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IDEAL EXHAUSTIVENESS, CONTINUITY AND (α)-CONVERGENCE FOR LATTICE GROUP-VALUED FUNCTIONS

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Abstract: We examine some fundamental properties of $\mathcal{I}$-exhaustiveness, previously studied in the real case in [9], in the context of (ℓ)-groups with respect to (D)-convergence and we answer to an open problem posed by V. Gregoriades and N. Papanastassiou in [4].

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1. Introduction

The concept of *α-convergence,* or *continuous convergence* for real-valued functions has been known in the literature since the beginning of last century (see for instance [3], [6], [11]). This notion was extended to the case of an ordered structure by E. Wolk in 1975 (see [12]).

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In [4] there are some investigations and comparisons concerning the notions and main properties of α-convergence, equicontinuity and exhaustiveness in the real setting and some Ascoli-type theorems are given.

These results are extended in [9] for functions taking values in metric spaces in the context of $\mathcal{I}$-convergence introduced in [8].

In this paper we extend these concepts to functions with values in Dedekind complete weakly σ-distributive (ℓ)-groups, dealing with (D)-convergence (see [2]) and investigating, in this setting, the interrelations between exhaustiveness, equicontinuity, even continuity and α-convergence involving admissible ideals of the set of the natural numbers. The use of (D)-convergence could seem apparently more difficult, but it often simplifies the proofs and allows us to replace a countable family of regulators with one (D)-sequence without assuming further additional hypotheses on the involved (ℓ)-group.

We give the concept of weak (ideal) exhaustiveness for sequences of functions, by means of which we present a characterization of continuity of the limit function. Moreover we introduce axiomatically some different types of convergences and relate them with (α)- and pointwise convergence.

2. Preliminaries

Definitions 2.1. a) Let T be any infinite set. A family of sets $\mathcal{I} \subset \mathcal{P}(T)$ is called an *ideal* of T iff $A \cup B \in \mathcal{I}$ whenever $A, B \in \mathcal{I}$ and for each $A \in \mathcal{I}$ and $B \subset A$ we get $B \in \mathcal{I}$.

b) An ideal $\mathcal{I}$ is said to be *non-trivial* iff $\mathcal{I} \neq \emptyset$ and $T \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I}$ is said to be *admissible* iff it contains all singletons of elements of T.

c) We say that an admissible ideal $\mathcal{I}$ is a *P-ideal* iff for any sequence $(A_j)_j$ in $\mathcal{I}$ there are sets $B_j \subset T$, $j \in \mathbb{N}$, such that the symmetric difference $A_j \Delta B_j$ is finite for all $j \in \mathbb{N}$ and $\displaystyle\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$ (see also [8]).

d) Let $T = \mathbb{N}$. We say that an admissible ideal $\mathcal{I}$ of $\mathcal{P}(\mathbb{N})$ is *good* iff for every sequence $(P_k)_k$ of subsets of $\mathbb{N}$ such that $P_k \notin \mathcal{I}$ for all $k \in \mathbb{N}$, there is a disjoint sequence $(B_k)_k$ with the property that $B_k \subset P_k$, $B_k \in \mathcal{I}$ for every $k \in \mathbb{N}$ and $\cup_{k=1}^{\infty} B_k \notin \mathcal{I}$ (see [9]).

It is easy to check that the ideal $\mathcal{I}_{\text{fin}}$ of all finite subsets of $\mathbb{N}$ is good.

Remark 2.2. Let $\mathcal{I}_d$ be the ideal of all subsets of $\mathbb{N}$ having zero asymptotic density, where the asymptotic density of a set $A \subset \mathbb{N}$ is defined as

$$d(A) = \lim_n \frac{\mathrm{card}(A \cap \{1, \dots, n\})}{n}$$

(if this limit exists) and card denotes the cardinality of the set in brackets. It is known that $\mathcal{I}_d$ is a P-ideal but it is not good (see [9]).

Now, let $\mathbb{N} = \cup_{j=1}^{\infty} \Delta_j$ be a partition of $\mathbb{N}$ such that Δ_j is an infinite set for every $j \in \mathbb{N}$, and consider the ideal consisting of all sets $E \subset \mathbb{N}$ which intersect only a finite number of Δ_j's. Obviously this ideal is good but it is not a P-ideal (see [8]).

Definitions 2.3. a) An (ℓ)-group R is said to be *Dedekind complete iff every nonempty subset of R, bounded from above, has supremum in R.*

b) A bounded double sequence $(a_{i,l})_{i,l}$ in R is called (D)-sequence or *regulator iff for all $i, l \in \mathbb{N}$ we have $a_{i,l} \geq a_{i,l+1}$ and $\wedge_l a_{i,l} = 0$ for all $i \in \mathbb{N}$, where the symbol $\wedge$ denotes the lattice infimum.*

c) A sequence $(x_n)_n$ in R is said to be (D)-convergent to $x \in R$ (and we write $(D) \lim_n x_n = x$) iff there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds $n_0 \in \mathbb{N}$ such that $|x_n - x| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for all $n \in \mathbb{N}$, $n \geq n_0$.

