International Journal of Pure and Applied Mathematics
Volume 70 No. 2 2011, 151-166

ITERATION OF DIFFERENTIABLE FUNCTIONS UNDER m-MODAL MAPS

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Abstract: We consider the dynamical system $(\mathcal{A}, T)$, where $\mathcal{A}$ is a class of differentiable real functions defined on some interval and $T : \mathcal{A} \to \mathcal{A}$ is an operator $T\phi := f \circ \phi$, where f is a function on the real line. In this work we introduce and develop some techniques of symbolic dynamics for the dynamical system $(\mathcal{A}, T)$. We analyze in detail the case in which T arises from a differentiable m-modal map f. In this case we obtain a combinatorial description of the orbits of $(\mathcal{A}, T)$ which depends on the combinatorial description of the orbits of one dimensional dynamical system induced by f on the interval.

AMS Subject Classification: 37B10, 39B12, 37C05, 26A18
Key Words: symbolic dynamics, infinite dimensional dynamical systems, m-modal maps, iteration theory

1. Introduction and Preliminaries

One-dimensional maps of the interval have had an important role for the understanding of many properties and phenomena found in nonlinear dynamics and complex systems. Part of its success is due to symbolic dynamics, which allowed the simplification of the dynamical description to its essential features, usually concerning topological dynamics. Some subsets of the state space are identified, represented symbolically and treated as equivalent. Then, instead of orbits, we deal with sequences of symbols. Important results on classification and computation of invariants, such topological entropy, was obtained for one-dimensional iterated maps, using symbolic dynamics. Many efforts have been made to extend the techniques and results obtained for iterated maps of

Received: February 10, 2011

the interval to higher dimensional dynamical systems, however the difficulties appear immediately even for dimension two. In infinite dimensional systems important progresses was obtained in the study of boundary value problems for partial differentiable equations reducible to difference equations, as those studied in [4], [7] and [8]. An approach, in that context, using symbolic dynamics was made in [5], [6], [9] and [10].

The main objective of our work is to develop techniques of symbolic dynamics for a certain type of infinite dimensional dynamical systems which arise from iterated differentiable interval maps. Similarly to the one-dimensional case the critical points of the iterated interval map play a decisive role in the overall dynamical behaviour. We start with a class of differentiable functions, $\mathcal{A}$, defined on an interval (eventually could be defined on $\mathbb{R}$). Several conditions could be imposed, however, at this point we simply assume that $\phi \in \mathcal{A}$ means that $\phi \in C^1([0,1])$, with the additional conditions $\phi'(0) = \phi'(1) = 0$ and $\mathrm{Im}(\phi) \subset I$, for a given interval I. The use of the domain interval $[0,1]$ is a simplifying assumption. The important point is that ϕ has zero derivative on the boundary points of the interval domain. The nature of this class $\mathcal{A}$ in terms of topological, metrical or algebraic closure, for now, is not discussed. The discrete time evolution, in order to obtain a discrete dynamical system, is implemented by the operator T defined by the rule $\phi \mapsto f \circ \phi$, where $f : I \to I$ is a fixed map in $C^1(I)$. Therefore, we obtain a discrete dynamical system $(\mathcal{A}, T)$, in the sense that we have a set $\mathcal{A}$ (with additional structure, a topology or a metric) and a self map T, which gives the discrete time evolution. This dynamical system has infinite dimension, although induced by a one-dimensional discrete dynamical system (I, f). From the topological point of view, the dynamical system (I, f) is formally contained in $(\mathcal{A}, T)$, since the constant functions, $\phi \equiv c$, belong to $\mathcal{A}$ and $T(c) = f(c)$. Moreover, a monotone function $\phi \in \mathcal{A}$ (nontrivial) determine a signed interval, $([\phi(0), \phi(1)], +)$ if ϕ is an increasing function and $([\phi(1), \phi(0)], -)$ if ϕ is a decreasing function. Then the dynamics of intervals under iteration of an interval map f is also formally contained in $(\mathcal{A}, T)$. Since a differentiable function has many different characteristics like critical points, critical values, oscillation $([\min(\phi), \max(\phi)])$, spectral decomposition, energy, moments, etc., we can analyze the behaviour of every such characteristics under iteration of T. Our work is a development of the results obtained in [1] and [2], where we analyzed the behaviour and evolution of the spectrum and the symbolic dynamics of functions in $\mathcal{A}$ under iteration of the quadratic family $f_\mu(x) = 1 - \mu x^2$.

Here, we do not consider the asymptotic behaviour or asymptotic properties of the iterates $T^k \phi$, instead we analyze how the particular features or attributes, as referred above, of a function $\phi \in \mathcal{A}$ change under iteration of T in a combinatorial and topological perspective. In the particular case of f being a m-modal map the combinatorial description is particularly interesting.

Now, in what follows we describe some preliminaries on symbolic dynamics, in particular, aspects concerning to the m-modal maps on an interval I.

Let $I \subset \mathbb{R}$ be an interval. A map $f : I \to I$ is called *m-modal* map if is in $C^1(I)$ and has m critical points.

