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FRACTIONAL DIFFERENCE INEQUALITIES OF VOLTERRA TYPE

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Abstract: In this work, fractional difference equation of Volterra type is considered and some fundamental inequalities and comparison results are obtained.

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1. Introduction

Fractional calculus has gained importance during the past three decades due to its applicability in diverse fields of science and engineering and lot of literature is available on the applications of fractional calculus in modeling mechanical and electrical properties of real materials. But the study of Theory of fractional differential equations was initiated and existence and uniqueness of solutions for different types of fractional differential equations have been established recently [8]. Much of literature is not available on fractional integro differential equations also, though theory of integro differential equations [9] has been almost all

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developed parallel to theory of differential equations. Very little progress has been made to develop the theory of analogous fractional difference equations, particularly of Volterra type.

In this direction, Diaz and Osler [5] defined the fractional difference by the rather natural approach of allowing the index of differencing, in the standard expression for the n^{th} difference, to be any real or complex number. Later, R.Hirota [12], defined the difference operator ∇^{α} using Taylor's series. G.V.S.R.Deekshitulu and J.Jagan Mohan [3] modified the definition of Atsushi Nagai [1] for $0 < \alpha \leq 1$ in such a way that the expression for ∇^{α} does not involve any difference operator and derived some basic inequalities and comparison theorems.

The inequalities and comparison principles which provide explicit bounds on unknown functions play a very important role in the theory of finite difference equations. In the present paper, the authors considered an initial value problem of volterra type of fractional order and some basic difference inequalities and comparison results are obtained.

2. Preliminaries

Let u_n be any function defined for $n \in \mathbb{N}_0^+$ where $\mathbb{N}_a^+ = \{a, a+1, a+2, ...\}$ for $a \in \mathbb{Z}$. Hirota [12] took the first n terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-B)^{\alpha}$ and gave the following definition.

Definition 1. Let $\alpha \in \mathbb{R}$. Then difference operator of order α is defined by

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \binom{\alpha}{j} (-1)^j u_{n-j}, & \alpha \neq 1,2,..., \\ \varepsilon^{-m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j u_{n-j}, & \alpha = m \in \mathbb{Z}_{>0}. \end{cases} \tag{1}$$

Here $\binom{a}{n}$, $(a \in \mathbb{R}, n \in \mathbb{Z})$ stands for a binomial coefficient defined by

$$\binom{a}{n} = \begin{cases} \frac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)}, & n > 0, \\ 1, & n = 0, \\ 0, & n < 0. \end{cases} \tag{2}$$

In 2002, Atsushi Nagai [1] introduced another definition of fractional difference which is a slight modification of Hirota's fractional difference operator.

Definition 2. Let $\alpha \in \mathbb{R}$ and m be an integer such that $m-1 < \alpha \leq m$. The difference operator $\Delta_{*,-n}$ of order α is defined as

$$\Delta^{\alpha}_{*,-n} u_n = \Delta^{\alpha-m}_{-n} \Delta^{m}_{-n} u_n = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \binom{\alpha-m}{j} (-1)^j \Delta^{m}_{-(n-j)} u_{n-j}. \quad (3)$$

G.V.S.R.Deekshitulu and J.Jagan Mohan [3] gave a more convenient form of the above definition by taking $m = 1$ and reorganizing the terms as follows.

Definition 3. Let $\alpha \in \mathbb{R}$ such that $0 < \alpha \leq 1$. The difference operator ∇ of order α is defined as

$$\nabla^{\alpha} u_n = \sum_{j=0}^{n-1} \binom{j-\alpha}{j} \nabla u_{n-j}. \quad (4)$$

Remark 1. *For any* $\alpha \in \mathbb{R}$ $(0 < \alpha \leq 1)$,

$$\nabla^{-\alpha} u_n = \sum_{j=0}^{n-1} \binom{j+\alpha}{j} \nabla u_{n-j}. \quad (5)$$

