\mathbb{R})$, *with* $\|\psi\|_2 = 1$, *is a wavelet if and only if*

(i) $\sum_{n\in\mathbb{Z}} |\hat{\psi}(2^n\xi)|^2 = 1$, *for a.e.* $\xi \in \mathbb{R}$.

(ii) $\sum_{n=0}^{\infty} \hat{\psi}(2^n\xi)\overline{\hat{\psi}(2^n(\xi + 2m\pi))} = 0$, *for a.e.* $\xi \in \mathbb{R}$ *and for each* $m \in 2\mathbb{Z}+1$.

Recognizing underlying concept in the theory of wavelets Mallat [9] developed the notion of 'Multiresolution Analysis' which proved to be an elegant method to construct wavelets.

Definition 2. (see [9]) A *multiresolution analysis*, or simply an MRA, is a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ satisfying following conditions:

(1) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$,

(2) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$, for all $j \in \mathbb{Z}$,

(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,

(4) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(5) There exists a function $\varphi \in V_0$, such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

The following characterizes those wavelets ψ which arise from an MRA.

Theorem 3. (see [8], Chapter 7, Theorem 3.2) *A wavelet* $\psi \in L^2(\mathbb{R})$ *is associated with an MRA if and only if*

$$D_\psi(\xi) = \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{Z}} \left|\hat{\psi}\left(2^n\left(\xi + 2m\pi\right)\right)\right|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R}.$$

In [8], it has been shown that if $\psi \in L^2(\mathbb{R})$ is a wavelet, then $|\text{supp}(\hat{\psi})| \geq 2\pi$. Wavelets, whose Fourier transforms have minimal support of measure 2π are called *minimally supported frequency* (MSF) *wavelets* by Fang and Wang [5]. Independent of Fang and Wang, Dai and Larson [4], studied the same class of wavelets and introduced the concept of wavelet sets.

A measurable set $W \subset \mathbb{R}$ is said to be a *wavelet set* if $|\hat{\psi}| = \chi_W$ for some wavelet ψ on $L^2(\mathbb{R})$. Such a wavelet ψ is called an *s-elementary wavelet*, see [4]. MSF wavelets are indeed those wavelets which are associated with wavelet sets. One of the earliest examples of wavelet sets is the Shannon or Littlewood-Paley wavelet set $[-2\pi, -\pi) \cup [\pi, 2\pi)$.

Let $\tau : \mathbb{R} \to [0, 2\pi)$, denotes the map sending an element x in $\mathbb{R}$ to $x + 2n_x\pi$, and δ denotes the map from $\mathbb{R} \setminus \{0\}$ to $[-2\pi, -\pi) \cup [\pi, 2\pi)$ which sends x to $2^{m_x}x$, where n_x, $m_x \in \mathbb{Z}$ are unique integers. Then, we have

Theorem 4. (see [4]) *Let $E \subseteq \mathbb{R}$ be a measurable set. Then E is a wavelet set iff $\tau_{|E}$ and $\delta_{|E}$ both are measurable bijections. Equivalently, E is a wavelet set iff*

(i) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} (E + 2n\pi)$, *a.e., and*

(ii) $\mathbb{R} = \dot{\bigcup}_{n\in\mathbb{Z}} 2^n E$, *a.e.,*

where $\dot{\bigcup}$ *denotes the disjoint union.*

3. Non-MSF Wavelets from Two-Interval Wavelet Sets

In this Section, we employ two-interval wavelet sets to construct non-MSF wavelets. Further, we find that these wavelets are associated with MRA.

Wavelet sets with two intervals are $[2a - 4\pi,\ a - 2\pi] \cup [a,\ 2a]$, for some $0 < a < 2\pi$. Let $I_a \equiv [2a - 4\pi,\ a - 2\pi]$ and $J_a \equiv [a,\ 2a]$.