Let c_i, with $i = 1, 2, \ldots, m$, be the m critical points of the m-modal map f such that $c_0 < c_1 < \cdots < c_m < c_{m+1}$, where c_0 and c_{m+1} represent the boundary points of the interval I. In these conditions, consider the partition of the interval I into disjoint subsets

$$I = I_1 \cup I_{C_1} \cup I_2 \cup I_{C_2} \cup \cdots \cup I_{C_m} \cup I_{m+1},$$

where I_{C_j} is the set $\{c_j\}$, $j = 1, 2, \ldots, m$, and I_j, $j = 1, 2, \ldots, m+1$, are given by

$$I_1 = [c_0, c_1[,\ I_2 =]c_1, c_2[, \ldots,\ I_{m+1} =]c_m, c_{m+1}].$$

Each interval I_j, $j = 1, 2, \ldots, m+1$, is a maximal interval of monotonicity of f. Next, to each point x in I_j, $j = 1, 2, \ldots, m+1$, we assign the symbol j, $j = 1,\ 2, \ldots,\ m+1$, or C_j, $j = 1,\ 2, \ldots,\ m$, if the point x is the critical point c_j, $j = 1, 2, \ldots, m$, of f. This assignment is called the *address* of x and it is denoted by $ad(x)$. The address of the point x, $ad(x)$, is thus given by

$$ad(x) = \begin{cases} j & if \quad x \in I_j, \quad j = 1, 2, \ldots, m+1, \\ C_j & if \quad x \in I_{C_j}, \quad j = 1, 2, \ldots, m. \end{cases}$$

As usual, we get a correspondence between orbits of points and symbolic sequences of the alphabet $\{1, C_1, 2, \ldots, m+1\}$, the *itinerary* of x under f is defined by

$$it(x) := ad(x)\ ad(f(x))\ ad\left(f^2(x)\right) \cdots \in \{1, C_1, 2, \ldots, m+1\}^{\mathbb{N}}.$$

The orbits, under f, of the critical points are of special importance, in particular, their itineraries. Following Milnor and Thurston in [3], for each critical point, the *kneading sequence* is given by $\mathcal{K}_i := it(f(c_i))$, $i = 1, 2, \ldots m$, and the collection of symbolic sequences $\mathcal{K}_f := (\mathcal{K}_1, \ldots, \mathcal{K}_m)$ is called the *kneading invariant* of f.

A symbolic sequence $(i_k)_{k \geq 1}$ in $\{1, C_1, 2, \ldots, m+1\}^{\mathbb{N}_0}$ is called *admissible*, with respect to f, if it occurs as an itinerary for some point x in I. The set of all admissible sequences in $\{1, C_1, 2, \ldots, m+1\}^{\mathbb{N}_0}$ is denoted by Σ.

In the sequence space Σ, we define the usual *shift map* $\sigma : \Sigma \to \Sigma$ by

$$\sigma(i_1 i_2 i_3 \dots) = i_2 i_3 \dots .$$

Moreover, the following relation with f and the itinerary map is satisfied

$$\sigma\left(it\left(x\right)\right) = it\left(f\left(x\right)\right). \tag{1}$$

Therefore, we obtain the symbolic system (Σ, σ) associated with the discrete dynamical system (I, f).

An *admissible word* is a finite sub-sequence occurring in an admissible sequence. The set of admissible words of size k occurring in some sequence from Σ is denoted by $\mathcal{W}_k = \mathcal{W}_k(f)$.

We define the *cylinder* set $I_{i_0 i_1 \dots i_k} \subset I$, $i_0 i_1 \dots i_k \in \mathcal{W}_{k+1}$, by

$$\begin{aligned} I_{i_0 i_1 \dots i_k} &= \left\{ x \in I : x \in I_{i_0}, f(x) \in I_{i_1}, \dots, f^k(x) \in I_{i_k} \right\} = \\ &= I_{i_0} \cap f^{-1}(I_{i_1}) \cap \cdots \cap f^{-k}(I_{i_k}). \end{aligned}$$

In other words, $x \in I_{i_0 i_1 \dots i_k}$ means that $ad\left(x\right) = i_0$, $ad\left(f\left(x\right)\right) = i_1, \dots,$ $ad\left(f^k\left(x\right)\right) = i_k$.

Consider the sign function $\varepsilon : \cup_{k=1}^{\infty} \mathcal{W}_k \to \{-1, 0, 1\}$ defined by

$$\varepsilon\left(i_1 \dots i_k\right) = \prod_{j=1}^{k} \varepsilon\left(j\right),$$

with $i_1 \dots i_k \in \mathcal{W}_k$, $\varepsilon\left(C_j\right) = 0$ and $\varepsilon\left(j\right) = +1$ or $\varepsilon\left(j\right) = -1$, according f restricted to I_j is increasing or decreasing.

The parity, with respect to the map f, of a given admissible word $i_1 \dots i_k \in \mathcal{W}_k$ is *even* if $\varepsilon\left(i_1 \dots i_k\right) = 1$ and *odd* if $\varepsilon\left(i_1 \dots i_k\right) = -1$. From the order relation $1 \prec C_1 \prec 2 \prec \cdots \prec m+1$, inherited from the order of the intervals of the partition of the interval I, we introduce an order relation between symbolic sequences as follows: given any distinct sequences $(i_k)_{k \geq 1}$, $(j_k)_{k \geq 1} \in \{1, C_1, 2, \dots, m+1\}^{\mathbb{N}_0}$, admitting that they have a common initial subsequence, *i.e.*, there is a natural $r \geq 0$ such that $i_1 \dots i_r = j_1 \dots j_r$ and $i_{r+1} \neq j_{r+1}$, we will say that $(i_k)_{k \geq 1} \prec (j_k)_{k \geq 1}$ if and only if $i_{r+1} \prec j_{r+1}$ and $\varepsilon(i_1 \dots i_r) = 1$ or $j_{r+1} \prec i_{r+1}$ and $\varepsilon(i_1 \dots i_r) = -1$.

2. Iteration on a Class of Differentiable Functions

Now, consider a m-modal map f in the class $C^1(I)$, for a certain interval I, and the class of differentiable functions

$$\mathcal{A} = \left\{ \varphi \in C^1([0,1]) : \varphi'(0) = \varphi'(1) = 0 \text{ and } Im\,(\varphi) \subset I \right\}.$$

Let T be the operator

$$\begin{array}{rccc} T: & \mathcal{A} & \to & \mathcal{A} \\ & \varphi & \mapsto & f \circ \varphi \end{array}.$$

Note that this operator is well defined since $(f \circ \varphi)'(0) = (f \circ \varphi)'(1) = 0$. Moreover, if $\phi \in \mathcal{A}$ and $Im(\phi) \subset I$ then $Im(T^k \phi) \subset I$ for every $k \in \mathbb{N}$. Therefore, we obtain a discrete dynamical system $(\mathcal{A}, T)$ in the sense that we have a set $\mathcal{A}$ (eventually with additional structure, a topology or a metric, for now not specified) and a self map T, which gives the discrete time evolution.