Further $\nabla^{\alpha} u_n$ and $\nabla^{-\alpha} u_n$ can be expressed in the terms of the arguments of u_n as for any $\alpha \in \mathbb{R}$ $(0 < \alpha \leq 1)$,

$$\nabla^{\alpha} u_n = u_n - \binom{n-1-\alpha}{n-1} u_0 - \alpha \sum_{j=1}^{n-1} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} u_{n-j} \quad (6)$$

and

$$\nabla^{-\alpha} u_n = u_n - \binom{n-1+\alpha}{n-1} u_0 + \alpha \sum_{j=1}^{n-1} \frac{1}{(j+\alpha)} \binom{j+\alpha}{j} u_{n-j} \quad (7)$$

i.e.

$$\nabla^{-\alpha} u_n = \sum_{j=1}^{n} \binom{n-j+\alpha-1}{n-j} u_j - \binom{n-1+\alpha}{n-1} u_0. \quad (8)$$

Remark 2. *The difference operator* ∇ *of order* α *satisfies the following properties.*

i *For any real numbers* α *and* β, $\nabla^{\alpha}\nabla^{\beta} u_n = \nabla^{\alpha+\beta} u_n$.

ii *For any constant 'c', $\nabla^\alpha[cu_n+v_n] = c\nabla^\alpha u_n + \nabla^\alpha v_n$ where v_n be any function defined for $n \in \mathbb{N}_0^+$.*

iii *For $\alpha \in \mathbb{R}$, $\nabla^\alpha(u_n v_n) = \sum_{m=0}^{n-1} \binom{\alpha}{m} [\nabla^{\alpha-m} u_{n-m}][\nabla^m v_n]$.*

iv $\nabla^\alpha u_0 = 0$ *and* $\nabla^\alpha u_1 = u_1 - u_0 = \nabla u_1$.

v $\nabla^\alpha \nabla^{-\alpha} u_n = \nabla^{-\alpha}\nabla^\alpha u_n = u_n - u_0$.

vi $\nabla^\alpha \nabla^{-\alpha}(u_n - u_0) = \nabla^{-\alpha}\nabla^\alpha (u_n - u_0) = u_n - u_0$.

Definition 4. Let $f(n, r, s)$ be a function defined for $n \in \mathbb{N}_0^+$, $0 \le r < \infty$, $0 \le s < \infty$ and $g(n, m, r)$ be a function defined for n, $m \in \mathbb{N}_0^+$, $m \le n$, $0 \le r < \infty$. Let v_n be a function defined for $n \in \mathbb{N}_0^+$. Then a nonlinear fractional difference equation of Volterra type is of the form

$$\nabla^\alpha v_{n+1} = f(n, v_n, \sum_{m=0}^{n-1} g(n, m, v_m)), \;\; v(0) = v_0. \tag{9}$$

3. Main Results

In this section we deal with a nonlinear fractional difference equation of Volterra type of the form (9) where $f(n, r, s)$ is a nonnegative and nondecreasing function with respect to r and s, $0 \le r, s < \infty$ for any fixed $n \in \mathbb{N}_0^+$ and $g(n, m, r)$ is a nonnegative and nondecreasing function with respect to r, $0 \le r < \infty$ for any fixed n, $m \in \mathbb{N}_0^+$.

Theorem 5. *Let v_n and w_n be any two nonnegative functions defined for $n \in \mathbb{N}_0^+$. Suppose that for $n \in \mathbb{N}_0^+$ and $0 < \alpha \le 1$, the inequalities*

$$\nabla^\alpha v_{n+1} \le f(n, v_n, \sum_{m=0}^{n-1} g(n, m, v_m)), \tag{10}$$

$$\nabla^\alpha w_{n+1} \ge f(n, w_n, \sum_{m=0}^{n-1} g(n, m, w_m)) \tag{11}$$

hold. Then $v_0 \leq w_0$ *implies*

$$v_n \leq w_n \tag{12}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Suppose that (12) is not true. Then, because of $v_0 \leq w_0$, there exists a $k \in \mathbb{N}_0^+$ such that $v_m \leq w_m$ for $m \leq k$ and $v_{k+1} > w_{k+1}$. From the monotone properties of f and g, for $m \leq k$,

$$\begin{aligned} v_m \leq w_m \quad &\Rightarrow \quad g(k,m,v_m) \leq g(k,m,w_m) \\ &\Rightarrow \quad \sum_{m=0}^{k-1} g(k,m,v_m) \leq \sum_{m=0}^{k-1} g(k,m,w_m) \text{ (since } g \text{ is nonnegative)} \\ &\Rightarrow \quad f(k, v_k, \sum_{m=0}^{k-1} g(k,m,v_m)) \leq f(k, w_k, \sum_{m=0}^{k-1} g(k,m,w_m)). \end{aligned}$$