Lemma 5. *Let $E_a^r = (2^{-r}J_a) \cup J_a \cup (J_a - 2^{r+1}\pi) \cup I_a$, where $r \in \mathbb{N}$ and $0 < a \leq \frac{2^{r+1}\pi}{2^{r+1}-1}$.*

(i) If $\xi \in 2^{-r}J_a$, then $\xi + 2k\pi \in E_a^r$ if and only if $k = 0, -1$, and $2^k\xi \in E_a^r$ if and only if $k = 0, r$.

(ii) If $\xi \in J_a$, then $\xi + 2k\pi \in E_a^r$ if and only if $k = 0, -2^r$, and $2^k\xi \in E_a^r$ if and only if $k = 0, -r$.

(iii) If $\xi \in J_a - 2^{r+1}\pi$, then $\xi + 2k\pi \in E_a^r$ if and only if $k = 0, 2^r$, and $2^k\xi \in E_a^r$ if and only if $k = 0, -r$.

(iv) *If* $\xi \in 2^{-r}J_a - 2\pi$, *then* $\xi + 2k\pi \in E_a^r$ *if and only if* $k = 0,\ 1$, *and* $2^k\xi \in E_a^r$ *if and only if* $k = 0,\ r$.

(v) *If* $\xi \in I_a \setminus (2^{-r}J_a - 2\pi)$, *then* $\xi + 2k\pi \in E_a^r$ *if and only if* $k = 0$, *and* $2^k\xi \in E_a^r$ *if and only if* $k = 0$.

Proof. It is easy to see that for every measurable subset $E \subset \mathbb{R}$, $\tau(E) = \tau(E + 2k\pi)$ and $\delta(E) = \delta(2^kE)$, where $k \in \mathbb{Z}$. Therefore,

$$\tau(2^{-r}J_a) = \tau(2^{-r}J_a - 2\pi), \quad \tau(J_a) = \tau(J_a - 2^{r+1}\pi), \qquad \text{and} \tag{1}$$

$$\delta(2^{-r}J_a) = \delta(J_a), \quad \delta(2^{-r}J_a - 2\pi) = \delta(J_a - 2^{r+1}\pi). \tag{2}$$

Further, if W is a wavelet set and E, F are two subsets of W such that $E \cap F = \phi$, then $\tau(E) \cap \tau(F) = \phi$ and $\delta(E) \cap \delta(F) = \phi$. Therefore,

$$\tau(J_a) \cap \tau(2^{-r}J_a - 2\pi) = \phi, \qquad \text{and} \tag{3}$$

$$\tau(I_a \setminus (2^{-r}J_a - 2\pi)) \cap \tau(2^{-r}J_a - 2\pi) = \phi. \tag{4}$$

From (1), (3) and (4), we get

$$\tau(2^{-r}J_a) \cap \tau(I_a \setminus (2^{-r}J_a - 2\pi)) = \phi,$$

$$\tau(2^{-r}J_a) \cap \tau(J_a) = \phi, \quad \tau(2^{-r}J_a) \cap \tau(J_a - 2^{r+1}\pi) = \phi.$$

Therefore, if $\xi \in 2^{-r}J_a$, then $\xi + 2k\pi \in E_a^r$ if and only if $k = 0,\ -1$. Similarly,

$$\delta(J_a) \cap \delta(2^{-r}J_a - 2\pi) = \phi, \qquad \text{and} \tag{5}$$

$$\delta(I_a \setminus (2^{-r}J_a - 2\pi)) \cap \delta(J_a) = \phi. \tag{6}$$

From (2), (5) and (6), we get

$$\delta(2^{-r}J_a) \cap \delta(I_a \setminus (2^{-r}J_a - 2\pi)) = \phi,$$

$$\delta(2^{-r}J_a) \cap \delta(2^{-r}J_a - 2\pi) = \phi, \quad \delta(2^{-r}J_a) \cap \delta(J_a - 2^{r+1}\pi) = \phi.$$

Thus, if $\xi \in 2^{-r}J_a$, then $2^k\xi \in E_a^r$ if and only if $k = 0,\ r$. This proves (i) of Lemma. Similarly, we can prove (ii)-(v). □

Lemma 6. *Let* $E_a^{'r} = (2^{-r}I_a) \cup I_a \cup (I_a + 2^{r+1}\pi) \cup J_a$, *where* $r \in \mathbb{N}$ *and* $\frac{2(2^r-1)\pi}{2^{r+1}-1} \le a < 2\pi$.