Let us consider the following results about the localization of the critical values and critical points of the iterates of ϕ_0 in $\mathcal{A}$ under the map f.

Proposition 1. Let f be a m-modal map and let $\phi \in \mathcal{A}$ and $J \subset [0,1]$ be an interval and $x \in J$. If $\phi_{|J}(x)$ is a maximal (resp. minimal) value in $I_{i_0 i_1 \ldots i_k}$, so that $\varepsilon(i_0) = -1$, then $f \circ \phi_{|J}(x)$ is a minimal (resp. maximal) value in $I_{i_1 \ldots i_k}$. If $\phi_{|J}(x)$ is a maximal (resp. minimal) value in $I_{j_0 j_1 \ldots j_k}$, so that $\varepsilon(j_0) = +1$, then $f \circ \phi_{|J}(x)$ is a maximal (resp. minimal) value in $I_{j_1 \ldots j_k}$.

Proof. First, by the definition of $I_{i_0 i_1 \ldots i_k}$, we have $f(I_{i_0 i_1 \ldots i_k}) = I_{i_1 \ldots i_k}$ for any admissible word $i_0 \ldots i_k$. Next, since $f_{|I_{i_0}}$ is decreasing it reverses the order, therefore maximal (*resp.* minimal) points for $\phi_{|J}$ in some J, subinterval of $[0,1]$, correspond to minimal (*resp.* maximal) points for $f \circ \phi_{|J}$. The same reasoning, noting that $f_{|I_{j_0}}$ is increasing and preserves the order, leads to the claimed result. □

Let $cp(\phi)$ denote the set of the critical points of $\phi \in \mathcal{A}$, $cv(\phi)$ denote the set of the critical values of $\phi \in \mathcal{A}$ and $s(\phi)$ denote the set of points $x \in [0,1]$ whose image under ϕ is a critical point of f, that is, $\phi(x) \in cp(f)$. Note that $cv(\phi) = \phi(cp(\phi))$.

Proposition 2. Let f be a m-modal map and let $\phi \in \mathcal{A}$. Then

$$cp(f \circ \phi) = s(\phi) \cup cp(\phi) \;\; and \;\; cv(f \circ \phi) = f\left(\phi\left(s(\phi)\right)\right) \cup f\left(cv(\phi)\right).$$

Proof. From the chain rule $(f \circ \phi)'(x) = f'(\phi(x))\phi'(x) = 0$. Therefore, either $\phi'(x) = 0$ or $f'(\phi(x)) = 0$, and the first claim follows. The critical values of $f \circ \phi$ are naturally the images under $f \circ \phi$ of the critical points of $f \circ \phi$. Moreover, $f(cv(\phi)) = f \circ \phi(cp(\phi))$ and the result follows. □

A consequence of this result is the following:

Corollary 1. *Let f be a m-modal map, $\phi_0 \in \mathcal{A}$ and $\phi_{k+1} = f \circ \phi_k$. Then*

$$cp(\phi_{k+1}) = \cup_{j=0}^{k} s(f^j(\phi_0)) \cup cp(\phi_0)$$

and

$$cv(\phi_{k+1}) = \cup_{j=0}^{k} f^{k+1-j}(\phi_j(s(\phi_j)) \cup f^{k+1}(cv(\phi_0)).$$

This last result means that the maxima and the minima of the iterates $\phi_k = f^k(\phi_0)$ of some initial function $\phi_0 \in \mathcal{A}$ arise from the orbits of the maxima and minima of ϕ_0 and also from the appearance of points where $\phi_0, \phi_1, \ldots, \phi_k$, attained the critical points of f. The preimages, under f, of the critical points c_j, $j = 1, 2, \ldots, m$, that is, points $y \in I$ such that $f^k(y) = c_j$, for some natural k and some $j = 1, 2, \ldots, m$, can be labeled by admissible words in $\mathcal{W}_k$. For example, the admissible word $i_1 i_2 \ldots i_k$ together with the symbol C_j (for a fixed $j = 1, \ldots, m$), $i_1 i_2 \ldots i_k C_j$, represents the k-preimage of c_j, that is, $ad(x) = i_1$, $ad(f(x)) = i_2, \ldots, ad\left(f^{k-1}(x)\right) = i_k$ and $f^k(x) = c_j$. An admissible word gives the prefix of the itinerary of a preimage y. If we consider the graphics of the constant functions equal to the preimages of the critical points of f we obtain a grid with each line is labeled by an admissible word. If one of such line, labeled by a certain admissible word $i_1 i_2 \ldots i_k C_j$, intersects ϕ_0 in a point $x \in [0, 1]$, then $T^k(\phi_0)$ will have a new critical point in x with critical value localized in the cylinder $I_{i_1 i_2 \ldots i_k}$. Note that x is not necessarily a critical point for $T^{k-1}(\phi_0)$. In order to illustrate the results given above consider the Example 1.