Thus

$$\nabla^\alpha v_{k+1} \leq \nabla^\alpha w_{k+1}. \tag{13}$$

Now consider

$$\begin{aligned} \nabla^\alpha w_{k+1} &= w_{k+1} - \binom{k-\alpha}{k} w_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} \\ &< v_{k+1} - \binom{k-\alpha}{k} v_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} v_{k+1-j} \\ &= \nabla^\alpha v_{k+1} \end{aligned}$$

which is a contradiction to (13). Hence the proof.

Corollary 6. *Let f(n,r) be a nonnegative function defined for* $n \in \mathbb{N}_0^+$, $0 \leq r < \infty$ *and* $g(n,m,r)$ *be as above. Let* v_n *and* w_n *be any two nonnegative functions defined for* $n \in \mathbb{N}_0^+$. *Suppose that for* $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$, *the inequalities*

$$\nabla^\alpha v_{n+1} \leq f(n, v_n) + \sum_{m=0}^{n-1} g(n,m,v_m), \tag{14}$$

$$\nabla^\alpha w_{n+1} \geq f(n, w_n) + \sum_{m=0}^{n-1} g(n,m,w_m) \tag{15}$$

hold. If for any $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$,

$$f(n, v_n) - f(n, w_n) \leq -\alpha(v_n - w_n) \tag{16}$$

then $v_0 \leq w_0$ *implies*

$$v_n \leq w_n \tag{17}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Suppose that (17) is not true. Then because of $v_0 \leq w_0$ there exists a $k \in \mathbb{N}_0^+$ such that $v_m \leq w_m$ for $m \leq k$ and

$$v_{k+1} > w_{k+1}. \tag{18}$$

From the monotone property of g, for $m \leq k$,

$$\begin{aligned} v_m \leq w_m \quad &\Rightarrow \quad g(k, m, v_m) \leq g(k, m, w_m) \\ &\Rightarrow \quad \sum_{m=0}^{k-1} g(k, m, v_m) \leq \sum_{m=0}^{k-1} g(k, m, w_m). \end{aligned}$$

Now using Remark (1) and (16),

$$\begin{aligned} & v_{k+1} \leq \\ & \binom{k-\alpha}{k} v_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} v_{k+1-j} + f(k, v_k) + \sum_{m=0}^{k-1} g(k, m, v_m) \\ =& \binom{k-\alpha}{k} v_0 + \alpha \sum_{j=2}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} v_{k+1-j} + \alpha v_k + f(k, v_k) + \sum_{m=0}^{k-1} g(k, m, v_m) \\ \leq& \binom{k-\alpha}{k} w_0 + \alpha \sum_{j=2}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} + \alpha w_k + f(k, w_k) \\ & + \sum_{m=0}^{k-1} g(k, m, w_m) \\ =& \binom{k-\alpha}{k} w_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} + f(k, w_k) + \sum_{m=0}^{k-1} g(k, m, w_m) \\ =& w_{k+1}, \end{aligned}$$

which is a contradiction to (18). Hence the proof.

Theorem 7. *Let w_n be solution of the difference equation*

$$\nabla^\alpha w_{n+1} = f(n, w_n, \sum_{m=0}^{n-1} g(n, m, w_m)),\ w(0) = w_0 \tag{19}$$

for all n, $m \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. Suppose that the inequality

$$\nabla^\alpha v_{n+1} \leq f(n, v_n, \sum_{m=0}^{n-1} g(n, m, v_m)) \tag{20}$$

satisfied for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$, where v_n is a nonnegative function defined for $n \in \mathbb{N}_0^+$ such that $v_0 \leq w_0$. Then

$$v_n \leq w_n \tag{21}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Consider (19) and (20). Applying Theorem 5, since $v_0 \leq w_0$ we obtain $v_n \leq w_n$.