d) Let $\Lambda \neq \emptyset$ be any set, $(x_n^{(\lambda)})_n, \lambda \in \Lambda$, be a family of sequences of elements of R and $x^{(\lambda)}$, $\lambda \in \Lambda$, be elements of R. We say that $(D) \lim_n x_n^{(\lambda)} = x^{(\lambda)}$ *uniformly with respect to $\lambda \in \Lambda$ iff there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R, such that for any $\varphi \in \mathbb{N}^{\mathbb{N}}$ there is $n_0 \in \mathbb{N}$ with $|x_n^{(\lambda)} - x^{(\lambda)}| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ for all $n \in \mathbb{N}$, $n \geq n_0$ and $\lambda \in \Lambda$.*

e) A sequence $(x_n)_n$ is said to be (D)-Cauchy iff $(D) \lim_n (x_n - x_{n+p}) = 0$ uniformly with respect to $p \in \mathbb{N}$. We say that an (ℓ)-group is (D)-*complete iff every (D)-Cauchy sequence in R is (D)-convergent. We recall that every Dedekind complete (ℓ)-group is (D)-complete (see also [2, Chapter 2]).*

f) An (ℓ)-group R is said to be weakly σ-distributive iff for every (D)-sequence $(a_{i,l})_{i,l}$ we have:

$$\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0.$$

Throughout this paper we always assume that $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ is an admissible ideal, (X, d) is any complete metric space and R is a Dedekind complete weakly

σ-distributive (ℓ)-group. Moreover, given $x \in X$ and $\delta > 0$, we denote by $B(x, \delta)$ the set $\{z \in X : d(z, x) < \delta\}$.

We now recall the Fremlin Lemma (see [2] [10]) which allows us to replace countably many regulators by one (D)-sequence.

Lemma 2.4. *Let $\{(a_{i,l}^{(n)})_{i,l} : n \in \mathbb{N}\}$ be any countable family of regulators. Then for each fixed positive element u of R there is a regulator $(a_{i,l})_{i,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ one has:*

$$u \wedge \left(\sum_{n=1}^{\infty} \left[\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right] \right) \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Definitions 2.5. a) A sequence $(x_n)_n$ in X *$\mathcal{I}$-converges to $x \in X$ iff* $\{n \in \mathbb{N} : d(x_n, x_0) > \delta\} \in \mathcal{I}$ *for any $\delta > 0$. In this case we write $\mathcal{I} - \lim_n x_n = x$.*

b) A sequence $(x_n)_n$ in R $(D\mathcal{I})$-converges to $x \in R$ iff there exists a regulator $(a_{i,l})_{i,l}$ with the property that

$$\{n \in \mathbb{N} : |x_n - x| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\} \in \mathcal{I}$$

for all $\varphi \in \mathbb{N}^{\mathbb{N}}$ (see [1]). In this case we write $(D\mathcal{I}) - \lim_n x_n = x$.

c) Let $f : X \to R$ be a function and $x \in X$. Then f is called *continuous at x iff there exists a regulator $(a_{i,l})_{i,l}$ (depending on x) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a $\delta > 0$ (depending on φ and x) such that*

$$|f(x) - f(z)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $z \in B(x, \delta)$.

d) A function $f : X \to R$ is called continuous on X iff f is continuous at every point $x \in X$.

e) A function $f : X \to R$ is called *globally continuous on X iff there exists a regulator $(a_{i,l})_{i,l}$ such that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ there is $\delta > 0$ (depending on φ and on x) with*

$$|f(x) - f(z)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $z \in B(x, \delta)$.

Remarks 2.6. a) Observe that in Definition 2.5 e) the regulator $(a_{i,l})_{i,l}$ is independent of the choice of the point $x \in X$, in contrast to Definitions 2.5 c) and 2.5 d).

b) If X is a singleton, then Definitions 2.5 c), d) and e) coincide.

Definitions 2.7. a) Let X and R be as above and Φ be a (possibly infinite) family of functions from X to R. We say that Φ is *equicontinuous at* $x \in X$ *iff there is a regulator* $(a_{i,l})_{i,l}$ *(depending on* x*) such that to every* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *there corresponds a* $\delta > 0$ *(depending on* φ *and* x*) such that*

$$|f(y) - f(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $y \in B(x, \delta)$ *and* $f \in \Phi$.

b) Let X*,* R *and* Φ *be as in a). The family* Φ *is called* equicontinuous on X iff Φ is equicontinuous at x for every $x \in X$.

c) Let X, R and Φ be as in a). The family Φ is called *globally equicontinuous on* X *iff there exists a regulator* $(a_{i,l})_{i,l}$ *such that to every* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *and* $x \in X$ *there corresponds a* $\delta > 0$ *(depending on* φ *and on* x*) such that*

$$|f(y) - f(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $y \in B(x, \delta)$ *and* $f \in \Phi$.

Remarks 2.8. a) Observe that in Definition 2.7 c) the regulator $(a_{i,l})_{i,l}$ is independent of the choice of the point $x \in X$, in contrast to Definitions 2.7 a) and 2.7 b).

b) If X is a singleton, then Definitions 2.7 a), b) and c) coincide.