Example 1. *Let us consider the bimodal map $f : [-1, 1] \to [-1, 1]$ given by $f(x) = 4x^3 - 3x$, with the critical points $c_1 = -1/2$ and $c_2 = 1/2$. The partition of the interval I is given by $I = I_1 \cup I_{C_1} \cup I_2 \cup I_{C_2} \cup I_3$, with $I_1 = [-1, -1/2[\,, I_{C_1} = \{-1/2\}$, $I_2 =]-1/2, 1/2[\,, I_{C_2} = \{1/2\}$, $I_3 =]1/2, 1]$. Therefore we consider the alphabet set $\{1, C_1, 2, C_2, 3\}$. The Figures 1, 2 and 3 illustrate the Propositions 1 and 2. In the Figure 4. a), each horizontal line is a k-preimage under f of each critical point of f, c_1 and c_2, in particular, the line indicated is the first preimage of c_2 belonging to I_2. Thus, for any initial function which intersect the indicated line in a point, we get a new critical point at the second iterate and not before that. In the Figure 4. b), we* *consider* $\phi_0(x) = a_0 + a_1 \cos(2\pi x)$, *with* $a_0 = -0.2$ *and* $a_1 = 0.06$. *Both* ϕ_0 *and* $\phi_1 = T(\phi_0)$ *have one critical point inside* $[0,1]$. *However the second iterate* $\phi_2 = T^2(\phi_0) = T(T(\phi_0))$ *has two new critical points precisely in the points where* ϕ_0 *intersects the line labeled by* $2C_2$.

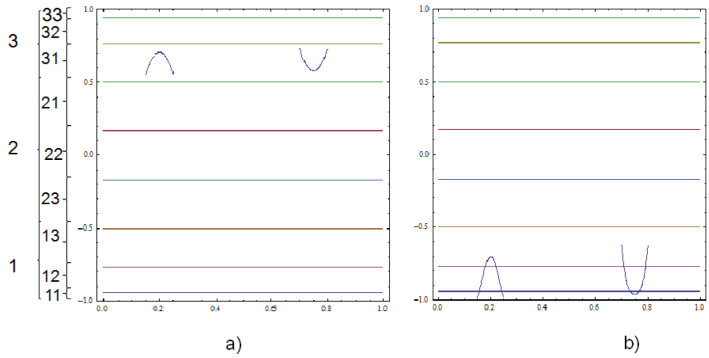

Figure 1: a) Graph of the restriction of a function ϕ, in two subintervals J_1, J_2, with a maximum (resp. minimum) in the region I_{31}. b) Graph of the restriction of $f \circ \phi$, in the same subintervals J_1, J_2, which has a maximum (resp. minimum) in the region I_1.

3. Symbolic Dynamics for the Infinite Dynamical System $(\mathcal{A}, T)$

In order to introduce a symbolic dynamics description for the discrete dynamical system $(\mathcal{A}, T)$, let us consider the decomposition of $\mathcal{A}$ into the following classes, as in [2]:

$$\mathcal{A}_c = \{\phi \in \mathcal{A} : \phi(x) \text{ is constant in } [0,1]\},$$

$$\mathcal{A}_0 = \{\phi \in \mathcal{A} : \phi \text{ has no critical points in }]0,1[\}$$

and

$$\mathcal{A}_j = \{\phi \in \mathcal{A} : \phi \text{ has } j \text{ critical points in }]0,1[\}, \quad j = 1, 2, \ldots .$$

Let $\phi \in \mathcal{A}$ and let $\eta(\phi)$ be the number of non-trivial critical points of ϕ (inside $[0,1]$). In this case if $\phi \in \mathcal{A}_j$, $j \in \mathbb{N}_0$, then $\eta(\phi) = j$ and the total number

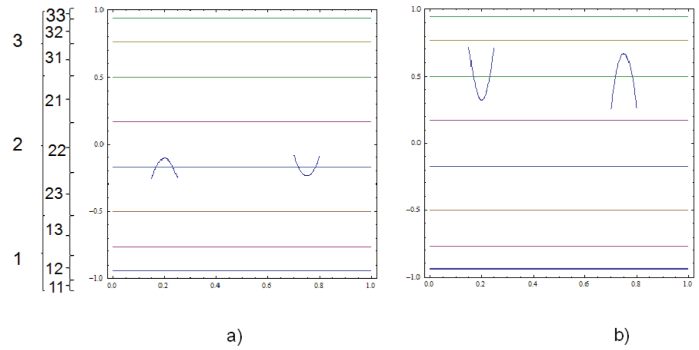

Figure 2: a) Graph of the restriction of a function ϕ, two subintervals J_1, J_2, with a maximum (resp. minimum) in the region I_{22} (resp. I_{23}). b) Graph of the restriction of $f \circ \phi$, in the same subintervals J_1, J_2, which has a minimum (resp. maximum) in the region I_2 (resp. I_3).

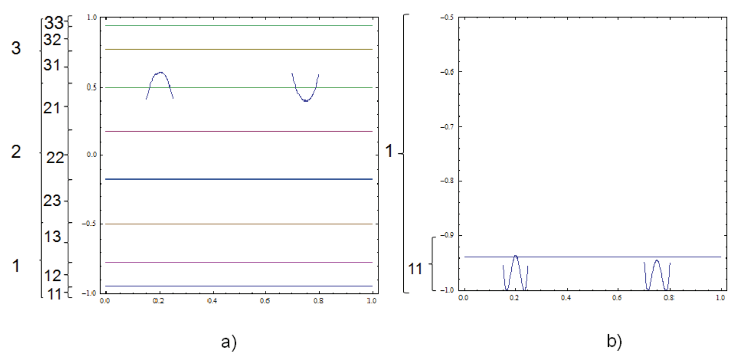

Figure 3: a) Graph of the restriction of a function ϕ, in subintervals J_1, J_2, which intersect the horizontal line corresponding to a critical point of f at some points $x_1, x_1' \in J_1$, $x_2, x_2' \in J_2$ and have a maximum (resp. minimum) in the region I_{31} (resp. I_{21}). b) Graph of the restriction of the function $f \circ \phi$, in the same subintervals J_1, J_2, with new minimal values at $x_1, x_1' \in J_1$, $x_2, x_2' \in J_2$ and a maximum in the region I_1. Note that in b) the vertical scale is changed.