Theorem 8. *Let $f_1(n, r, s)$ and $f_2(n, r, s)$ be two nonnegative and nondecreasing function with respect to r and s, $0 \leq r, s < \infty$ for any fixed $n \in \mathbb{N}_0^+$ and $g_1(n, m, r)$ and $g_2(n, m, r)$ be a nonnegative and nondecreasing function with respect to r, $0 \leq r < \infty$ for any fixed n, $m \in \mathbb{N}_0^+$. Let y_n be a function defined for $n \in \mathbb{N}_0^+$ and that*

$$f_1(n, x_n, \sum_{m=0}^{n-1} g_1(n, m, x_m)) \leq \nabla^\alpha y_{n+1} \leq f_2(n, y_n, \sum_{m=0}^{n-1} g_2(n, m, y_m)) \tag{22}$$

for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. Let v_n and w_n be the solutions of the difference equations

$$\nabla^\alpha v_{n+1} = f_1(n, x_n, \sum_{m=0}^{n-1} g_1(n, m, x_m)),\ v(0) = v_0, \tag{23}$$

$$\nabla^\alpha w_{n+1} = f_2(n, y_n, \sum_{m=0}^{n-1} g_2(n, m, y_m)),\ w(0) = w_0. \tag{24}$$

and suppose that $v_0 \leq y_0 \leq w_0$. Then

$$v_n \leq y_n \leq w_n, \qquad n \in \mathbb{N}_0^+. \tag{25}$$

Proof. Consider the second part of (22) and (24). i.e.

$$\begin{aligned}\nabla^\alpha y_{n+1} &\leq f_2(n, y_n, \sum_{m=0}^{n-1} g_2(n, m, y_m)),\\ \nabla^\alpha w_{n+1} &= f_2(n, y_n, \sum_{m=0}^{n-1} g_2(n, m, y_m)).\end{aligned}$$

Applying Theorem 7, since $y_0 \leq w_0$ we obtain the right half of the inequality in (25) i.e $y_n \leq w_n$. A similar argument yields the left half of the inequality (25).

Theorem 9. *Let x_n and y_n be solutions of the difference equations*

$$\nabla^\alpha x_{n+1} = f_1(n, x_n, \sum_{m=0}^{n-1} g_1(n, m, x_m)), \; x(0) = x_0, \tag{26}$$

$$\nabla^\alpha y_{n+1} = f_2(n, y_n, \sum_{m=0}^{n-1} g_2(n, m, y_m)), \; y(0) = y_0 \tag{27}$$

where the functions x_n, y_n, $g_1(n, m, r)$, $g_2(n, m, r)$, $f_1(n, r, s)$ and $f_2(n, r, s)$ are defined for n, $m \in \mathbb{N}_0^+$, $m \leq n$, $0 \leq r < \infty$, $0 \leq s < \infty$ and satisfy the conditions

$$|g_1(n, m, x_m) - g_2(n, m, y_m)| \leq g(n, m, |x_m - y_m|), \tag{28}$$

$$|f_1(n, x_m, u_m) - f_2(n, y_m, v_m)| \leq f(n, |x_m - y_m|, |u_m - v_m|) \tag{29}$$

for all n, $m \in N_0^+$ and $m \leq n$. Here u_n and v_n are any nonnegative functions defined for $n \in \mathbb{N}_0^+$. Let w_n be solution of the difference equation

$$\nabla^\alpha w_{n+1} = f(n, w_m, \sum_{m=0}^{n-1} g(n, m, w_m), \; w(0) = w_0 \tag{30}$$

for all n, $m \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. If

$$|x_0 - y_0| \leq w_0 \tag{31}$$

then

$$|x_n - y_n| \leq w_n \tag{32}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Define a function z_n by $z_n = |x_n - y_n|$. Then $z_0 = |x_0 - y_0| \leq w_0$. On account of the monotonicity of $f(n, r, s)$, we obtain, using Remark 2(v),

$$\begin{aligned} z_1 &= |x_1 - y_1| \;=\; |x_0 + f_1(0, x_0, 0) - y_0 - f_2(0, y_0, 0)| \\ &\leq |x_0 - y_0| + |f_1(0, x_0, 0) - f_2(0, y_0, 0)| \\ &\leq |x_0 - y_0| + f(0, |x_0 - y_0|, 0) \\ &\leq w_0 + f(0, w_0, 0) \\ &= w_0 + \nabla^\alpha w_1 \;=\; w_1. \end{aligned}$$

