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ORBITAL HAUSDORFF CONTINUOUS DEPENDENCE OF THE SOLUTIONS OF IMPULSIVE DIFFERENTIAL EQUATIONS WITH RESPECT TO IMPULSIVE PERTURBATIONS

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Abstract: The impulsive nonlinear autonomous systems of differential equations with non fixed moments of impulsive perturbation are the fundamental objects of investigation in the present paper. The impulses are realized when the trajectory of the solution falls over the so called "impulsive set", situated in the phase space of the system. For such type of problems are introduced the concepts orbital Hausdorff continuous dependence with respect to the initial point and the impulsive perturbations. Sufficient conditions are found out under which the solutions possess this property. The results are applied to the generalized mathematical model of the evolution dynamics of a prey-predator type co association, which is subjected to short term external influences.

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1. Introduction

The initial problems for impulsive nonlinear autonomous systems of differential equations with non fixed moments of impulsive perturbations are the main objects for investigation in the present paper. The impulsive systems with fixed moments of impulsive effects [29], [30] are created first from historical point of view. The systems with non fixed moments of impulsive influences [1], [3], [7], [10], [31], [32], are relatively more complicated. The investigations devoted to the specific problems of the systems of the impulsive equations provoke great interest. We will point at [35] and [37], as the examples, where is considered the absence of the phenomenon "beating" for this type of impulsive equations; also [3] and [12], where are studied the properties of continuous dependence and stability of the solutions of impulsive systems of differential equations with regard to the impulsive sets (in the indicated considerations they coincide with hypersurfaces, situated in the extended phase space) and to the magnitudes of impulsive perturbations etc. In connection with numerous applications of impulsive differential equations (see [1], [2], [5], [6], [8], [9], [13]-[25], [36], [37]-[40], etc), their qualitative theory has been developed relatively intensively during the last twenty years. To this new mathematical theory are devoted more than twenty monographs of which we will cite [4], [9], [35], and [37]. Different aspects of evolution dynamics of a "prey-predator" type co-association with external impulsive perturbations are considered in [26], [27], [28], [31], [33], and [34].

2. Statement of the Problem and Preliminary Remarks

The basic object of investigation in this article is the following initial problem:

$$\frac{dx}{dt} = f(x), \qquad \varphi(x(t)) \neq 0, \tag{1}$$

$$x(t+0) = x(t) + I(x(t)), \quad \varphi(x(t)) = 0, \tag{2}$$

$$x(0) = x_0, \tag{3}$$

where: $f : D \to \mathrm{R}^n$; $\varphi : D \to \mathrm{R}$; $n \in \mathrm{N}, n \geq 2$; D is a domain in R^n; $x_0 \in D$. The set of points $x \in D$, satisfying the equality $\varphi(x) = 0$, is named an impulsive set (in the case this is called an impulsive surface, situated in D). This set we denote by Φ, i.e. $\Phi = \{x \in D; \varphi(x) = 0\}$. The function $I : \Phi \to \mathrm{R}^n$ is called an impulsive function. We assume that $(Id + I) : \Phi \to D$ is satisfied, where Id is the identity in R^n. The moments in which the trajectory of the above problem meets consecutively the impulsive surface are denoted by $t_1, t_2, ..., 0 < t_1 < t_2 < ...$

The solution $x(t; x_0)$ of the considered problem is a piecewise continuous function. We have:

1.a. For $0 \leq t < t_1$ the solution of problem (1), (2), (3) coincides with the solution of problem (without impulses) (1), (3) and the inequality $\varphi(x(t; x_0)) \neq 0$ is valid;

1.b. At the moment t_1 the equalities $x(t_1; x_0) = x(t_1 - 0; x_0) = x_1$ and $\varphi(x(t_1; x_0)) \quad = \varphi(x_1) = 0$ are true;

1.c. $x(t_1 + 0; x_0) = x(t_1; x_0) + I(x(t_1; x_0)) = (Id + I)(x(t_1; x_0)) = x_1^+$;

2.a. For $t_1 < t < t_2$ the solution of problem (1), (2), (3) coincides with the solution of the system (1) with the initial condition $x(t_1 + 0) = x_1^+$ and the inequality $\varphi(x(t; x_0)) \neq 0$ is valid;

2.b. At the moment t_2 the equalities $x(t_2; x_0) = x(t_2 - 0; x_0) = x_2$ and $\varphi(x(t_2; x_0)) = \varphi(x_2) = 0$ hold;

2.c. $x(t_2 + 0; x_0) = x(t_2; x_0) + I(x(t_2; x_0)) = x_2^+$ etc.

With the problem (1), (2), (3) we consider also the corresponding perturbed problem

$$\frac{dx^*}{dt} = f(x^*), \qquad \varphi(x^*(t)) \neq 0, \tag{4}$$

$$x^*(t+0) = x^*(t) + I^*(x^*(t)), \quad \varphi(x^*(t)) = 0, \tag{5}$$

$$x^*(0) = x_0^*, \tag{6}$$

where $I^* : \Phi \to \mathrm{R}^n$ and $(Id + I^*) : \Phi \to D$, $x_0^* \in D$. As can be seen the difference between problems (1), (2), (3) and (4), (5), (6) is in the initial point and the impulsive function. The solution of the perturbed problem (4), (5), (6) we will denote by $x^*(t; x_0^*)$ and the moments at which the trajectory of this problem meets the impulsive surface Φ we will denote by $t_1^*, t_2^*, ..., 0 < t_1^* < t_2^* <$ ... We will use the notations:

$$\begin{aligned} x_i &= x(t_i; x_0) = x(t_i - 0; x_0), \\ x_i^* &= x^*(t_i^*; x_0^*) = x^*(t_i^* - 0; x_0^*), \\ x_i^+ &= x(t_i; x_0) + I(x(t_i; x_0)) = x(t_i + 0; x_0), \\ x_i^{*+} &= x^*(t_i^*; x_0^*) + I^*(x^*(t_i^*; x_0^*)) = x^*(t_i^* + 0; x_0^*), \quad i = 1, 2, \ldots . \end{aligned}$$