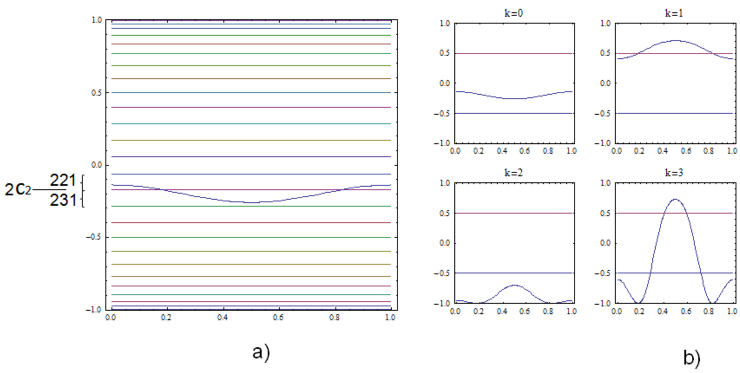

Figure 4: a) Graph of $\phi_0(x) = a_0 + a_1 \cos(2\pi x)$, with $a_0 = -0.2$ and $a_1 = 0.06$, so that $\phi_2 = f^2 \circ \phi_0$ has two new critical points. The line indicated is the first preimage of c_2 belonging to I_2. b) Graphs of the $k-$iterations under f of $\phi_0(x) = a_0 + a_1 \cos(2\pi x)$, with $a_0 = -0.2$ and $a_1 = 0.06$.

of critical points is $\eta(\phi)+2 = j+2$. We are interested in the symbolic description of the dynamical evolution of a function ϕ under iteration of f. Moreover, we are interested that this symbolic description has essentially a topological meaning, therefore the important point is to distinguish and codify the critical points and values of ϕ. Given $\phi \in \mathcal{A}$, we identify its critical points and collect the addresses and itineraries of the corresponding critical values. Our generalized symbolic space will be $\underline{\Sigma} := \cup_{j\in\mathbb{N}_0}\Sigma^{j+1}$, where $\Sigma^{j+1} = \Sigma \times \Sigma \times \cdots \times \Sigma$ ($j+1$ times). We now define the generalized address, itinerary and shift maps for the space $\mathcal{A}$,

$$\begin{aligned} \underline{ad}: \quad \mathcal{A} &\rightarrow \{1, C_1, 2, \ldots, m+1\} \\ \phi &\mapsto \underline{ad}(\phi) := \left(ad(d_0), ad(d_1), \ldots, ad\left(d_{\eta(\phi)}\right), ad\left(d_{\eta(\phi)+1}\right)\right), \end{aligned}$$

$$\begin{aligned} \underline{it}: \quad \mathcal{A} &\rightarrow \underline{\Sigma} \\ \phi &\mapsto \underline{it}(\phi) = (it(d_0), it(d_1), \ldots, it\left(d_{\eta(\phi)}\right), it\left(d_{\eta(\phi)+1}\right)), \end{aligned}$$

where $d_i = \phi(a_i)$, $i = 0, 1, \ldots \eta(\phi)+1$, are the critical values of ϕ in the interval $[-1, 1]$ (with $d_0 = \phi(0)$ and $d_{\eta(\phi)+1} = \phi(1)$). As an example, consider f given approximately by the analytical expression $f(x) = -2.15x^3+1.15x$ and ϕ given approximately by the analytical expression $\phi(x) = 0.261\cos(\pi x) + 0.711$ (see the Example 2 and the Figure 5 for more details). In this case we have

$$\underline{ad}\,(\phi) = (ad(\phi(0)), ad(\phi(1))) = (3,3)$$

and

$$\underline{it}\,(\phi) = (it(\phi(0)), it(\phi(1))) = (312^{\infty}, 32^{\infty})\,.$$

Let $i^{(j)} = i_1^{(j)} i_2^{(j)} \ldots,\ j = 0,1,2,\ldots,\eta\,(\phi)+1$. The extended shift map is then defined by

$$\underline{\sigma}(i^{(0)},\ldots,i^{(j)},i^{(j+1)},\ldots,i^{(\eta(\phi)+1)}) :=$$

$$= \begin{cases} (\sigma(i^{(0)}),\ldots,\sigma(i^{(j)}),\sigma(i^{(j+1)}),\ldots,\sigma(i^{(\eta(\phi)+1)})) \text{ if } \ i_1^{(j)} = i_1^{(j+1)}, \\ (\sigma(i^{(0)}),\ldots,\sigma(i^{(j)}),\mathcal{K}_{i_1^{(j)}},\ldots,\mathcal{K}_{i_1^{(j+1)}-1},\sigma(i^{(j+1)}),\ldots,\sigma(i^{(\eta(\phi)+1)})) \\ \qquad\qquad\qquad\qquad\qquad \text{if } i_1^{(j)} \neq i_1^{(j+1)} \text{ and } i_1^{(j)} \prec i_1^{(j+1)}, \\ (\sigma(i^{(0)}),\ldots,\sigma(i^{(j)}),\mathcal{K}_{i_1^{(j)}-1},\ldots,\mathcal{K}_{i_1^{(j+1)}},\sigma(i^{(j+1)}),\ldots,\sigma(i^{(\eta(\phi)+1)})) \\ \qquad\qquad\qquad\qquad\qquad \text{if } i_1^{(j)} \neq i_1^{(j+1)} \text{ and } i_1^{(j+1)} \prec i_1^{(j)}, \end{cases}$$

where $\mathcal{K}_{i_1^{(j)}}$ with $i_1^{(j)} \in \{1,\ldots,m\}$ is the kneading sequence corresponding to the critical point of f, $c_{i_1^{(j)}}$, localized between d_j and $d_{j+1},\ j = 0,1,2,\ldots,\eta\,(\phi)$.