Definitions 2.9. a) With the same notations as in Definition 2.7, we say that Φ is *evenly continuous at* $x \in X$ *iff for each* (D)*-sequence* $(b_{i,l})_{i,l}$ *there is a regulator* $(a_{i,l})_{i,l}$ *(depending on* x*) such that for every* $y \in R$ *and* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *there are a positive real number* δ *and* $\psi \in \mathbb{N}^{\mathbb{N}}$ *with the property that, if* $f \in \Phi$*,* $|f(x) - y| \leq \bigvee_{i=1}^{\infty} b_{i,\psi(i)}$ *and* $d(z, x) < \delta$*, then* $|f(z) - y| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$.

b) Let X*,* R *and* Φ *be as in a). The family* Φ *is said to be* evenly continuous on X iff Φ is evenly continuous at x, for every $x \in X$.

c) The family Φ is called *globally evenly continuous at* $x \in X$ *iff for every regulator* $(b_{i,l})_{i,l}$ *there exists a* (D)*-sequence* $(a_{i,l})_{i,l}$ *such that for each* $y \in R$ *and* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *there exist a positive real number* δ *and* $\psi \in \mathbb{N}^{\mathbb{N}}$ *such that, if* $f \in \Phi$*,* $|f(x) - y| \leq \bigvee_{i=1}^{\infty} b_{i,\psi(i)}$ *and* $d(z, x) < \delta$*, then* $|f(z) - y| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$.

Remarks 2.10. a) Note that in Definition 2.9 c) the regulator $(a_{i,l})_{i,l}$ is independent of the choice of the point $x \in X$, in contrast to Definitions 2.9 a) and 2.9 b).

b) If X is a singleton, then Definitions 2.9 a), b) and c) coincide.

c) Observe that, proceeding analogously as in [7, Theorem 7.22], it is easy to check that a [globally] equicontinuous family in R^X is [globally] evenly continuous. Moreover, note that, if Φ is [globally] evenly continuous, then every element of Φ is [globally] continuous (see also [9]). In general the notion of even continuity is strictly weaker than the notion of equicontinuity (see [5]).

Definitions 2.11. a) Let X and R be as above and Φ be a (possibly infinite) family of functions from X to R. We say that Φ is *exhaustive at* $x \in X$ *iff there exists a* (D)*-sequence* $(a_{i,l})_{i,l}$ *(depending on* x*) such that for every* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *there are a* $\delta > 0$ *and a finite subset* A *of* Φ *(depending on* φ *and* x*) such that*

$$|f(y) - f(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $y \in B(x, \delta)$ *and* $f \in \Phi \setminus A$.

b) Let X, R *and* Φ *be as in a). The family* Φ *is called* exhaustive on X iff Φ is exhaustive at every $x \in X$.

c) Let X, R and Φ be as in a). The family Φ is said to be *globally exhaustive on* X *iff there exists a* (D)*-sequence* $(a_{i,l})_{i,l}$ *such that to each* $\varphi \in \mathbb{N}^{\mathbb{N}}$ *and* $x \in X$ *there correspond a* $\delta > 0$ *and a finite subset* A *of* Φ *(depending on* φ *and on* x*) such that*

$$|f(y) - f(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $y \in B(x, \delta)$ *and* $f \in \Phi \setminus A$.

Remarks 2.12. a) Observe that in Definition 2.11 c) the (D)-sequence $(a_{i,l})_{i,l}$ is independent of the choice of the point $x \in X$, in contrast to Definitions 2.11 a) and b).

b) If X is a singleton, then Definitions 2.11 a), b) and c) coincide.

c) If in Definitions 2.11 a), b), c) the involved finite set A is the empty set, then these definitions coincide with Definitions 2.7 a), b) and c) respectively.

We now introduce α-convergence in the context of (ℓ)-groups. We give the following:

Definitions 2.13. Let X, R and $(f_n)_n$ be as in the previous definitions.

a) We say that $(f_n)_n$ *α-converges to the function $f : X \to R$ iff to every $x \in X$ there corresponds a regulator $(a_{i,l})_{i,l}$ with the property that for each sequence $(x_n)_n$ in X with $\lim_n x_n = x$ we get $(D)\lim_n f_n(x_n) = f(x)$ (with respect to the regulator $(a_{i,l})_{i,l}$).*

b) The sequence $(f_n)_n$ is said to be globally α-convergent to $f : X \to R$ iff a (D)-sequence $(a_{i,l})_{i,l}$ can be found, such that for every $x \in X$ and for all sequences $(x_n)_n$ in X with $\lim_n x_n = x$ we have $(D)\lim_n f_n(x_n) = f(x)$ (with respect to the (D)-sequence $(a_{i,l})_{i,l}$).

Remark 2.14. Observe that in Definition 2.13 b) the regulator $(a_{i,l})_{i,l}$ is independent of the choices both of $x \in X$ and of the sequence $(x_n)_n$, while in Definition 2.13 a) the involved regulator $(a_{i,l})_{i,l}$ may depend on $x \in X$, but it is required to be independent of $(x_n)_n$.

Let now $\mathcal{I}$ be any fixed admissible ideal of $\mathbb{N}$. We introduce the following:

Definitions 2.15. Let X, R, $(f_n)_n$ and f be as above.

a) We say that $(f_n)_n$ *$(\mathcal{I}\alpha)$-converges to $f : X \to R$ iff for every $x \in X$ there exists a regulator $(a_{i,l})_{i,l}$ such that for each sequence $(x_n)_n$ in X with $\mathcal{I} - \lim_n x_n = x$ we get $(D\mathcal{I}) - \lim_n f_n(x_n) = f(x)$ with respect to the regulator $(a_{i,l})_{i,l}$.*

b) The sequence $(f_n)_n$ is said to be globally $(\mathcal{I}\alpha)$-convergent to $f : X \to R$ iff a (D)-sequence $(a_{i,l})_{i,l}$ can be found, such that for any $x \in X$ and for each sequence $(x_n)_n$ in X with $\mathcal{I} - \lim_n x_n = x$ we have $(D\mathcal{I}) - \lim_n f_n(x_n) = f(x)$ with respect to the (D)-sequence $(a_{i,l})_{i,l}$.