We denote with $X\,(t;x_0)$ and $X^*\,(t;x_0^*)$ the solutions of the problems without impulses (1), (3) and (4), (6) respectively. For the trajectories of the problems (1), (2), (3) and (4), (5), (6) we introduce the notations:

$$\gamma\,(x_0;[0,T]) = \{x\,(t;x_0)\,,0\le t\le T\}\,,\quad \gamma^*\,(x_0^*;[0,T]) = \{x^*\,(t;x_0^*)\,,0\le t\le T\}\,,$$

where T is a positive constant.

If the points $a\,(a_1,a_2,...,a_n)\,,b\,(b_1,b_2,...,b_n)\in \mathrm{R}^n$, then their dot product, the Euclidean norm and the Euclidean distance between them are denoted respectively by:

$$\langle a,b\rangle = a_1b_1+a_2b_2+...+a_nb_n,\quad \|a\| = \langle a,a\rangle^{\frac{1}{2}} = \sqrt{a_1^2+a_2^2+...+a_n^2},$$

$$\rho_E\,(a,b) = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2+...+(a_n-b_n)^2}.$$

It is clear that equality $\|a-b\| = \rho_E\,(a,b)$ is valid. If the non empty sets $A,B\subset \mathrm{R}^n$, then the Euclidean and the Hausdorff distances between them are denoted respectively by:

$$\rho_E\,(A,B) = \inf\left\{\inf\left\{\rho_E\,(a,b)\,,b\in B\right\},a\in A\right\},$$

$$\rho_H\,(A,B) = \max\Big\{\sup\left\{\inf\left\{\rho_E\,(a,b)\,,b\in B\right\},a\in A\right\},\; \sup\left\{\inf\left\{\rho_E\,(a,b)\,,a\in A\right\},b\in B\right\}\Big\}.$$

The inequality $\rho_E\,(A,B)\le \rho_H\,(A,B)$ is obviously true. The closure of the set A is denoted by $\bar{A}$ and ∂A is the boundary of A.

Remark 1. We have:

$$\rho_E(\gamma(x_0;[0,T]),\gamma^*(x_0^*;[0,T])) = \inf\{\inf\{\rho_E(x^*(t^*;x_0^*),x(t;x_0)), 0\le t\le T\}, 0\le t^*\le T\};\ \rho_H(\gamma(x_0;[0,T]),\gamma^*(x_0^*;[0,T]))$$
$$= \max\{\sup\{\inf\{\rho_E(x^*(t^*;x_0^*),x(t;x_0)),0\le t\le T\},0\le t^*\le T\}, \sup\{\inf\{\rho_E(x^*(t^*;x_0^*),x(t;x_0)),0\le t^*\le T\},0\le t\le T\}\};$$
$$\rho_E(\gamma(x_0;[0,T]),\gamma^*(x_0^*;[0,T]))\le \rho_H(\gamma(x_0;[0,T]),\gamma^*(x_0^*;[0,T]))$$
$$\le \sup\{\rho_E(x^*(t;x_0^*),x(t;x_0)),0\le t\le T\} = \sup\{\|x^*(t;x_0^*)-x(t;x_0)\|, 0\le t\le T\}.$$

Definition 1. We shall say that the solution of problem (1), (2), (3) depends orbital Hausdorff continuously on the initial point x_0 and on the impulsive function I, if:

$$(\forall \varepsilon > 0)\,(\forall T > 0)\,(\forall x_0 \in D)\,(\forall I : \Phi \to \mathrm{R}^n)\,(\exists \delta = \delta\,(\varepsilon, T, x_0, I) > 0) :$$
$$(\forall x_0^* \in D,\quad \rho_E\,(x_0^*, x_0) < \delta)\,(\forall I^* : \Phi \to \mathrm{R}^n,\quad \rho_E\,(I^*\,(x)\,, I\,(x)) < \delta \text{ for } \ x \in \Phi)$$
$$\Rightarrow \rho_H\,(\gamma^*\,(x_0^*; [0,T])\,, \gamma\,(x_0; [0,T])) < \varepsilon.$$

The fundamental aim of the present article is to find sufficient conditions of orbital Hausdorff continuous dependence with respect to the initial point and the impulsive function of the solution of problem (1), (2), (3).

Further we use the following conditions:

H1. The function $f \in C\,[D, \mathrm{R}^n]$.

H2. It is true $\lim\limits_{i\to\infty} t_i = \infty$.

H3. For every point $x_0 \in D$ the problem without impulses (1), (3) possesses a unique solution defined for $t \geq 0$.

H4. The impulsive set $\Phi = \{x \in D; \varphi\,(x) = 0\}$ is such that $\bar{\Phi}\backslash\Phi \subset \bar{D}\backslash D$.

H5. The function $\varphi \in C^1\,[D, \mathrm{R}]$ and the inequality $|\langle \operatorname{grad}\varphi\,(x)\,, f\,(x)\rangle| > 0, x \in \Phi$, is valid.

H6. There exists a positive constant C_f such that $\|f\,(x)\| \leq C_f, x \in D$.

H7. The function $I \in C\,[\Phi, \mathrm{R}^n]$.

H8. The inequality $\rho_E\,(\Phi, (Id + I)\,(\Phi)) > 0$ is satisfied.