Therefore, we obtain a symbolic system $(\underline{\Sigma},\ \underline{\sigma})$ associated to $(\mathcal{A}, T)$. Similarly to the finite dimensional discrete dynamical systems we obtained the following result:

Theorem 1. *Let $\phi,\ \tilde{\phi} \in \mathcal{A}$ with $\phi \neq \tilde{\phi}$ so that $\underline{it}(\phi) = \underline{it}(\tilde{\phi})$, then*

$$\underline{it}(T^k\phi) = \underline{it}(T^k\tilde{\phi}),\ k \in \mathbb{N}_0.$$

Moreover,

$$\underline{\sigma} \circ \underline{it} = \underline{it} \circ T.$$

Proof. We first prove the last assertion. Let $d_i = \phi(a_i),\ i = 1,\ldots\eta\,(\phi)$, where $a_i,\ i = 1,\ldots\eta\,(\phi)$, are the non-trivial critical points of ϕ in the interval $[0,1]$ (with $d_0 = \phi(0)$ and $d_{\eta(\phi)+1} = \phi(1)$).

If $i_1^{(j)} = i_1^{(j+1)}$, for every $j = 0,1,\ldots,\eta\,(\phi)$, and by (1) we have

$$\begin{aligned} &\underline{\sigma}(\underline{it}(\phi)) = \\ &\underline{\sigma}(it(d_0),\ldots,it(d_i),it(d_{i+1}),\ldots,it(d_{\eta(\phi)+1})) = \\ &= (\sigma(it(d_0)),\ldots,\sigma(it(d_i)),\sigma(it(d_{i+1})),\ldots,\sigma(it(d_{\eta(\phi)+1}))) = \\ &= (it(Td_0),\ldots,it(T(d_i)),it(T(d_{i+1})),\ldots,it(T(d_{\eta(\phi)+1})) = \\ &= \underline{it}(T\phi), \end{aligned}$$

if $i_1^{(j)} \neq i_1^{(j+1)}$ and $i_1^{(j)} \prec i_1^{(j+1)}$, for some $j \in \{0, 1, \ldots, \eta(\phi)\}$ and by (1) we obtain

$$\begin{aligned}
&\underline{\sigma}(\underline{it}(\phi)) = \\
&= \underline{\sigma}(it(d_0), \ldots, it(d_i), it(d_{i+1}), \ldots, it(d_{\eta(\phi)+1})) = \\
&= (\sigma(it(d_0)), \ldots, \sigma(it(d_i)), \mathcal{K}_{i_1^{(j)}}, \ldots, \mathcal{K}_{i_1^{(j+1)}-1}, \sigma(it(d_{i+1})), \ldots, \sigma(it(d_{\eta(\phi)+1}))) = \\
&= (\sigma(T(d_0)), \ldots, it(T(d_i)), \mathcal{K}_{i_1^{(j)}}, \ldots, \mathcal{K}_{i_1^{(j+1)}-1}, it(T(d_{i+1})), \ldots, \sigma(T(d_{\eta(\phi)+1}))) = \\
&= \underline{it}(T\phi),
\end{aligned}$$

and finally, in the last case, if $i_1^{(j)} \neq i_1^{(j+1)}$ and $i_1^{(j+1)} \prec i_1^{(j)}$, for some $j \in \{0, 1, \ldots, \eta(\phi)\}$ and by (1) we obtain

$$\begin{aligned}
&\underline{\sigma}(\underline{it}(\phi)) = \\
&= \underline{\sigma}(it(d_0), \ldots, it(d_i), it(d_{i+1}), \ldots, it(d_{\eta(\phi)+1})) = \\
&= (\sigma(it(d_0)), \ldots, \sigma(it(d_i)), \mathcal{K}_{i_1^{(j)}-1}, \ldots, \mathcal{K}_{i_1^{(j+1)}}, \sigma(it(d_{i+1})), \ldots, \sigma(it(d_{\eta(\phi)+1}))) = \\
&= (\sigma(T(d_0)), \ldots, it(T(d_i)), \mathcal{K}_{i_1^{(j)}-1}, \ldots, \mathcal{K}_{i_1^{(j+1)}}, it(T(d_{i+1})), \ldots, \sigma(T(d_{\eta(\phi)+1}))) = \\
&= \underline{it}(T\phi).
\end{aligned}$$

For $k = 0$, we obtain $\underline{it}(\phi) = \underline{it}(\tilde{\phi})$ (by assumption). We assume the result is true for $k = n$, i.e., assume $\underline{it}(T^n\phi) = \underline{it}(T^n\tilde{\phi})$. For $k = n + 1$,

$$\underline{it}(T^{n+1}\phi) = \underline{it}(T(T^n\phi)) = \underline{\sigma}(\underline{it}(T^n\phi)) = \underline{\sigma}(\underline{it}(T^n\tilde{\phi})) = \underline{it}(T^{n+1}\tilde{\phi}),$$

which is the desired result for $k = n + 1$. □

Next, we give the following examples to illustrate the previous definitions and results.

Example 2. *Let us consider again the bimodal map $f : [-1, 1] \to [-1, 1]$, with the alphabet $\{1, C_1, 2, C_2, 3\}$, it is given approximately by the analytical expression $f(x) = -2.15x^3 + 1.15x$. Different functions can have the same itinerary, consequently, in terms of symbolic dynamics they represent the same class. For example, ϕ and $\widetilde{\phi}$ given approximately by the analytical expressions $\phi(x) = 0.261\cos(\pi x) + 0.711$ and $\widetilde{\phi}(x) = 0.2885\cos(0.8\pi x) + 0.6834$, respectively, have equal itinerary $\underline{it}(\phi) = (312^\infty, 32^\infty) = \underline{it}(\widetilde{\phi})$. Consider also ψ given approximately by the analytical expression $\psi(x) = 0.261\cos(3\pi x) + 0.711$, which has the same critical values as ϕ and $\widetilde{\phi}$. Nevertheless, its itinerary is*

$$\underline{it}(\psi) = (312^\infty, 32^\infty, 312^\infty, 32^\infty)$$

and the function ψ belongs to a different class of ϕ and $\widetilde{\phi}$, see the Figure 5.