Remark 2.16. If $\mathcal{I} = \mathcal{I}_{fin}$ then Definitions 2.15 a) and b) coincide with Definitions 2.13 a) and b) respectively. Also Remark 2.14 is valid in this case.

The following result holds:

Proposition 2.17. *If $(f_n)_n$ is a function sequence, α-convergent to the function f, then every subsequence $(f_{k_n})_n$ of $(f_n)_n$, α-converges to f.*

Moreover, we have

$$(D)\lim_n f_{k_n}(x_n) = f(x) \tag{1}$$

with respect to a same regulator, independent of the choice of $(k_n)_n$. Finally, if we have global α-convergence of $(f_n)_n$ to f, then (1) holds with respect to a same regulator, independent also of the choice of x.

Proof. Fix arbitrarily an $x \in X$, let $(x_n)_n$ be convergent to x and $(k_n)_n$ be a strictly increasing sequence in $\mathbb{N}$. Set $y_n = x_i$ if $n = k_i$ for some $i \in \mathbb{N}$

and $y_n = x$ otherwise. Obviously, $\lim_n y_n = x$. By α-convergence of $(f_n)_n$ to f, we get that the sequence $(f_n(y_n))_n$ (D)-converges to $f(x)$ with respect to a regulator, depending only on x and not on the sequence $(y_n)_n$. Hence, $(D)\lim_n f_{k_n}(y_{k_n}) = f(x)$, that is $(D)\lim_n f_{k_n}(x_n) = f(x)$ (and with respect to the same regulator, independent of $(k_n)_n$). From this and Definition 2.13 b), also the last part of the assertion follows. □

Let now $\mathcal{I}$ be any fixed admissible ideal of $\mathbb{N}$. We state the following:

Definitions 2.18. Let X, R be as in Definitions 2.11.

a) A function sequence $f_n : X \to R$, $n \in \mathbb{N}$, is called *$\mathcal{I}$-exhaustive at $x \in X$ iff there exists a regulator $(a_{i,l})_{i,l}$ (depending on x) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there correspond a $\delta > 0$ and an element $A \in \mathcal{I}$ (depending on φ and x) such that*

$$|f_n(z) - f_n(x)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $z \in B(x,\delta)$ and $n \in \mathbb{N} \setminus A$.

b) Let X, R and $(f_n)_n$ be as in a). The sequence $(f_n)_n$ is said to be $\mathcal{I}$-exhaustive on X iff it is $\mathcal{I}$-exhaustive at x, for every $x \in X$.

c) Let X, R and $(f_n)_n$ be as in a). The sequence $(f_n)_n$ is called globally $\mathcal{I}$-exhaustive on X iff there exists a regulator $(a_{i,l})_{i,l}$ (independent on the choice of $x \in X$) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ we can associate a $\delta > 0$ and an element $A \in \mathcal{I}$ (depending on φ and x) such that

$$|f_n(z) - f_n(x)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $z \in B(x,\delta)$ and $n \in \mathbb{N} \setminus A$.

d) A sequence of functions $f_n : X \to R$, $n \in \mathbb{N}$, is said to be weakly $\mathcal{I}$-exhaustive at $x \in X$ iff there is a (D)-sequence $(a_{i,l})_{i,l}$ (depending on x) with the property that for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there are a $\delta > 0$ (depending on φ and x) such that to every $z \in B(x,\delta)$ there corresponds a set $A \in \mathcal{I}$ (depending on φ, x and z) with

$$|f_n(z) - f_n(x)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for all $n \in \mathbb{N} \setminus A$.

e) Let X, R and $(f_n)_n$ be as in d). The sequence $(f_n)_n$ is called weakly $\mathcal{I}$-exhaustive on X iff it is weakly $\mathcal{I}$-exhaustive at x for every $x \in X$.

f) Let X, R and $(f_n)_n$ be as in d). The sequence $(f_n)_n$ is called globally weakly $\mathcal{I}$-exhaustive on X iff there exists a regulator $(a_{i,l})_{i,l}$ (independent on the choice of $x \in X$) such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ and $x \in X$ we can associate a $\delta > 0$ (depending on φ and x) such that to every $z \in B(x, \delta)$ there corresponds an $A \in \mathcal{I}$ (depending on φ, z and x) with

$$|f_n(z) - f_n(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

for any $n \in \mathbb{N} \setminus A$.

Remarks 2.19. a) Observe that in Definitions 2.18 c) and f) the regulator $(a_{i,l})_{i,l}$ is independent of the choice of $x \in X$ in contrast to Definitions 2.18 a), b), d) and e).

b) If X is a singleton, then Definitions 2.18 a), b) and c) [d), e) and f)] coincide.

c) If $\mathcal{I} = \mathcal{I}_{fin}$, the ideal of all finite subsets of $\mathbb{N}$, then Definitions 2.18 a), b), c) coincide with Definitions 2.11 a), b), c) respectively.

d) Note that, in general, weak $\mathcal{I}$-exhaustiveness is strictly weaker than $\mathcal{I}$-exhaustiveness (see also [4, Example 4.2.5 (2)], [9]).