3. Main Results

Theorem 1. *Let the conditions H1, H2 and H3 hold.*

Then for any point $x_0 \in D$ the solution of the problem with impulses (1), (2), (3) exists and it is unique for $t \geq 0$.

Theorem 2. *Let the conditions H1–H8 hold.*

Then the solution of problem (1), (2), (3) depends orbital Hausdorff continuously on the initial point x_0 and on the impulsive function I.

Proof. Let ε and T be arbitrary positive constants. The following cases are possible:

Case 1. The trajectory $\gamma\left(x_0;[0,T]\right)=\left\{x\left(t;x_0\right),0\le t\le T\right\}$ does not intersect the impulsive set Φ, i.e. $\gamma\left(x_0;[0,T]\right)\cap\Phi=\emptyset$. Assume that

$$\rho_E\left(\gamma\left(x_0;[0,T]\right),\bar{\Phi}\right)=0.$$

Then, since $\gamma\left(x_0;[0,T]\right)$ is a compact set and $\bar{\Phi}$ is a close set, then we conclude that $\gamma\left(x_0;[0,T]\right)\cap\bar{\Phi}\neq\emptyset$, whence we find out that $\gamma\left(x_0;[0,T]\right)\cap\bar{\Phi}\backslash\Phi\neq\emptyset$. With the condition H4 we reach to the conclusion $\gamma\left(x_0;[0,T]\right)\cap\bar{D}\backslash D\neq\emptyset$. In other words, the trajectory of problem (1), (3) meets the contour of the domain D. By this means we obtain that the solution of the problem without impulses is not continuable from some place (the moment of the meeting with ∂D) further. This situation contradicts condition H3. In this way we find out that $\rho_E\left(\gamma\left(x_0;[0,T]\right),\bar{\Phi}\right)>0$. Therefore, there exists a constant $d>0$, such that

$$\rho_E\left(\gamma\left(x_0;[0,T]\right),\Phi\right)=d>0. \tag{7}$$

Using the theorem of the continuous dependence of the solution of the initial problem without impulses on the initial condition (see theorem 7.1, §7, I., [7], for brevity we will name a continuous dependence theorem) it follows that

$$\begin{aligned}&\left(\exists\delta_1=\delta_1\left(x_0,T\right)>0\right):\ \left(\forall x_0^*\in D,\ \ \left\|x_0^*-x_0\right\|<\delta_1\right)\\&\qquad\Rightarrow\left\|x^*\left(t;x_0^*\right)-x\left(t;x_0\right)\right\|<\min\left\{\varepsilon,d\right\},\ \ 0\le t\le T\\&\qquad\Leftrightarrow\sup\left\{\left\|x^*\left(t;x_0^*\right)-x\left(t;x_0\right)\right\|,\ 0\le t\le T\right\}<\min\left\{\varepsilon,d\right\}.\end{aligned} \tag{8}$$

From the above estimate, taking into consideration remark 1, it follows that

$$\rho_E\left(\gamma^*\left(x_0^*;[0,T]\right),\gamma\left(x_0;[0,T]\right)\right)<d.$$

Using (7) we get the inequality $\rho_E\left(\gamma^*\left(x_0^*;[0,T]\right),\Phi\right)>0$, which means that the trajectory $\gamma^*\left(x_0^*;[0,T]\right)$ (as well as the trajectory $\gamma\left(x_0;[0,T]\right)$) does not intersect the impulsive set Φ. Again from (8) yields

$$\rho_H\left(\gamma^*\left(x_0^*;[0,T]\right),\gamma\left(x_0;[0,T]\right)\right)<\varepsilon,$$

which proves the theorem in this case.

Case 2. The trajectory $\gamma\left(x_0;[0,T]\right)$ intercepts (it is possible many times) the impulsive set Φ. The proof of the theorem in this case we will split into several parts:

Part 2.1. We shall show that the trajectory $\gamma^* (x_0^*; [0,T])$ of the perturbed problem (4), (5), (6) intersects also the impulsive set Φ for $0 < t < T$. Let the moment of the first meeting of the trajectory $\gamma (x_0; [0,T])$ with the impulsive set Φ be t_1, $0 < t_1 < T$. Let:

$$F(t) = \varphi (X(t; x_0)), F^*(t) = \varphi (X^*(t; x_0^*)), 0 < t < T,$$

where $X(t; x_0)$ and $X^*(t; x_0^*)$ are the solutions for the problems without impulses (1), (3) and (4), (6) respectively. Then it is satisfied

$$F(t_1) = \varphi (X(t_1; x_0)) = \varphi (x(t_1; x_0)) = 0.$$

In view of the condition H5 we assume that $\langle \operatorname{grad} \varphi(x), f(x) \rangle > 0, x \in \Phi$ (by analogy we regard the case $\langle \operatorname{grad} \varphi(x), f(x) \rangle < 0, x \in \Phi$). We have

$$\frac{d}{dt} F(t_1) = \frac{d}{dt} \varphi (X(t_1; x_0)) = \langle \operatorname{grad} \varphi (X(t_1; x_0)), f(X(t_1; x_0)) \rangle > 0,$$

whence we deduce that there exists a constant $\Delta t > 0$ such that:

$$\begin{aligned} F(t_1 - \tau) &= \varphi (X(t_1 - \tau; x_0)) < 0, \\ F(t_1 + \tau) &= \varphi (X(t_1 + \tau; x_0)) > 0, \end{aligned} \qquad 0 < \tau < \Delta t.$$

The magnitude of Δt we will clarify further. From the continuity of the function φ it follows that there exists a constant $d > 0$, such that:

$$(\forall x \in D, \rho_E (x, X(t_1 - \Delta t; x_0)) < d) \Rightarrow \varphi(x) < 0,$$

$$(\forall x \in D, \rho_E (x, X(t_1 + \Delta t; x_0)) < d) \Rightarrow \varphi(x) > 0.$$