(i) *If* $\xi \in 2^{-r}I_a$, *then* $\xi + 2k\pi \in E_a^{'r}$ *if and only if* $k = 0,\ 1$, *and* $2^k\xi \in E_a^{'r}$ *if and only if* $k = 0,\ r$.

(ii) *If* $\xi \in I_a$, *then* $\xi + 2k\pi \in E_a^{'r}$ *if and only if* $k = 0,\ 2^r$, *and* $2^k\xi \in E_a^{'r}$ *if and only if* $k = 0,\ -r$.

(iii) *If* $\xi \in I_a + 2^{r+1}\pi$, *then* $\xi + 2k\pi \in E_a^{'r}$ *if and only if* $k = 0,\ -2^r$, *and* $2^k\xi \in E_a^{'r}$ *if and only if* $k = 0,\ -r$.

(iv) *If* $\xi \in 2^{-r}I_a + 2\pi$, *then* $\xi + 2k\pi \in E_a^{'r}$ *if and only if* $k = 0,\ -1$, *and* $2^k\xi \in E_a^{'r}$ *if and only if* $k = 0,\ r$.

(v) *If* $\xi \in J_a \setminus (2^{-r}I_a + 2\pi)$, *then* $\xi + 2k\pi \in E_a^{'r}$ *if and only if* $k = 0$, *and* $2^k\xi \in E_a^{'r}$ *if and only if* $k = 0$.

Proof. It is analogous to that of Lemma 5. □

Theorem 7. *For* $r \in \mathbb{N}$, *and* $0 < a \le \frac{2^{r+1}\pi}{2^{r+1}-1}$, ψ_a^r *defined by*

$$\hat{\psi}_a^r(\xi) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \in J_a \cup (2^{-r}J_a) \cup (2^{-r}J_a - 2\pi), \\ -\frac{1}{\sqrt{2}}, & \text{if } \xi \in J_a - 2^{r+1}\pi, \\ 1, & \text{if } \xi \in I_a \setminus (2^{-r}J_a - 2\pi), \\ 0, & \text{otherwise}, \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$. *In addition, it is an MRA wavelet.*

Proof. To prove the theorem we make use of Lemma 5 and show that ψ_a^r satisfies all the three conditions required for a wavelet given in Theorem 1.

(1) Since

$$\left\|\hat{\psi}_a^r\right\|_2^2 = \int_{\mathbb{R}} |\hat{\psi}_a^r(\xi)|^2\, d\xi$$
$$= \frac{1}{2}\left(\frac{1}{2^r}|J_a| + |J_a| + \frac{1}{2^r}|J_a| + |J_a|\right) + |I_a| - \frac{1}{2^r}|J_a|$$
$$= |I_a| + |J_a| = 2\pi,$$

we have $\|\psi_a^r\|_2 = 1$.

(2) To show that

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}_a^r(2^n\xi)|^2 = 1, \qquad \text{for a.e. } \xi \in \mathbb{R},$$

we write

$$\sum_{n\in\mathbb{Z}} |\hat{\psi}_a^r(2^n\xi)|^2 = \rho(\xi),$$

and observe that $\rho(2\xi) = \rho(\xi)$ for every $\xi \in \mathbb{R}$. Therefore, it is sufficient to prove that $\rho(\xi) = 1$ on any subset E of $\mathbb{R}$ such that $\delta(E) = [-2\pi, -\pi) \cup [\pi, 2\pi)$. For if $\xi \in \mathbb{R}$, then $2^k\xi \in E$ for some $k \in \mathbb{Z}$. Then $\rho(2^k\xi) = 1$, and hence $\rho(\xi) = 1$. We may take

$$E = I_a \cup J_a = (I_a \setminus (2^{-r}J_a - 2\pi)) \cup (2^{-r}J_a - 2\pi) \cup J_a.$$