3. Ideal Exhaustiveness and $(\mathcal{I}\alpha)$-Convergence

We begin with giving a characterization of continuity of the limit function in terms of weak exhaustiveness.

Theorem 3.1. *Assume that $f_n : X \to R$, $n \in \mathbb{N}$, $(D\mathcal{I})$-converges (pointwise) to $f : X \to R$ with respect to a same regulator $(a^*_{i,l})_{i,l}$ and fix $x_0 \in X$. Then the following are equivalent:*

(i) $(f_n)_n$ is weakly $\mathcal{I}$-exhaustive at x_0;

(ii) f is continuous at x_0.

Proof. (i) $\Longrightarrow$ (ii): Let $(a_{i,l})_{i,l}$ be a regulator associated to weak $\mathcal{I}$-exhaustiveness of f at x_0, and pick $\varphi \in \mathbb{N}^{\mathbb{N}}$. By hypothesis there exists a $\delta > 0$ satisfying condition of weak $\mathcal{I}$-exhaustiveness. Fix arbitrarily $z \in B(x_0, \delta)$: there is a set $A_1 \in \mathcal{I}$ with

$$|f_n(z) - f_n(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{2}$$

for any $n \in \mathbb{N} \setminus A_1$. Moreover there exists a set $A_2 \in \mathcal{I}$ with

$$|f_n(x_0) - f(x_0)| \vee |f_n(z) - f(z)| \leq \bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)}$$

whenever $n \in \mathbb{N} \setminus A_2$. So for each $n \in \mathbb{N} \setminus (A_1 \cup A_2)$ we have

$$\begin{aligned} |f(z) - f(x_0)| &\leq |f(z) - f_n(z)| + |f_n(z) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \\ &\leq 2 \bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}. \end{aligned}$$

Hence, in correspondence with φ there is a $\delta > 0$ such that for all $z \in B(x_0, \delta)$ we get

$$|f(z) - f(x_0)| \leq 2 \bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

Thus the assertion follows.

(ii) $\Longrightarrow$ (i): Since f is continuous at x_0, there exists a regulator $(b_{i,l})_{i,l}$ such that to every $\varphi \in \mathbb{N}^{\mathbb{N}}$ there corresponds a positive real number δ with

$$|f(z) - f(x_0)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \tag{3}$$

for all $z \in B(x_0, \delta)$. By $\mathcal{I}$-pointwise convergence of $(f_n)_n$ to f with respect to the regulator $(a^*_{i,l})_{i,l}$, in correspondence with x_0 and z there is a set $A^* \in \mathcal{I}$ with

$$|f_n(z) - f(z)| \vee |f_n(x_0) - f(x_0)| \leq \bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} \tag{4}$$

whenever $n \in \mathbb{N} \setminus A^*$. For every $n \notin A^*$, from (3) and (4) we get

$$\begin{aligned} 0 &\leq |f_n(z) - f_n(x_0)| \leq |f_n(z) - f(z)| + |f_n(x_0) - f(x_0)| + |f(z) - f(x_0)| \\ &\leq 2 \bigvee_{i=1}^{\infty} a^*_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}. \end{aligned} \tag{5}$$

By (5), we obtain the existence of a regulator $(c_{i,l})_{i,l}$, such that to every $z \in B(x_0, \delta)$ there is a set $A^* \in \mathcal{I}$ with

$$|f_n(z) - f(x_0)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}$$

for all $n \in \mathbb{N} \setminus A^*$. Thus, $(f_n)_n$ is weakly $\mathcal{I}$-exhaustive at x_0. This ends the proof. □

If we deal with global continuity, a version of Theorem 3.1, with a similar proof, can be formulated as follows:

Theorem 3.2. *Assume that $(f_n)_n$ $(D\mathcal{I})$-converges (pointwise) to f with respect to a same regulator. Then the following are equivalent:*
(i) $(f_n)_n$ is globally weakly $\mathcal{I}$-exhaustive;
(ii) f is globally continuous.

We now prove the following relation between $(\mathcal{I}\alpha)$-convergence and $\mathcal{I}$-exhaustiveness in the context of (D)-convergence. For the case $R = \mathbb{R}$, see [4, Theorem 2.6].

Theorem 3.3. *If $(f_n)_n$ is $\mathcal{I}$-convergent to f at x_0 and $(f_n)_n$ is $\mathcal{I}$-exhaustive at x_0, then $(f_n)_n$ is $(\mathcal{I}\alpha)$-convergent to f at x_0.*
Conversely, if $\mathcal{I}$ is good and $(f_n)_n$ is $(\mathcal{I}\alpha)$-convergent to f at x_0, then

$$(D\mathcal{I}) - \lim_n f_n(x_0) = f(x_0) \tag{6}$$

and $(f_n)_n$ is $\mathcal{I}$-exhaustive at x_0.

Proof. We begin with the first part. Assume that $\mathcal{I} - \lim_n x_n = x_0$. Let $(a_{i,l})_{i,l}$ be a regulator related with $\mathcal{I}$-exhaustiveness of f at x_0, and take $\varphi \in \mathbb{N}^{\mathbb{N}}$. Then there are $\delta > 0$ and $A_1 \in \mathcal{I}$ with the property that

$$|f_n(z) - f_n(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \tag{7}$$

whenever $z \in B(x_0, \delta)$ and $n \in \mathbb{N} \setminus A_1$.