Let ε_1 be an arbitrary positive number, which we will specify further. The theorem of continuous dependence yields

$$\begin{aligned} &(\exists \delta_2 = \delta_2 (x_0, d, \varepsilon_1, \Delta t), \quad 0 < \delta_2 < \delta_1) : \ (\forall x_0^* \in D, \quad \|x_0^* - x_0\| < \delta_2) \\ &\quad \Rightarrow \|X^*(t_1 - \Delta t; x_0^*) - X(t_1 - \Delta t; x_0)\| < d, \\ &\quad \quad \|X^*(t_1 + \Delta t; x_0^*) - X(t_1 + \Delta t; x_0)\| < d, \\ &\quad \quad \|X^*(t; x_0^*) - X(t; x_0)\| < \frac{\varepsilon_1}{2}, \quad 0 \le t \le t_1 + \Delta t. \end{aligned} \tag{9}$$

From the first inequality of (9) it follows

$$\rho_E (X^*(t_1 - \Delta t; x_0^*), X(t_1 - \Delta t; x_0)) < d,$$

from where we find

$$F^*(t_1 - \Delta t) = \varphi(X^*(t_1 - \Delta t; x_0^*)) < 0.$$

Also

$$\rho_E(X^*(t_1 + \Delta t; x_0^*), X(t_1 + \Delta t; x_0)) < d$$

is satisfied, whence we get the inequality

$$F^*(t_1 + \Delta t) = \varphi(X^*(t_1 + \Delta t; x_0^*)) > 0.$$

Then from the continuity of the function F^* it follows that there exists a point t_1^*, $t_1 - \Delta t < t_1^* < t_1 + \Delta t$ such that

$$F^*(t_1^*) = 0 \Leftrightarrow \varphi(X^*(t_1^*; x_0^*)) = 0,$$

i.e. the trajectory $\gamma^*(x_0^*; [0, \infty))$ intercepts the impulsive surface Φ in the point t_1^*, where $|t_1^* - t_1| < \Delta t$.

Part 2.2. We will find the estimate for the distance

$$\rho_H(\gamma^*(x_0^*; [0, t_1^*]), \gamma(x_0; [0, t_1])).$$

We denote by $t_i^{\min} = \min\{t_i, t_i^*\}$, $t_i^{\max} = \max\{t_i, t_i^*\}$, $i = 1, 2, ...$ From the last estimate in (9) it follows that

$$\|x^*(t; x_0^*) - x(t; x_0)\| = \|X^*(t; x_0^*) - X(t; x_0)\| < \frac{\varepsilon_1}{2}, \quad 0 \le t \le t_1^{\min}. \quad (10)$$

Without loss of generality we can assume $t_1 \le t_1^*$. Then using condition H6, for any $t^*, t_1 \le t^* \le t_1^*$, the following estimate is valid

$$\|x^*(t^*; x_0^*) - x(t_1; x_0)\| \le \|x^*(t^*; x_0^*) - x^*(t_1; x_0^*)\| + \|x^*(t_1; x_0^*) - x(t_1; x_0)\|$$
$$< \int_{t_1}^{t^*} \|f(x^*(\tau; x_0^*))\| d\tau + \frac{\varepsilon_1}{2} \le (t^* - t_1) C_f + \frac{\varepsilon_1}{2} \le \Delta t.C_f + \frac{\varepsilon_1}{2}.$$

Now we specify the magnitude of the constant Δt. We suppose that $\Delta t.C_f < \frac{\varepsilon_1}{2}$ is true. The last inequality yields

$$\|x^*(t^*; x_0^*) - x(t_1; x_0)\| < \varepsilon_1, t_1^{\min} = t_1 \le t^* \le t_1^* = t_1^{\max}. \quad (11)$$

In particular, from the above inequality for $t = t_1^{\max} = t_1^*$ w get

$$\|x^*(t_1^*; x_0^*) - x(t_1; x_0)\| < \varepsilon_1. \quad (12)$$

Using (10) we have

$$\begin{aligned}&\sup\left\{\inf\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t;x_0\right)\right),0\le t^*\le t_1^*\right\},0\le t\le t_1\right\}\\&\quad\le\sup\left\{\inf\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t;x_0\right)\right),0\le t^*\le t_1\right\},0\le t\le t_1\right\}\\&\quad\le\sup\left\{\rho_E\left(x^*\left(t;x_0^*\right),x\left(t;x_0\right)\right),0\le t\le t_1\right\}\\&\quad=\max\left\{\left\|x^*\left(t;x_0^*\right)-x\left(t;x_0\right)\right\|,0\le t\le t_1\right\}\le\frac{\varepsilon_1}{2}.\end{aligned}\tag{13}$$

Again, by inequalities (10) and (11) we obtain

$$\begin{aligned}&\sup\left\{\inf\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t;x_0\right)\right),0\le t\le t_1\right\},0\le t^*\le t_1^*\right\}\\&=\max\Big\{\sup\Big\{\inf\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t;x_0\right)\right),0\le t\le t_1\right\},0\le t^*\le t_1=t_1^{\min}\Big\},\\&\quad\sup\Big\{\inf\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t;x_0\right)\right),0\le t\le t_1\right\},t_1^{\min}=t_1\le t^*\le t_1^*\Big\}\Big\}\\&\le\max\left\{\sup\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t^*;x_0\right)\right),0\le t^*\le t_1\right\},\right.\\&\quad\left.\sup\left\{\rho_E\left(x^*\left(t^*;x_0^*\right),x\left(t_1;x_0\right)\right),t_1\le t^*\le t_1^*\right\}\right\}\\&\le\max\left\{\max\left\{\left\|x^*\left(t^*;x_0^*\right)-x\left(t^*;x_0\right)\right\|,0\le t^*\le t_1\right\},\right.\\&\quad\left.\max\left\{\left\|x^*\left(t^*;x_0^*\right)-x\left(t_1;x_0\right)\right\|<\varepsilon_1,t_1\le t^*\le t_1^*\right\}\le\max\left\{\frac{\varepsilon_1}{2},\varepsilon_1\right\}\right.\\&=\varepsilon_1.\end{aligned}\tag{14}$$