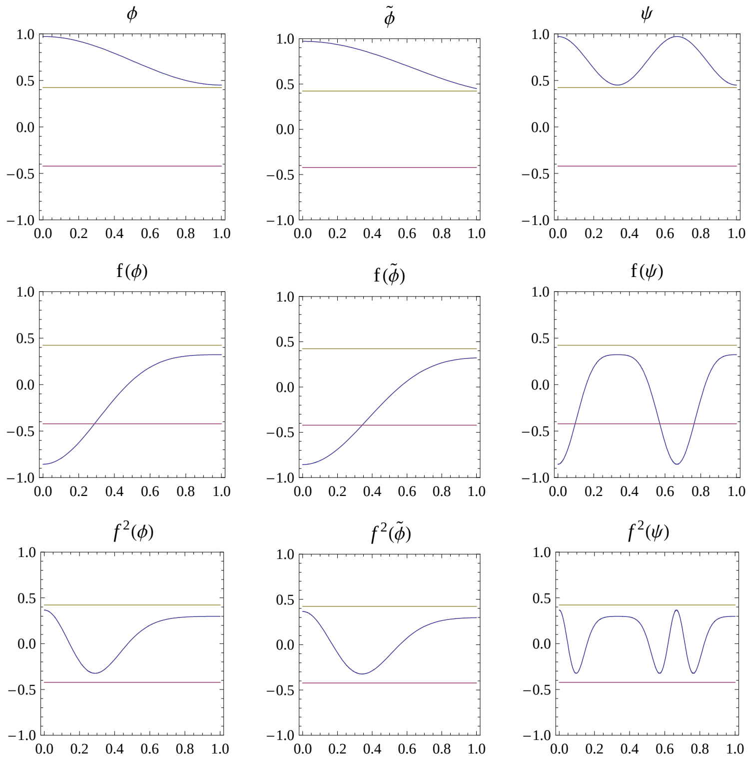

Figure 5: Graphs of the functions ϕ, $\tilde{\phi}$, ψ and of the iterations $f(\phi)$, $f(\tilde{\phi})$, $f(\psi)$, $f^2(\phi)$, $f^2(\tilde{\phi})$ and $f^2(\psi)$, where ϕ, $\widetilde{\phi}$, ψ and f are given approximately by the analytical expressions $\phi(x) = 0.261\cos(\pi x) + 0.711$, $\widetilde{\phi}(x) = 0.2885\cos(0.8\pi x) + 0.6834$, $\psi(x) = 0.261\cos(3\pi x) + 0.711$ and $f(x) = -2.15x^3 + 1.15x$, respectively.

Example 3. *Let $f : [-1,1] \to [-1,1]$ be the bimodal map with the alphabet $\{1, C_1, 2, C_2, 3\}$. The analytical formula of f is approximately given by $f(x) = 3.9818x^3 - 2.9818x$, with critical points given approximately by $c_1 = -0.503052$ and $c_2 = 0.503052$. The itineraries of the two critical points c_1 and c_2 are $\mathcal{K}_1 = it(f(c_1)) = (33C_2 11C_1)^\infty$ and $\mathcal{K}_2 = it(f(c_2)) = (11C_1 33C_2)^\infty$, respectively, see the Figure 6.*

Now, we consider, in $\mathcal{A}$, the function given approximately by the analytical expression

$$\phi(x) = 0.05\cos(2\pi x) - 0.92.$$

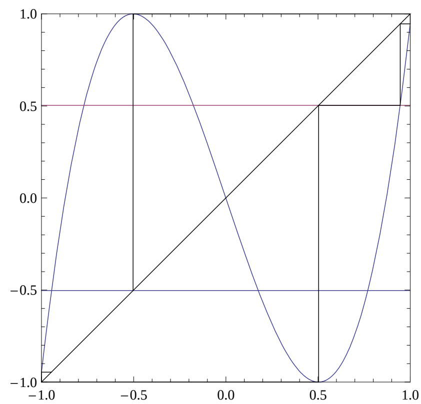

Figure 6: Graph of the map f given approximately by the analytical expression $f(x) = 3.9818x^3 - 2.9818x$, with the alphabet $\{1, C_1, 2, C_2, 3\}$.

We have

$$\underline{it}(\phi) = (122223\ldots, 113133\ldots, 122223\ldots) \in \Sigma_f^3,$$

then

$$\underline{\sigma}(\underline{it}(\phi)) = (22223\ldots, 13133\ldots, 22223\ldots) \in \Sigma^3,$$

$$\begin{aligned}\underline{\sigma}^2(\underline{it}(\phi)) &= (2223\ldots, \mathcal{K}_1, 3133\ldots, \mathcal{K}_1, 2223\ldots) = \\ &= (2223\ldots, (33C_211C_1)^\infty, 3133\ldots, (33C_211C_1)^\infty, 2223\ldots) \in \Sigma^5,\end{aligned}$$

$$\begin{aligned}\underline{\sigma}^3(\underline{it}(\phi)) &= (223\ldots, \mathcal{K}_2, (3C_211C_13)^\infty, 133\ldots, (3C_211C_13)^\infty, \mathcal{K}_2, 223\ldots) = \\ &= (223\ldots, (11C_133C_2)^\infty, (3C_211C_13)^\infty, 133\ldots, (3C_211C_13)^\infty, \\ &\quad (11C_133C_2)^\infty, 223\ldots) \in \Sigma^7,\end{aligned}$$