By $\mathcal{I}$-convergence of $(f_n)_n$ to f at x_0 there are a regulator $(\alpha_{i,l})_{i,l}$ and $A_2 \in \mathcal{I}$ such that

$$|f_n(x_0) - f(x_0)| \leq \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \tag{8}$$

for any $n \in \mathbb{N} \setminus A_2$. As $\mathcal{I} - \lim_n x_n = x_0$, there is an element $A_3 \in \mathcal{I}$ with $d(x_n, x_0) < \delta$ whenever $n \in \mathbb{N} \setminus A_3$. From this, (7) and (8) it follows that for each $n \in \mathbb{N} \setminus (A_1 \cup A_2 \cup A_3)$ we get:

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f_n(x_0)| + |f_n(x_0) - f(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)}.$$

Thus the assertion of the first part follows.

We now turn to the second part. Property (6) is an immediate consequence of $(\mathcal{I}\alpha)$-convergence of $(f_n)_n$ to f at x_0.

Let now $(a_{i,l})_{i,l}$ and $(b_{i,l})_{i,l}$ be two regulators, related with $(\mathcal{I}\alpha)$-convergence of $(f_n)_n$ to f at x_0 and (6) respectively, and set $c_{i,l} = 2(a_{i,l} + b_{i,l})$, $i, l \in \mathbb{N}$. Let us show that $(c_{i,l})_{i,l}$ is the requested regulator for $\mathcal{I}$-exhaustiveness of $(f_n)_n$ at the point x_0.

Otherwise, there is $\varphi \in \mathbb{N}^{\mathbb{N}}$ with the property that in correspondence with any $k \in \mathbb{N}$ and $A \in \mathcal{I}$ there are $z \in B(x_0, 1/k)$ and $n \in \mathbb{N} \setminus A$ such that

$$|f_n(x_0) - f_n(z)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}. \tag{9}$$

In particular, (9) holds when $A = \emptyset$. Thus there exist $n_0^k \in \mathbb{N}$ and $z_0^k \in B(x_0, 1/k)$ with

$$|f_{n_0^k}(x_0) - f_{n_0^k}(z_0^k)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

Assume that we have chosen n_β^k and z_β^k for $\beta < \alpha$, where α is a not limit ordinal, and that $A_\alpha^k = \{n_\beta^k : \beta \leq \alpha\} \in \mathcal{I}$. Then, an integer $n_{\alpha+1}^k \notin A_\alpha^k$ and an element $z_{\alpha+1}^k \in B(x_0, 1/k)$ can be found, with

$$|f_{n_{\alpha+1}^k}(x_0) - f_{n_{\alpha+1}^k}(z_{\alpha+1}^k)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.$$

Set $A_{\alpha+1}^k = A_\alpha^k \cup \{n_{\alpha+1}^k\}$.

If we have chosen n_β^k and z_β^k for $\beta < \alpha$, where α is a limit ordinal, then define $A_\alpha^k = \cup_{\beta<\alpha} A_\beta^k$. Since there is a countable ordinal α_k with $A_{\alpha_k}^k \notin \mathcal{I}$, then our procedure ends, because the indexes n_α were chosen from a countable set N. It is not difficult to check that α_k turns out to be a limit ordinal. Set $P_k = A_{\alpha_k}^k$, $k \in \mathbb{N}$. Since the ideal $\mathcal{I}$ is good, then there is a disjoint sequence $(B_k)_k$ in $\mathcal{I}$ with $B_k \subset P_k$ for all $k \in \mathbb{N}$ and $\cup_{k=1}^\infty B_k \notin \mathcal{I}$.

Now for $n \in \mathbb{N}$ set $z_n = x_0$ if $n \notin \cup_{k=1}^\infty B_k$ and $z_n = z_\beta^k$ if $n \in B_k$, $n = n_\beta^k$ (note that the integer k is uniquely determined, since the B_k's are disjoint and β is unique, because the n_β^k's are different for different choices of β).

It is not difficult to check that $\mathcal{I} - \lim_n z_n = x_0$: indeed to every $\delta > 0$ there corresponds $k_0 = \left[\frac{1}{\delta}\right] + 1$ with $\{n \in \mathbb{N} : d(z_n, x_0) \geq \delta\} \subset \cup_{k=1}^{k_0} B_k$ and

$\cup_{k=1}^{k_0} B_k \in \mathcal{I}$. By (6) we get

$$B = \{n \in \mathbb{N} : |f_n(x_0) - f(x_0)| \not\leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}\} \in \mathcal{I}. \tag{10}$$

Since

$$\{n \in \mathbb{N} : |f_n(z_n) - f_n(x_0)| \not\leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}\} \supset \bigcup_{k=1}^{\infty} B_k, \tag{11}$$

then from (10) and (11) we obtain

$$\{n \in \mathbb{N} : |f_n(z_n) - f(x_0)| \not\leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}\} \supset \bigcup_{k=1}^{\infty} B_k \setminus B \notin \mathcal{I},$$

getting to a contradiction with $(\mathcal{I}\alpha)$-convergence. This proves the theorem. □

A "global" version analogous to Theorem 3.3 can be formulated as follows.

Theorem 3.4. *If $(f_n)_n$ is $\mathcal{I}$-convergent to f with respect to a same regulator and $(f_n)_n$ is globally $\mathcal{I}$-exhaustive, then $(f_n)_n$ is globally $(\mathcal{I}\alpha)$-convergent to f.*

Conversely, if $\mathcal{I}$ is good and $(f_n)_n$ is globally $(\mathcal{I}\alpha)$-convergent to f, then $(f_n)_n$ is pointwise $\mathcal{I}$-convergent to f with respect to a same regulator and $(f_n)_n$ is globally $\mathcal{I}$-exhaustive.