Then, from (13) and (14) we deduce that

$$\rho_H\left(\gamma^*\left(x_0^*;[0,t_1^*]\right),\gamma\left(x_0;[0,t_1]\right)\right)<\varepsilon_1.$$

Part 2.3. We shall estimate $\rho_E\left(x_1^{*+},x_1^+\right)$. Let ε_2 be an arbitrary positive number. Then in accordance with condition H7 and inequality (12), it is fulfilled

$$\begin{aligned}&\left(\exists\varepsilon_1=\varepsilon_1\left(\varepsilon_2,I\right)>0\right):\left(\left\|x^*\left(t_1^*;x_0^*\right)-x\left(t_1;x_0\right)\right\|<\varepsilon_1\right)\\&\qquad\qquad\Rightarrow\left\|I\left(x^*\left(t_1^*;x_0^*\right)\right)-I\left(x\left(t_1;x_0\right)\right)\right\|<\varepsilon_2.\end{aligned}\tag{15}$$

Again, using (12), (15) and the additional assumption that for any $x\in\Phi$ the following estimate is valid

$$\left\|I^*\left(x\right)-I\left(x\right)\right\|=\rho_E\left(I^*\left(x\right),I\left(x\right)\right)<\delta_2,$$

we get

$$\begin{aligned}\rho_E\left(x_1^{*+},x_1^+\right)&=\left\|x_1^{*+}-x_1^+\right\|=\left\|x^*\left(t_1^*+0;x_0^*\right)-x\left(t_1+0;x_0\right)\right\|\\&=\left\|x^*\left(t_1^*;x_0^*\right)+I^*\left(x^*\left(t_1^*;x_0^*\right)\right)-x\left(t_1;x_0\right)-I\left(x\left(t_1;x_0\right)\right)\right\|\\&\le\left\|x^*\left(t_1^*;x_0^*\right)-x\left(t_1;x_0\right)\right\|+\left\|I^*\left(x^*\left(t_1^*;x_0^*\right)\right)-I\left(x\left(t_1;x_0\right)\right)\right\|\end{aligned}$$

$$
\begin{aligned}
&<\varepsilon_1 + \left\| I^* \left(x^* \left(t_1^*; x_0^* \right) \right) - I \left(x^* \left(t_1^*; x_0^* \right) \right) \right\| \\
&\quad + \left\| I \left(x^* \left(t_1^*; x_0^* \right) \right) - I \left(x \left(t_1; x_0 \right) \right) \right\| \\
&<\varepsilon_1 + \delta_2 + \varepsilon_2.
\end{aligned}
$$

Part 2.4. We shall estimate the difference $t_2 - t_1$ from below, i.e. the difference between the second and the first impulsive moment of the fundamental problem (1), (2), (3). According to condition H8 there exists a positive constant d such that $\rho_E \left(\Phi, \left(Id + I \right) \left(\Phi \right) \right) = d > 0$. Since $x_2 = x \left(t_2; x_0 \right) \in \Phi$ and $x_1^+ \in \left(I + Id \right) \left(\Phi \right)$, then

$$
\begin{aligned}
d \leq &\rho_E \left(x_2, x_1^+ \right) \leq \left\| x_2 - x_1^+ \right\| = \left\| \int_{t_1}^{t_2} f \left(x \left(\tau; x_1^+ \right) \right) d\tau \right\| \\
\leq &\int_{t_1}^{t_2} \left\| f \left(x \left(\tau; x_1^+ \right) \right) \right\| d\tau \\
\leq &\left(t_2 - t_1 \right) C_f
\end{aligned}
$$

whence we receive the estimate $t_2 - t_1 \geq d/C_f$.

Part 2.5. We shall show that the trajectory $\gamma \left(x_0; [0, T] \right)$ meets finite times the impulsive set Φ. Assume that the meetings are infinite numbers. By analogy with Part 2.4 we find the estimates

$$
t_{i+1} - t_i \geq d/C_f, i = 1, 2, ...
$$

The last inequalities show that $\lim_{i \to \infty} t_i = \infty$, hence finite numbers of these impulsive moments belong to the finite interval $[0, T]$, which proves the statement in this part. Let $0 < t_1 < t_2 < ... < t_k < T \leq t_{k+1}$.

Part 2.6. Using the previous parts we come to the following conclusions:

$$
\begin{aligned}
&\left(\forall \delta_1^* > 0 \right) \left(\exists \delta_0^*, \quad 0 < \delta_0^* < \delta_1^* \right) : \ \left(\forall x_0^* \in D, \quad \rho_E \left(x_0^*, x_0 \right) < \delta_0^* \right) \\
&\qquad\qquad \left(\forall I^* : D \to \mathrm{R}^n, \quad \rho_E \left(I^* \left(x \right), I \left(x \right) \right) < \delta_0^*, \quad x \in D \right).
\end{aligned}
$$

We have:

1.1. The trajectory $\gamma^* \left(x_0^*; [0, T] \right)$ meets the impulsive set Φ at the moment t_1^* and the inequality $\left| t_1^* - t_1 \right| < \delta_1^*$ is valid;