$$\begin{aligned}\underline{\sigma}^4(\underline{it}(\phi)) &= (23\ldots, \mathcal{K}_1, (1C_133C_21)^\infty, \mathcal{K}_1, \mathcal{K}_2, (C_211C_133)^\infty, \mathcal{K}_2, \mathcal{K}_1, 33\ldots, \\ &\quad \mathcal{K}_1, \mathcal{K}_2, (C_211C_133)^\infty, \mathcal{K}_2, \mathcal{K}_1, (1C_133C_21)^\infty, \mathcal{K}_1, 23\ldots) = \\ &= (23\ldots, (33C_211C_1)^\infty, (1C_133C_21)^\infty, (33C_211C_1)^\infty, (11C_133C_2)^\infty, \\ &\quad (C_211C_133)^\infty, (11C_133C_2)^\infty, (33C_211C_1)^\infty, 33\ldots, (33C_211C_1)^\infty, \\ &\quad (11C_133C_2)^\infty, (C_211C_133)^\infty, (11C_133C_2)^\infty, (33C_211C_1)^\infty, (1C_133C_21)^\infty, \\ &\quad (33C_211C_1)^\infty, 23\ldots) \in \Sigma^{17}\end{aligned}$$

and

$$\underline{it}(T\phi) = (22223\ldots, 13133\ldots, 22223\ldots),$$

$$\begin{aligned}
\underline{it}(T^2\phi) &= (2223\ldots,\mathcal{K}_1,3133\ldots,\mathcal{K}_1,2223\ldots),\\
\underline{it}(T^3\phi) &= (223\ldots,\mathcal{K}_2,(3C_211C_13)^\infty,133\ldots,(3C_211C_13)^\infty,\mathcal{K}_2,223\ldots),\\
\underline{it}(T^4\phi) &= (23\ldots,\mathcal{K}_1,(1C_133C_21)^\infty,\mathcal{K}_1,\mathcal{K}_2,(C_211C_133)^\infty,\mathcal{K}_2,\mathcal{K}_1,33\ldots,\\
&\qquad \mathcal{K}_1,\mathcal{K}_2,(C_211C_133)^\infty,\mathcal{K}_2,\mathcal{K}_1,(1C_133C_21)^\infty,\mathcal{K}_1,23\ldots).
\end{aligned}$$

We can verify that

$$\underline{\sigma}^k(\underline{it}(\phi)) = \underline{it}(T^k\phi), \quad k = 0,1,2,\ldots.$$

In the Figure 7, we show the graphs of ϕ, $T\phi$, $T^2\phi$, $T^3\phi$, $T^4\phi$ *and* $T^5\phi$ *and we compute their itineraries by the equality*

$$\underline{\sigma}^k(\underline{it}(\phi)) = \underline{it}(T^k\phi),\ k = 1,2,\ldots.$$

It is visually clear the accumulation of new critical points, in the last picture.

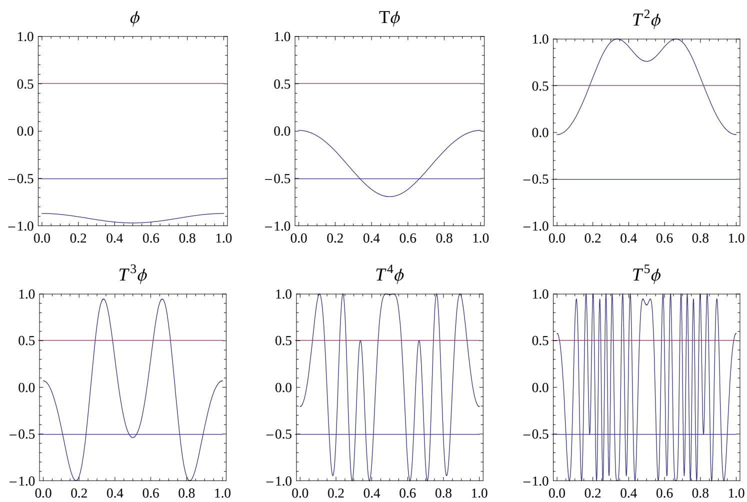

Figure 7: Graphs of $T^k\phi$, $k = 0,1,2,3,4,5$, with ϕ and f given approximately by the analytical expressions $\phi(x) = 0.05\cos(2\pi x) - 0.92$ and $f(x) = 3.9818x^3 - 2.9818x$, respectively.

We can make our symbolic analysis without an explicit function $\phi \in \mathcal{A}$, especially if the itineraries involved are periodic sequences, as we can see in the next example.

Example 4. *Consider a bimodal map f with kneading invariant*

$$\mathcal{K}_f = (\mathcal{K}_1, \mathcal{K}_2) = ((33C_2 11C_1)^\infty, (11C_1 33C_2)^\infty)$$

(which is in fact the only necessary information). Now, consider an address $((121)^\infty, (21)^\infty)$. Both sequences are admissible, therefore, this address represents a class of functions belonging to $\mathcal{A}$ which have two critical points necessarily 0 *and* 1 *(the boundary points) and the corresponding critical values have itineraries given by $it(\phi(0)) = (121)^\infty$ and $it(\phi(1)) = (21)^\infty$ (periodic points with respect to f). Using the Theorem 1 we know that the time evolution under T of any function $\phi \in \mathcal{A}$ which has address $((121)^\infty, (21)^\infty)$, is given by*

$$\begin{array}{c}
((121)^\infty, (21)^\infty) \\
\downarrow \\
((211)^\infty, \mathcal{K}_1, (12)^\infty) = \\
= ((211)^\infty, (33C_2 11C_1)^\infty, (12)^\infty) \\
\downarrow \\
((112)^\infty, \mathcal{K}_2, (3C_2 11C_1 3)^\infty, \mathcal{K}_2, \mathcal{K}_1, (21)^\infty) = \\
= ((112)^\infty, (11C_1 33C_2)^\infty, (3C_2 11C_1 3)^\infty, (11C_1 33C_2)^\infty, (33C_2 11C_1)^\infty, (21)^\infty) \\
\downarrow \\
((121)^\infty, (1C_1 33C_2 1)^\infty, \mathcal{K}_1, \mathcal{K}_2, (C_2 11C_1 33)^\infty, \mathcal{K}_2, \mathcal{K}_1, (1C_1 33C_2 1)^\infty, \mathcal{K}_1, \mathcal{K}_2, (3C_2 11C_1 3)^\infty, (12)^\infty) \\
\downarrow \\
\cdots
\end{array}$$

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