If $\xi \in J_a$, then from Lemma 5, $2^k\xi \in \operatorname{supp}(\hat{\psi}_a^r) = E_a^r$ iff $k = 0$ or $k = -r$. Therefore,

$$\begin{aligned}\rho(\xi) &= |\hat{\psi}_a^r(\xi)|^2 + |\hat{\psi}_a^r(2^{-r}\xi)|^2 \\ &= \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.\end{aligned}$$

Similarly, it can be shown that $\rho(\xi) = 1$, for every $\xi \in I_a$.

(3) Finally, we have to show that

$$t_q(\xi) \equiv \sum_{k=0}^{\infty} \hat{\psi}_a^r(2^k\xi)\overline{\hat{\psi}_a^r(2^k(\xi + 2q\pi))} = 0, \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and } q \in 2\mathbb{Z} + 1.$$

Since $t_{-q}(\xi) = \overline{t_q(\xi - 2q\pi)}$, it is enough to prove that $t_q = 0$, a.e., for all positive odd integers q. The term

$$\hat{\psi}_a^r(2^k\xi)\overline{\hat{\psi}_a^r(2^k(\xi + 2q\pi))}$$

is non-zero if and only if both $2^k\xi$ and $2^k\xi + 2.2^k q.\pi$ are in $\operatorname{supp}(\hat{\psi}_a^r)$. This is possible, if either

(a) $2^k\xi \in 2^{-r}J_a - 2\pi$ and $2^k q = 1$, by Lemma 5 (iv),

or

(b) $2^k\xi \in J_a - 2^{r+1}\pi$ and $2^k q = 2^r$, by Lemma 5 (iii).

If (a), then $2^{k+r}\xi \in J_a - 2^{r+1}\pi$, and

$$\begin{aligned}t_q(\xi) &= \hat{\psi}_a^r(2^k\xi)\overline{\hat{\psi}_a^r(2^k\xi + 2\pi)} + \hat{\psi}_a^r(2^{k+r}\xi)\overline{\hat{\psi}_a^r(2^{k+r}\xi + 2^{r+1}\pi)} \\ &= \frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right).\frac{1}{\sqrt{2}} = 0.\end{aligned}$$

Similar is the situation if (b) is true.

Next, we prove that ψ_a^r is an MRA wavelet. Writing

$$\hat{\psi}_a^r(\xi) = \frac{1}{\sqrt{2}}\chi_{J_a \cup (2^{-r}J_a) \cup (2^{-r}J_a - 2\pi)}(\xi) - \frac{1}{\sqrt{2}}\chi_{J_a - 2^{r+1}\pi}(\xi) + \chi_{I_a \setminus (2^{-r}J_a - 2\pi)}(\xi),$$

we have

$$|\hat{\psi}_a^r(\xi)|^2 = \frac{1}{2}\chi_{J_a\cup(2^{-r}J_a)\cup(2^{-r}J_a-2\pi)}(\xi) + \frac{1}{2}\chi_{J_a-2^{r+1}\pi}(\xi) + \chi_{I_a\setminus(2^{-r}J_a-2\pi)}(\xi)$$

$$= \chi_{I_a\cup J_a}(\xi) + \frac{1}{2}\chi_{(2^{-r}J_a)\cup(J_a-2^{r+1}\pi)}(\xi) - \frac{1}{2}\chi_{J_a\cup(2^{-r}J_a-2\pi)}(\xi),$$

and

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} |\hat{\psi}_a^r(2^j(\xi+2k\pi))|^2 = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{I_a\cup J_a}(2^j(\xi+2k\pi))$$

$$+\frac{1}{2}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{(2^{-r}J_a)\cup(J_a-2^{r+1}\pi)}(2^j(\xi+2k\pi))$$

$$-\frac{1}{2}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{J_a\cup(2^{-r}J_a-2\pi)}(2^j(\xi+2k\pi)).$$