1.2. The distance $\rho_H \left(\gamma^* \left(x_0^*; [0, t_1^*] \right), \gamma \left(x_0; [0, t_1] \right) \right) < \delta_1^*$;

1.3. The distance $\rho_E \left(x_1^{*+}, x_1^+ \right) < \delta_1^*$.

Analogically we get

$$(\forall \delta_2^* > 0) (\exists \delta_1^*, 0 < \delta_1^* < \delta_2^*) : \left(\forall x_1^{*+} \in D, \rho_E \left(x_1^{*+}, x_1^+\right) < \delta_1^*\right)$$
$$(\forall I^* : D \to \mathrm{R}^n, \rho_E (I^* (x), I (x)) < \delta_1^*, x \in D) \Rightarrow$$

2.1. The trajectory $\gamma^* (x_0^*; [0, T])$ meets the impulsive set Φ at the moment t_2^* and $|t_2^* - t_2| < \delta_2^*$;

2.2. $\rho_H (\gamma^* (x_0^*; [0, t_2^*]), \gamma (x_0; [0, t_2])) < \delta_2^*$;

2.3. $\rho_E \left(x_2^{*+}, x_2^+\right) < \delta_2^*$ etc.

At the end we have

$$\left(\forall \delta_{k+1}^* > 0\right) \left(\exists \delta_k^*, 0 < \delta_k^* < \delta_{k+1}^*\right) :$$

$$\left(\forall x_k^{*+} \in D, \rho_E \left(x_k^{*+}, x_k^+\right) < \delta_k^*\right) (\forall I^* : D \to \mathrm{R}^n, \rho_E (I^* (x), I (x)) < \delta_k^*, x \in D) \Rightarrow$$

k.1. The trajectory $\gamma^* (x_0^*; [0, T])$ does not meet the impulsive set Φ for $t_k^* < t < T$;

$$k.2. \quad \rho_H (\gamma^* (x_0^*; [0, T]), \gamma (x_0; [0, T])) < \delta_{k+1}^*.$$

Finally, if we substitute $\varepsilon = \delta_{k+1}^*$ and define consistently the monotonously decreasing sequence of constants: $\delta_k^*, \delta_{k-1}^*, ..., \delta_0^*$, w obtain

$$(\forall \varepsilon > 0) (\exists \delta_0^* = \delta_0^* (\varepsilon, T, x_0, I) > 0) :$$

$$(\forall x_0^* \in D, \rho_E (x_0^*, x_0) < \delta_0^*) (\forall I^* : D \to \mathrm{R}^n, \rho_E (I^* (x), I (x)) < \delta_0^*, x \in D) \Rightarrow$$

$$\rho_H (\gamma^* (x_0^*; [0, T]), \gamma (x_0; [0, T])) < \varepsilon.$$

The theorem is proved.

4. Application of the Main Results

The Lotka-Volterra mathematical model describes quite accurately the evolution dynamics of the predator-prey interactions of an isolated (without external influences) biosystem. A co-association of the above type which is put to the external influences (usually due to human interventions) is studied in the article. These effects are expressed in the removal or addition of certain quantities of biomass both from the prey and the predator. It is natural to require the following restrictions of the external influences:

• The duration of each of these perturbations to be negligible compared with the total duration of the process so that it is possible to assume that these effects are instantaneous in the form of impulses;

• The influences take place in the moments at which the biomasses of the prey and the predators reach the certain quantitative characteristics. From mathematical point of view the impulsive removals or additions of biomass are materialized at the meeting of the system trajectory with a pre-fixed set called "impulsive set", which is located in the phase space of the system. Usually the impulsive set is a smooth curve of the space of permitted states of the system;

• Since it is practically impossible to add or to remove biomass under a certain minimum, then we conclude that ,,the sizes" of the impulsive perturbations are limited from below;

• "The sizes" of the impulsive perturbations are limited from above because:

• If the influence is a biomass removal (the most frequent case which is being investigated in the present paper), it is impossible for the amount taken away to be more than the amount available in the moment of the impulsive effect;

• If the influence is a biomass addition (this is a rare case), then from an economical point of view it is not appropriate for the supplement amount of biomass to be above an economically justified maximum;

• Very often, for objective reasons, it is impossible to separate a prey and a predator. In these cases the removed (taken) of the co-association biomass is a mix of biomasses from both of the prey and the predator. Moreover, the removed volumes from both species are proportional to the amounts of their biomasses at the moment of the removal. Let the quantities of biomasses of the prey and the predator be m and M at the moment of the impulsive effects, then the quantities of the removed biomasses are $\Delta.m$ and $\Delta.M$ respectively. The function $\Delta = \Delta(m, M)$ is defined for any point of the impulsive set and it satisfies the inequalities: $0 \leq \Delta \leq 1$.

• It turns out that the exploitation of the considered above co-associations is facilitated if they possess a periodic law of evolution. Therefore, after a "removal" type impulsive perturbation it is desirable for the amounts of both species biomasses to be in such volumes that they "lie" on the same trajectory, which is described before the impulsive moment.