If the inequality $z_n \leq w_n$ is fulfilled for $n = 1, 2, ...k$, it follows by the monotonicity of $f(n, r, s)$ that

$$\begin{aligned} z_{k+1} &= |\binom{k-\alpha}{k} x_0 + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} x_{k+1-j} \\ &\quad + f_1(k, x_k, \sum_{m=0}^{k-1} g_1(k, m, x_m)) \\ &- \binom{k-\alpha}{k} y_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} y_{k+1-j} \\ &\quad - f_2(k, y_k, \sum_{m=0}^{k-1} g_2(k, m, y_m))| \\ &= |\binom{k-\alpha}{k} (x_0 - y_0) + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} (x_{k+1-j} - y_{k+1-j}) \\ &+ f_1(k, x_k, \sum_{m=0}^{k-1} g_1(k, m, x_m)) - f_2(k, y_k, \sum_{m=0}^{k-1} g_2(k, m, y_m))| \\ &\leq |\binom{k-\alpha}{k} (x_0 - y_0)| + |\alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} (x_{k+1-j} - y_{k+1-j})| \\ &+ |f_1(k, x_k, \sum_{m=0}^{k-1} g_1(k, m, x_m)) - f_2(k, y_k, \sum_{m=0}^{k-1} g_2(k, m, y_m))| \\ &\leq \binom{k-\alpha}{k} |x_0 - y_0| + \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} |x_{k+1-j} - y_{k+1-j}| \\ &+ f(k, |x_k - y_k|, |\sum_{m=0}^{k-1} g_1(k, m, x_m) - \sum_{m=0}^{k-1} g_2(k, m, y_m)|) \end{aligned}$$

$$
\begin{aligned}
&= \binom{k-\alpha}{k}|x_0 - y_0| + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}|x_{k+1-j} - y_{k+1-j}| \\
&+ f(k, |x_k - y_k|, |\sum_{m=0}^{k-1}(g_1(k,m,x_m) - g_2(k,m,y_m))| \\
&\leq \binom{k-\alpha}{k}|x_0 - y_0| + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}|x_{k+1-j} - y_{k+1-j}| \\
&+ f(k, |x_k - y_k|, \sum_{m=0}^{k-1}|(g_1(k,m,x_m) - g_2(k,m,y_m))| \\
&\leq \binom{k-\alpha}{k}|x_0 - y_0| + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}|x_{k+1-j} - y_{k+1-j}| \\
&+ f(k, |x_k - y_k|, \sum_{m=0}^{k-1} g(k,m,|x_m - y_m|) \\
&\leq \binom{k-\alpha}{k}w_0 + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}w_{k+1-j} \\
&\quad + f(k, |x_k - y_k|, \sum_{m=0}^{k-1} g(k,m,w_m)) \\
&\leq \binom{k-\alpha}{k}w_0 + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}w_{k+1-j} \\
&\quad + f(k, w_k, \sum_{m=0}^{k-1} g(k,m,w_m)) \\
&= \binom{k-\alpha}{k}w_0 + \alpha\sum_{j=1}^{k}\frac{1}{(j-\alpha)}\binom{j-\alpha}{j}w_{k+1-j} + \nabla^\alpha w_{k+1} \;=\; w_{k+1}.
\end{aligned}
$$

Hence by mathematical induction we obtain $|x_n - y_n| \leq w_n$ for all $n \in \mathbb{N}_0^+$.