Now, easily, we have

$$\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{(2^{-r}J_a)\cup(J_a-2^{r+1}\pi)}(2^j(\xi+2k\pi)) = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{J_a\cup(2^{-r}J_a-2\pi)}(2^j(\xi+2k\pi)),$$

for every $\xi \in \mathbb{R}$. Therefore,

$$D_{\psi_a^r}(\xi) = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} |\hat{\psi}_a^r(2^j(\xi+2k\pi))|^2 = \sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \chi_{I_a\cup J_a}(2^j(\xi+2k\pi))$$

and hence the nature of the non-MSF wavelet ψ_a^r constructed in this way is the same as that of the wavelet set through which it is constructed. Since the wavelet set by which ψ_a^r is constructed is a two-interval wavelet set and a two-interval wavelet set is associated with an MRA [7], it follows that ψ_a^r is an MRA wavelet. □

Theorem 8. *For $r \in \mathbb{N}$, and $\frac{2(2^r-1)\pi}{2^{r+1}-1} \le a < 2\pi$, $\psi_a'^r$ defined by*

$$\hat{\psi}_a'^r(\xi) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \in I_a \cup (2^{-r}I_a) \cup (2^{-r}I_a + 2\pi), \\ -\frac{1}{\sqrt{2}}, & \text{if } \xi \in I_a + 2^{r+1}\pi, \\ 1, & \text{if } \xi \in J_a \setminus (2^{-r}I_a + 2\pi), \\ 0, & \text{otherwise}, \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$. In addition, it is an MRA wavelet.

Proof. It is similar to that of Theorem 7. □

4. Non-MSF Wavelets from Three-Interval Wavelet Sets

In this Section, we use three-interval wavelet sets to construct non-MSF wavelets. Further it is noted that some of these wavelets are not associated with MRA.

Wavelet sets with three intervals are given by

$W(j,m) \equiv I_{j,m} \cup J_{j,m} \cup K_{j,m}$, where

$$I_{j,m} \equiv \left[-2\left(1 - \frac{2m+1}{2^{j+1}-1}\right)\pi\,,\, -\left(1 - \frac{2m+1}{2^{j+1}-1}\right)\pi \right],$$

$$J_{j,m} \equiv \left[\frac{2(m+1)\pi}{2^{j+1}-1}\,,\, \frac{2(2m+1)\pi}{2^{j+1}-1} \right],$$

$$K_{j,m} \equiv \left[\frac{2^{j+1}(2m+1)\pi}{2^{j+1}-1}\,,\, \frac{2^{j+2}(m+1)\pi}{2^{j+1}-1} \right],$$

and j, m are natural numbers such that $j \geq 2$ and $1 \leq m \leq 2^j - 2$.

We state following Lemmas which can be established in similar way as those of Lemmas 5 and 6.

Lemma 9. *Let* $E^r_{j,m} = I_{j,m} \cup (2^{-r} J_{j,m}) \cup J_{j,m} \cup (J_{j,m} + 2^{r+1}(m+1)\pi) \cup K_{j,m}$, *where* $r \in \mathbb{N}$.

(i) If $\xi \in I_{j,m}$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0$.

(ii) If $\xi \in 2^{-r} J_{j,m}$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0,\, m+1$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0,\, r$.

(iii) If $\xi \in J_{j,m}$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0,\, 2^r(m+1)$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0,\, -r$.

(iv) If $\xi \in J_{j,m} + 2^{r+1}(m+1)\pi$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0,\, -2^r(m+1)$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0,\, -r$.