The generalized Lotka-Volterra impulsive model with an initial condition, satisfying the above requirements, has a form:

$$\frac{dm}{dt} = \dot{m} = F_m(m, M) = m(r_1 - q_1 M), \qquad M(t) \neq k.m(t); \tag{16}$$

$$\frac{dM}{dt} = \dot{M} = F_M(m, M) = -M(r_2 - q_2 m), \quad M(t) \neq k.m(t); \tag{17}$$

$$m(t+0) = (1-\Delta)\, m(t), \qquad M(t) = k.m(t); \tag{18}$$

$$M(t+0) = (1-\Delta)\, M(t), \qquad M(t) = k.m(t); \tag{19}$$

$$m(0) = m_0;\ M(0) = M_0, \tag{20}$$

where:

• $m = m(t) > 0$ and $M = M(t) > 0$ are the quantities of biomasses of the prey and the predator respectively at the moment $t \geq 0$;

• The constants $r_1 > 0$ and $r_2 > 0$ are specific coefficients of the relative growth of the first species (prey) and of the second species (predator), respectively;

• The constants $q_1 > 0$ and $q_2 > 0$ are the coefficients reflecting interspecies competition for the prey and the predator, respectively;

• The set of points (m, M), belonging to the acceptable states of the co-association $\mathrm{R}^+ \times \mathrm{R}^+$ and satisfying the equality $M = k.m$, is called an impulsive set. (In this case the set is a ray with origin coinciding with the origin of the Cartesian system and a slope $k > 0$, which will be specified further);

• $\Delta.m(t)$ and $\Delta.M(t)$ are the quantities of biomass of the prey and the predator respectively which are removed in the form of impulses. The function $\Delta = \Delta(m, M)$ will be specified later. "The impulsive moments" coincide with the moments at which the ratio of the quantities of biomasses of the predator and prey reaches the value k.

• The constants $m_0 > 0$ and $M_0 > 0$ are the quantities of biomasses of both species at the initial moment $t = 0$.

It is known that the system (16), (17) possesses:

• Unstable stationary point $(0,0)$, (the origin is a saddle point);

• Stable stationary point $(m_{00}, M_{00}) = \left(\frac{r_2}{q_2}, \frac{r_1}{q_1}\right)$;

• A first integral of the following form

$$\begin{aligned} U(m, M) =& q_1 M + q_2 m - r_1 \ln M - r_2 \ln m + r_1 \left(\ln \frac{r_1}{q_1} - 1\right) + r_2 \left(\ln \frac{r_2}{q_2} - 1\right) \\ =& W(m, M) - W(m_{00}, M_{00}), \end{aligned}$$

where $W(m,M) = q_1 M + q_2 m - r_1 \ln M - r_2 \ln m$;

• For any point $(m,M) \in \mathrm{R}^+ \times \mathrm{R}^+, (m,M) \neq (m_{00}, M_{00})$ the inequality $U(m,M) > 0$ is valid. It is fulfilled $U(m_{00}, M_{00}) = 0$;

• For any constant $c \geq 0$ the implicitly given curve

$$\gamma_c = \{(m,M) : U(m,M) = c\}$$

is a trajectory of the system (16), (17) with a properly chosen initial condition (it is sufficient to assume that $U(m_0, M_0) = c$);

• For any constant $c > 0$ the set

$$D_c = \{(m,M) : U(m,M) < c\}$$

is a connected domain, located in $\mathrm{R}^+ \times \mathrm{R}^+$, with a contour $\partial D_c = \gamma_c$;

• For any constant $c > 0$ it is satisfied $(m_{00}, M_{00}) \in D_c$;

• If $0 < c_1 < c_2$, then $\gamma_{c_1} \in D_{c_2}$.

The obtained basic results are applicable to the model (16)–(20). For more clearness the suppositions and the corresponding considerations in this section will be classified into several parts:

Part 1. We shall assume that the domain D, in which is located the trajectory of the problem with impulses from the population dynamics, is situated "between" two "border" trajectories γ_{c_1} and γ_{c_2}, where the constants c_1 and c_2 satisfy the inequalities $0 < c_1 < c_2$,i.e. $D = D_{c_2} \backslash D_{c_1}$.

Part 2. We shall assume that the trajectory of the impulsive problem is a part of the trajectory γ_{c_0}, where the constant c_0 satisfies the inequalities $c_1 < c_0 < c_2$.

Part 3. The impulsive surface (in this case the impulsive segment) is a part of the straight line $l : M = km$. Hence it is fulfilled $\varphi(m,M) = M - km$. The slope k of the line l is specified as follows. Let the points $\left(m_{00}, M_{00}^1\right), \quad \left(m_{00}^1, M_{00}\right) \in \gamma_{c_1}$, as constants m_{00}^1 and M_{00}^1 are the larger solutions of the following equations respectively:

$$U(m, M_{00}) = c_1 \Leftrightarrow q_2 m - r_2 \ln m = c_1 - r_2 \left(\ln \frac{r_2}{q_2} - 1\right),$$

$$U(m_{00}, M) = c_1 \Leftrightarrow q_1 M - r_1 \ln M = c_1 - r_1 \left(\ln \frac{r_1}{q_1} - 1\right).$$

The solutions of the above equations are founded approximately. Suppose that

$$\frac{M_{00}}{m_{00}^{1}} \leq k \leq \frac{M_{00}^{1}}{m_{00}}.$$

Then the impulsive set (impulsive segment) is defined in the following way:

$$\Phi = \{(m, M); M - km = 0, m > m_{00}, M > M_{00}\} \cap D_{c_2} \backslash D_{c_1}.$$

Having in mind the choice of the slope k we conclude that the endpoints of Φ (i.e. the contour $\partial\Phi$ of Φ) belong to the trajectories γ_{c_1} and γ_{c_2} respectively, i.e. on the contour ∂D of the domain D. This means that $\bar{\Phi}\backslash\Phi = \partial\Phi \subset \partial D = \bar{D}\backslash D$, i.e. the condition H4 is valid.