This result is a relation between even continuity and $\mathcal{I}$-exhaustiveness.

Theorem 3.5. *Let $\mathcal{I}$ be a fixed admissible ideal; f, $f_n \in R^X$, $n \in \mathbb{N}$. If the family $\Phi = \{f_n : n \in \mathbb{N}\}$ is evenly continuous at $x \in X$ [*globally evenly continuous*] and the sequence $(f_n)_n$ $\mathcal{I}$-converges pointwise to f [*with respect to a same regulator*], then Φ is $\mathcal{I}$-exhaustive at x [*globally exhaustive*].

Proof. Without loss of generality, assume that $f_n \neq f_m$ for $n \neq m$. Fix arbitrarily $x \in X$. Let $(b_{i,l})_{i,l}$ be a (D)-sequence, related with $\mathcal{I}$-convergence of the sequence $(f_n(x))_n$. In correspondence with the regulator $(b_{i,l})_{i,l}$, let $(a_{i,l})_{i,l}$ be a (D)-sequence associated with even continuity. Pick now $\varphi \in \mathbb{N}^{\mathbb{N}}$. Since Φ is evenly continuous at x, to $y = f(x)$ there correspond $\delta > 0$ and $\psi \in \mathbb{N}^{\mathbb{N}}$ such that for every choice of n with the property that $|f_n(x) - f(x)| \leq \bigvee_{i=1}^{\infty} b_{i,\psi(i)}$ and of z with $d(z, x) < \delta$ we get: $|f_n(z) - f(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$.

Since the sequence $(f_n(x))_n$ $\mathcal{I}$-converges to $f(x)$, then in correspondence with ψ there is $A_x \in \mathcal{I}$ such that $|f_n(x) - f(x)| \leq \bigvee_{i=1}^{\infty} b_{i,\psi(i)}$ for all $n \in \mathbb{N} \setminus A_x$.

Let $z \in B(x, \delta)$ and $n \in \mathbb{N} \setminus A_x$. Then by even continuity we get

$$|f_n(z) - f_n(x)| \leq |f_n(z) - f(x)| + |f_n(x) - f(x)| \leq 2 \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.$$

This concludes the proof. □

Corollary 3.6. *If a function sequence $f_n : X \to R$, $n \in \mathbb{N}$, is evenly continuous at every $x \in X$ and converges $\mathcal{I}$-pointwise to f with respect to a same regulator, then $(f_n)_n$ $(\mathcal{I}\alpha)$-converges to f and f is continuous.*

Proof. Corollary 3.6 is an easy consequence of Theorems 3.1 and 3.5. □

We now give the following

Example 3.7. Let $R = \mathbb{R}$, $f_n : (1, +\infty) \to (1, +\infty)$, $f_n(x) = x^n$, $\Phi = \{f_n : n \in \mathbb{N}\}$ and $x = 2$. For each $\varepsilon > 0$ and $y \in (1, +\infty)$ let $r = \varepsilon/2$. The set $\{f \in \Phi : f(2) \in S(y, r)\}$ is finite and from continuity of each f_n it is possible to find a positive real number δ to verify that Φ is evenly continuous at 2. But it is easy to see that Φ is not $\mathcal{I}$-exhaustive.

We now investigate some other properties, related with α-convergence, equicontinuity and even continuity in the context of (ℓ)-groups endowed with (D)-convergence.

Let Φ be a family of R^X and σ be a symbol for a (fixed) convergence. We associate to Φ the set Φ^σ of all σ-limits of sequences in Φ, which are not eventually constant. Moreover, if σ_j, $j = 1, 2$, are the symbols for two convergences and Φ is a family of elements of R^X, then define
$(\Phi^{\sigma_1})^{\sigma_2} = \{f \in R^X : \exists\, (f_n)_n \subset \Phi^{\sigma_1}$, not definitely constant, such that $(f_n)_n$ σ_2-converges to $f\}$. Note that the following facts hold.

a) In general, it is not true that $\Phi \subset \Phi^\sigma$. For example, let σ denote the global α-convergence and Φ be a family of not globally continuous functions. We get: $\Phi \cap \Phi^\sigma = \emptyset$. Indeed, if $f \in \Phi^\sigma$, then there exists a sequence $(f_n)_n$ of functions of Φ, globally (α)-convergent to f. By Theorem 3.4 used with $\mathcal{I} = \mathcal{I}_{fin}$, $(f_n)_n$ is pointwise convergent to f with respect to a same regulator. By Theorem 3.2, f is globally continuous. This proves $\Phi \cap \Phi^\sigma = \emptyset$ and hence $\Phi \not\subset \Phi^\sigma$.

b) If Φ_1, Φ_2 are any two families of functions from X to R, then $(\Phi_1 \cup \Phi_2)^\sigma = \Phi_1^\sigma \cup \Phi_2^\sigma$.

c) If the symbols α, pw denote the α-convergence and the pointwise convergence respectively, then there exists a family Φ of functions in R^X such that $(\Phi^\alpha)^{pw} \neq (\Phi^{pw})^\alpha$.