Corollary 10. *Let v_n and p_n be two nonnegative functions defined for $n \in N_0^+$. Let $h(n,m)$ be a nonnegative function defined for n, $m \in \mathbb{N}_0^+$, $m \leq n$. If the inequality*

$$\nabla^\alpha v_{n+1} \leq p_n + \sum_{m=0}^{n-1} h(n,m) f(m, v_m, \sum_{j=0}^{m-1} g(m,j,v_j)) \tag{33}$$

is satisfied for all $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$. *Then* $v_0 \leq w_0$ *implies*

$$v_n \leq w_n \tag{34}$$

where w_n *is solution of the difference equation*

$$\nabla^\alpha w_{n+1} = p_n + \sum_{m=0}^{n-1} h(n,m) f(m, w_m, \sum_{j=0}^{m-1} g(m, j, w_j)), \ w_0 = p_0 \tag{35}$$

for all $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$.

Theorem 11. *Let* f *be a nonnegative and nondecreasing function with respect to its arguments. Let* w_n *be solution of the difference equation*

$$\nabla^\alpha w_{n+1} = f(w_n, \sum_{j=0}^{n-1} w_j, \sum_{j=0}^{n-2} w_j) \tag{36}$$

for all $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$. *Suppose that the inequality*

$$\nabla^\alpha v_{n+1} \leq f(v_n, \sum_{j=0}^{n-1} v_j, \sum_{j=0}^{n-2} v_j) \tag{37}$$

is satisfied for all $n \in \mathbb{N}_0^+$ *and* $0 < \alpha \leq 1$, *where* v_j $(j = 0, 1, 2...)$ *is a positive sequence of functions defined for* $n \in \mathbb{N}_0^+$ *such that* $v_0 \leq w_0$. *Then*

$$v_n \leq w_n \tag{38}$$

for all $n \in \mathbb{N}_0^+$.

Proof. The proof of the present theorem is by induction on n. For $n = 0$ the claim is true. Suppose the theorem is true for $n = k$. Then $v_j \leq w_j$ for $j \leq k$. If possible, suppose $v_{k+1} > w_{k+1}$. By the monotone property of f, we have

$$v_k \leq w_k, \qquad \sum_{j=0}^{k-1} v_j \leq \sum_{j=0}^{k-1} w_j, \qquad \sum_{j=0}^{k-2} v_j \leq \sum_{j=0}^{k-2} w_j$$
$$\Rightarrow f(v_k, \sum_{j=0}^{k-1} v_j, \sum_{j=0}^{k-2} v_j) \leq f(w_k, \sum_{j=0}^{k-1} w_j, \sum_{j=0}^{k-2} w_j).$$

Now

$$\nabla^\alpha v_{k+1} \leq f(v_k, \sum_{j=0}^{k-1} v_j, \sum_{j=0}^{k-2} v_j) \leq f(w_k, \sum_{j=0}^{k-1} w_j, \sum_{j=0}^{k-2} w_j) = \nabla^\alpha w_{k+1}. \tag{39}$$

Consider

$$\begin{aligned}\nabla^\alpha w_{k+1} &= w_{k+1} - \binom{k-\alpha}{k} w_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} w_{k+1-j} \\ &< v_{k+1} - \binom{k-\alpha}{k} v_0 - \alpha \sum_{j=1}^{k} \frac{1}{(j-\alpha)} \binom{j-\alpha}{j} v_{k+1-j} \\ &= \nabla^\alpha v_{k+1}\end{aligned}$$

which is a contradiction to (38). Hence the theorem is true for n = k+1. Thus by the property of mathematical induction the theorem is true for all $n \in \mathbb{N}_0^+$.

Corollary 12. *Let f be a nonnegative and nondecreasing function with respect to its arguments. Let w_n be solution of the difference equation*

$$\nabla^\alpha w_{n+1} = f(w_n, w_{n-1}, ..., w_{n-k}) \tag{40}$$

for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$. Suppose that the inequality

$$\nabla^\alpha v_{n+1} \leq f(v_n, v_{n-1}, ..., v_{n-k}) \tag{41}$$

is satisfied for all $n \in \mathbb{N}_0^+$ and $0 < \alpha \leq 1$, where v_j $(j = 0, 1, 2, ...)$ is a positive sequence of functions defined for $n \in N_0^+$ such that $v_j \leq w_j$, $j = 0, 1, ..., k$. Then

$$v_n \leq w_n \tag{42}$$

for all $n \in \mathbb{N}_0^+$.

Proof. Using the monotone property of f the proof is clear.

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