(v) If $\xi \in 2^{-r} J_{j,m} + 2(m+1)\pi$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0,\, -(m+1)$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0,\, r$.

(vi) If $\xi \in K_{j,m} \setminus (2^{-r} J_{j,m} + 2(m+1)\pi)$, *then* $\xi + 2k\pi \in E^r_{j,m}$ *if and only if* $k = 0$, *and* $2^k\xi \in E^r_{j,m}$ *if and only if* $k = 0$.

Lemma 10. *Let* $E'^r_{j,m} = (2^{-r} I_{j,m}) \cup I_{j,m} \cup (I_{j,m} + 2^{r+1}(m+1)\pi) \cup J_{j,m} \cup K_{j,m}$, *where* $r \in \mathbb{N}$.

(i) *If* $\xi \in 2^{-r}I_{j,m}$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ m+1$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ r$.

(ii) *If* $\xi \in I_{j,m}$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ 2^r(m+1)$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ -r$.

(iii) *If* $\xi \in I_{j,m} + 2^{r+1}(m+1)\pi$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ -2^r(m+1)$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ -r$.

(iv) *If* $\xi \in J_{j,m}$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0$.

(v) *If* $\xi \in 2^{-r}I_{j,m} + 2(m+1)\pi$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ -(m+1)$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0,\ r$.

(vi) *If* $\xi \in K_{j,m} \setminus (2^{-r}I_{j,m} + 2(m+1)\pi)$, *then* $\xi + 2k\pi \in E'^{r}_{j,m}$ *if and only if* $k = 0$, *and* $2^k\xi \in E'^{r}_{j,m}$ *if and only if* $k = 0$.

Theorem 11. *For* $j \geq 2$, $1 \leq m \leq 2^j - 2$, *and* $r \in \mathbb{N}$, $\psi^r_{j,m}$ *defined by*

$$\hat{\psi}^r_{j,m}(\xi) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \in J_{j,m} \cup (2^{-r}J_{j,m}) \cup (2^{-r}J_{j,m} + 2(m+1)\pi), \\ -\frac{1}{\sqrt{2}}, & \text{if } \xi \in J_{j,m} + 2^{r+1}(m+1)\pi, \\ 1, & \text{if } \xi \in I_{j,m} \cup (K_{j,m} \setminus (2^{-r}J_{j,m} + 2(m+1)\pi)), \\ 0, & \text{otherwise}, \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$.

Proof. It can be obtained in a similar way as the proof of Theorem 7. □

Using Lemma 10, we obtain the following Theorem in an analogous manner to that of Theorem 11.

Theorem 12. *For* $j \geq 2$, $1 \leq m \leq 2^j - 2$, *and* $r \in \mathbb{N}$, $\psi'^r_{j,m}$ *defined by*

$$\hat{\psi}'^r_{j,m}(\xi) = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } \xi \in I_{j,m} \cup (2^{-r}I_{j,m}) \cup (2^{-r}I_{j,m} + 2(m+1)\pi), \\ -\frac{1}{\sqrt{2}}, & \text{if } \xi \in I_{j,m} + 2^{r+1}(m+1)\pi, \\ 1, & \text{if } \xi \in J_{j,m} \cup (K_{j,m} \setminus (2^{-r}I_{j,m} + 2(m+1)\pi)), \\ 0, & \text{otherwise}, \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$.

Remark 13. In the proof of Theorem 7, we have seen that the nature of the non-MSF wavelet constructed is same as that of the underlying wavelet set in respect of multiresolution analysis. Since a three-interval wavelet set $W(j,m)$, where $j \geq 2$, $1 \leq m \leq 2^j - 2$ is non-MRA for odd m, see [7], Theorem 4.7, while it is MRA for $m = 2^j - 2$ (see [10]), therefore $\psi^r_{j,m}$ as well as $\psi'^r_{j,m}$ for $r \in \mathbb{N}$, and $j \geq 2$, is non-MRA if m is odd and MRA if $m = 2^j - 2$.

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