For example, let R be a Dedekind complete Riesz space and $u \in R$ be a strictly positive element of R. For $y \in \mathbb{R}$ and $n, k \geq 2$, set

$$f_n^y : [0,1] \to R \quad : \quad f_n^y(x) = \begin{cases} (-nyx + y)u, & \text{if } 0 \leq x \leq 1/n \\ 0, & \text{if } 1/n < x \leq 1 \end{cases}$$

$$f_{n,k}^y : [0,1] \to R \quad : \quad f_{n,k}^y(x) = \begin{cases} \left[-\left(\frac{nk}{n+k}\right) \cdot yx + y\right] u, & \\ & \text{if } 0 \leq x \leq 1/n + 1/k \\ 0, & \\ & \text{if } 1/n + 1/k < x \leq 1 \end{cases}$$

$$f_\infty^y : [0,1] \to R \quad : \quad f_\infty^y(x) = \begin{cases} y\,u, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1 \end{cases}$$

$$f^y : [0,1] \to R \quad : \quad f^y(x) = y\,u.$$

If $\Phi_1 = \{f_{n,k}^1 : n, k \geq 2\}$, then $(\Phi_1^{pw})^\alpha = \emptyset$ but $(\Phi_1^\alpha)^{pw} = \{f_\infty^1\}$. If $\Phi_2 = \{f_k^{1/(2n+1)} : k \geq 2, n \geq 1\} \cup \{f^{2n+1/k} : k \geq 2, n \geq 1\}$, then $(\Phi_2^{pw})^\alpha = \{f^0\}$ but $(\Phi_2^\alpha)^{pw} = \emptyset$. Finally, if we take $\Phi = \Phi_1 \cup \Phi_2$, then $(\Phi^{pw})^\alpha = \{f^0\}$ but $(\Phi^\alpha)^{pw} = \{f_\infty^1\}$.

We now prove the following relations between exhaustiveness, equicontinuity and even continuity.

Theorem 3.8. *Let Φ be a* [globally] *exhaustive family of functions in R^X, $\mathcal{I}$ be an admissible ideal and σ be a symbol for a convergence stronger than the $\mathcal{I}$-pointwise convergence with respect to a same regulator. Then Φ^σ is* [globally] *equicontinuous.*

Proof. Pick $x_0 \in X$ and fix arbitrarily $\varphi \in \mathbb{N}^\mathbb{N}$. Since Φ is exhaustive, there exist: a regulator $(a_{i,l})_{i,l}$, a finite element A and a positive real number δ such that, if $x \in B(x_0, \delta)$ and $f \in \Phi \setminus A$, then $|f(x) - f(x_0)| \leq \bigvee_{i=1}^\infty a_{i,\varphi(i)}$. From this it follows that the set $A' = \{n \in \mathbb{N} : f_n \in A\}$ is finite, and hence $A' \in \mathcal{I}$.

Let now $\phi \in \Phi^\sigma$. Then there is a sequence $(f_n)_n$ in Φ such that $f_n \neq f_m$ whenever $n, m \in \mathbb{N}$, $n \neq m$, and $(f_n)_n$ σ-converges to ϕ. By hypothesis it follows that the sequence $(f_n)_n$ $\mathcal{I}$-converges pointwise to ϕ with respect to a same regulator, say $(b_{i,l})_{i,l}$. So in correspondence with $x \in X$ there exists $A_x \in \mathcal{I}$ such that $|f_n(x) - \phi(x)| \leq \bigvee_{i=1}^\infty b_{i,\varphi(i)}$ for every $n \in \mathbb{N} \setminus A_x$. Analogously there exists $A_{x_0} \in \mathcal{I}$ with $|f_n(x_0) - \phi(x_0)| \leq \bigvee_{i=1}^\infty b_{i,\varphi(i)}$ for every $n \in \mathbb{N} \setminus A_{x_0}$. Note that $A_x \cup A_{x_0} \cup A' \in \mathcal{I}$, and thus $A_x \cup A_{x_0} \cup A' \neq \mathbb{N}$, since $\mathcal{I}$ is an admissible ideal. If $n \in \mathbb{N} \setminus (A_x \cup A_{x_0} \cup A')$, then

$$|\phi(x) - \phi(x_0)| \leq |\phi(x) - f_n(x)| + |f_n(x) - f_n(x_0)| +$$

$$+|f_n(x_0) - \phi(x_0)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} + 2 \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}.$$

Thus the assertion follows. □

Open Problem. With the same notations as above, let $\sigma_i, = 1, 2$ be two symbols of convergence. A function f is said to be *σ_1-σ_2-continuous at $x \in X$ iff* $(\sigma_2) \lim_n f(x_n) = f(x)$ *whenever* $(\sigma_1) \lim_n x_n = x$. *A function f is called* σ_1-σ_2-continuous on X iff f is σ_1-σ_2-continuous at every $x \in X$.

A family Φ of functions is *σ_1-σ_2-equicontinuous at $x \in X$ iff* $(\sigma_2) \lim_n f(x_n) = f(x)$ *uniformly in $f \in \Phi$ whenever* $(\sigma_1) \lim_n x_n = x$. *A family Φ is called* σ_1-σ_2-equicontinuous on X iff f is σ_1-σ_2-equicontinuous at every $x \in X$.

Find some results similar to the ones given above when the concepts of continuity and equicontinuity are replaced with the corresponding ones of σ_1-σ_2-continuity and σ_1-σ_2-equicontinuity respectively.

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