Part 4. We assume that the trajectory of impulsive problem (16)–(20) of the population dynamics is periodical. This assumption will be satisfied if a "depicted point of the process" (m, M), which is moving on the trajectory of the considered problem after an impulsive effect "falls" again on the trajectory γ_{c_0}. Thus we attain the conclusion that the impulsive vector has a form $I(m, M) = I(m, km) = -(\Delta_m, k\Delta_m)$, where m and Δ_m are the solutions of the following system:

$$U(m, km) = c_0, U(m - \Delta_m, k(m - \Delta_m)) = c_0, \quad \Delta_m > 0.$$

The above equations are similar (they have only different variables). We have

$$q_1 km + q_2 m - r_1 \ln km - r_2 \ln m = c_0 - r_1 \left(\ln \frac{r_1}{q_1} - 1\right) - r_2 \left(\ln \frac{r_2}{q_2} - 1\right)$$
$$\Leftrightarrow (q_1 k + q_2) m - (r_1 + r_2) \ln m = c_{00},$$

where

$$c_{00} = c_0 - r_1 \left(\ln {}^{r_1}\!/_{kq_1} - 1\right) - r_2 \left(\ln {}^{r_2}\!/_{q_2} - 1\right).$$

Let the approximate solutions be $m_0' = m_0'(c_0)$ and $m_0'' = m_0''(c_0)$, as $m_0' < m_0''$. Then $m = m_0''$ and $M = km_0''$ are the approximate quantities of the prey and the predator biomasses respectively in the moment of impulsive perturbation. The sizes of impulsive effects (removals) according to the prey and predator are $\Delta_m = \Delta_m(c_0) = m_0''(c_0) - m_0'(c_0)$ and $k\Delta_m = k\Delta_m(c_0) = k\left(m_0''(c_0) - m_0'(c_0)\right)$. Clearly the impulsive function

$$I = -(\Delta_m(c_0), k\Delta_m(c_0))$$

is continuous on c_0, where $c_1 < c_0 < c_2$, i.e. on the impulsive set Φ. Consequently the solution H7 is valid.

Part 5. According to the choice of the impulsive function in the previous section it follows that

$$(Id + I)(\Phi) = \{(m, M); M - km = 0, m < m_{00}, M < M_{00}\} \cap D_{c_2} \backslash D_{c_1}.$$

Then $\rho_E(\Phi, (Id + I)(\Phi)) = \left| m_1^{''} - m_1^{'} \right| \sqrt{1 + k^2} > 0$, where $m_1^{'}$ and $m_1^{''}$ are two different solutions of the equation

$$q_1 km + q_2 m - r_1 \ln km - r_2 \ln m = c_1 - r_1 \left(\ln {}^{r_1}\!/_{q_1} - 1 \right) - r_2 \left(\ln {}^{r_2}\!/_{q_2} - 1 \right)$$

$$\Leftrightarrow (q_1 k + q_2) m - (r_1 + r_2) \ln m = c_{11},$$

where

$$c_{11} = c_1 - r_1 \left(\ln {}^{r_1}\!/_{kq_1} - 1 \right) - r_2 \left(\ln {}^{r_2}\!/_{q_2} - 1 \right).$$

This means that the condition H8 is fulfilled.

Part 6. Suppose that for $t = 0$ it is satisfied:

$$m(0) = m_0 = m_0^{'}, M(0) = M_0 = km_0^{'}.$$

The given assumption is negligible. The unique reason for its imposition is our desire the initial point (m_0, M_0) to belong to $(Id + I)(\Phi)$.

Part 7. Since the solution is a periodic, then there exists a positive constant Δ_t (equals to the period) such that: $t_i = i.\Delta_t, i = 1, 2, ...,$ are true. Therefore, $\lim\limits_{i \to \infty} t_i = \infty$,i.e. condition H2 is fulfilled.

Part 8. The condition H1 follows by the form of the right hand side of the equations (16) and (17). It is known that the problem without impulses (16), (17), (20) possesses a unique solution which trajectory coincides with γ_{c_0}, where $c_0 = U(m_0, M_0)$. Consequently Theorem 1 is valid, i.e. the solution of problem (16)–(20) exists and it is unique for every $t \geq 0$.

Part 9. Since $\varphi(m, M) = M - km$ and $M - km \in C^1(D_{c_2} \backslash D_{c_1}, \mathrm{R})$, then we deduce that $\varphi \in C^1(D, \mathrm{R})$. On the other hand we have

$$\operatorname{grad} \varphi(m, M) = \left(\frac{\partial}{\partial m}(M - km), \frac{\partial}{\partial M}(M - km) \right) = (-k, 1).$$

Hence

$$\langle \operatorname{grad} \varphi(m, M), f(m, km) \rangle = \langle (-k, 1), (m(r_1 - q_1 km), -km(r_2 - q_2 m)) \rangle$$

$$=km\left(q_2\left(m-{}^{r_2}\!/_{q_2}\right)+q_1\left(km-{}^{r_1}\!/_{q_1}\right)\right)$$
$$=M\left(q_2\left(m-m_{00}\right)+q_1\left(M-M_{00}\right)\right). \tag{21}$$

From Part 3 it follows that for every point $(m, M) \in \Phi$ the inequalities $m > m_{00}$ and $M > M_{00}$ are fulfilled. Therefore, having in mind (21), it follows

$$\langle \operatorname{grad} \varphi(m, M), f(m, km)\rangle > 0,$$

i.e. condition H5 is valid.

Part 10. It is trivial to verify the condition H6.

Part 11. The above considerations persuaded that the Lotka-Volterra generalized impulsive model satisfies the conditions of Theorem 2. Therefore, the periodical solution of problem (16)–(20) depends orbital Hausdorff continuously on the initial point (m_0, M_0) and the impulsive function I.

The following interpretation from the received results is possible. Let:

• The initial quantities of the biomasses of the prey and the predator in the perturbed co-association are not significantly different from the initial quantities of the biomasses of the prey and the predator in the original co association;

• The impulsive removals in the perturbed and the original co-association are approximately equal.

Then the regular and the perturbed problem possess neighboring trajectories for a finite time interval.

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