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OSCILLATION CRITERIA FOR SECOND ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS WITH POSITIVE AND NEGATIVE COEFFICIENTS

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Abstract: In this paper, the authors established some sufficient conditions for the oscillation of all solutions the second order nonlinear neutral delay differential equations with positive and negative coefficients

$$\Big[r(t)[x(t) \pm c(t)h(x(t-\tau))]'\Big]' + p(t)f(x(t-\delta)) - q(t)g(x(t-\sigma)) = 0.$$

They also extended these results to certain types of forced equations. Examples illustrating the main results are given.

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1. Introduction

In this paper, we consider second order nonlinear neutral delay differential equations of the form

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$$\Big[r(t)[x(t)+c(t)h(x(t-\tau))]'\Big]'+p(t)f(x(t-\delta))-q(t)g(x(t-\sigma))=0 \quad (1.1)$$

and

$$\Big[r(t)[x(t)-c(t)h(x(t-\tau))]'\Big]'+p(t)f(x(t-\delta))-q(t)g(x(t-\sigma))=0, \quad (1.2)$$

for $t>0$, subject to the following conditions:

(B_0) $f,\ g,\ h\in C(\mathbb{R},\mathbb{R})$;

(B_1) τ, δ, and σ are nonnegative constants with $\delta\geq\sigma\geq\tau$;

(B_2) $r\in C^2([0,\infty);(0,\infty))$ and $c,\ p,\ q\in C([0,\infty);[0,\infty))$;

(B_3) $\int_{t_0}^{\infty}\frac{1}{r(t)}dt=\infty$ for some $t_0>0$;

(B_4) there exist $\alpha\geq 1$ that is the ratio of odd positive integers and a positive constant M_1 such that $\frac{f(u)}{u^{\alpha}}\geq M_1$ for $u\neq 0$;

(B_5) there are positive constants M_2, M, N_1, and N_2 such that $0\leq\frac{g(u)}{u}\leq M_2$, $0<\frac{g(u)}{f(u)}\leq M$, and $N_1\leq\frac{h(u)}{u}\leq N_2$ for $u\neq 0$;

(B_6) there is a constant k such that $p(t)-Mq(t-\delta+\sigma)\geq k>0$ for all $t\geq t_0>0$.

By a *solution* of equation (1.1) (or (1.2)), we mean a continuous function $x(t)$ defined for $t\geq t_0-\delta$ with $\sup\{|x(t)|:t\geq t_1\}>0$ for all $t_1\geq t_0>0$ and satisfying equation (1.1) (or (1.2)).

In [9], Parhi and Chand considered equations (1.1) and (1.2) with $r(t)\equiv 1$ and $f\equiv g$ and obtained conditions for all bounded solutions of equations (1.1) and (1.2) to be oscillatory. Manojlovic et al. [6] considered the equations (1.1) and (1.2) with $f(u)=g(u)=h(u)=u$ and obtained sufficient conditions for every solution to be oscillatory. Yildiz et al. [12] gave sufficient conditions for the oscillation of all solutions of (1.1) and (1.2) with $r(t)\equiv 1$ and $f\equiv g$.

Motivated by these results, in this paper we establish sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2) without these types of restrictions. Our results improve and generalize some of the results of [5, 7, 10], and the references cited there in.

In Section 2, we present our main results for equations (1.1) and (1.2), in Section 3 we extend these results to certain types of forced equations, and in Section 4 we present some examples to illustrate our theorems.

2. Oscillation Results

In this section, we obtain oscillation criteria for equations (1.1) and (1.2). We begin with the following lemma given in [4]. Its proof, which is simple, is given for the sake of completeness.

Lemma 2.1. *If a_1 and a_2 are nonnegative and $\alpha \geq 1$, then $(a_1 + a_2)^\alpha \leq 2^{\alpha-1}({a_1}^\alpha + {a_2}^\alpha)$.*

Proof. From Hölder's inequality, we have

$$\sum_{i=1}^{n} a_i b_i \leq \Big(\sum_{i=1}^{n} {a_i}^\alpha\Big)^{\frac{1}{\alpha}} \Big(\sum_{i=1}^{n} {b_i}^\beta\Big)^{\frac{1}{\beta}} \quad \text{where} \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1 \quad \text{and} \quad \alpha \geq 1.$$

That is,

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^\alpha \leq \Big(\sum_{i=1}^{n} {a_i}^\alpha\Big) \Big(\sum_{i=1}^{n} {b_i}^\beta\Big)^{\alpha-1}.$$

If we take $n = 2$ and $b_1 = b_2 = 1$, we obtain the result. □

Theorem 2.2. *Assume that there are constants c_1 and c_2 such that $0 \leq c_1 \leq c(t) \leq c_2$. If*

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds < \infty, \tag{2.1}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq t_1 \geq t_0 + \delta > 0$. The proof in case $x(t) < 0$ eventually is similar and is omitted. Choose $T \geq t_1$ so that

$$\int_{T}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds \leq \frac{N_1 c_1}{M_2}. \tag{2.2}$$

Define

$$z(t) = x(t) + c(t)h(x(t-\tau)) - \int_{T}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) g(x(\xi-\sigma)) d\xi ds; \tag{2.3}$$

then from equation (1.1) and conditions (B_4) and (B_5), we have that

$$(r(t)z'(t))' \leq -p(t)f(x(t-\delta)) + q(t-\delta+\sigma)g(x(t-\delta))$$

$$\begin{aligned} &\le \left[-p(t) + q(t-\delta+\sigma)\frac{g(x(t-\delta))}{f(x(t-\delta))}\right] f(x(t-\delta)) \\ &\le -M_1[p(t) - Mq(t-\delta+\sigma)]x^\alpha(t-\delta) \\ &\le -kM_1 x^\alpha(t-\delta) \le 0 \end{aligned} \tag{2.4}$$

for $t \ge T$. Hence $r(t)z'(t)$ is eventually nonincreasing so either $z'(t) < 0$ or $z'(t) \ge 0$ for $t \ge T_1$ for some $T_1 \ge T$.

If $z'(t) < 0$ for all $t \ge T_1$, then (2.4) and (B_3) imply $\lim_{t\to\infty} z(t) = -\infty$. We claim that $x(t)$ is bounded from above. If this is not the case, then there exists a $T_2 \ge T_1 + \tau$ such that

$$z(T_2) < 0 \quad \text{and} \quad \max_{T_1 \le t \le T_2 - \tau} x(t) = x(T_2 - \tau). \tag{2.5}$$

Hence, from (2.3),

$$\begin{aligned} 0 > z(T_2) &= x(T_2) + c(T_2)h(x(T_2-\tau)) - \int_{t_0}^{T_2} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)g(x(\xi-\sigma))d\xi ds \\ &\ge N_1 c_1 x(T_2-\tau) - M_2 x(T_2-\tau) \int_{t_0}^{T_2} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds \\ &\ge \left[N_1 c_1 - M_2 \int_{t_0}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\right] x(T_2-\tau) \ge 0. \end{aligned}$$

This contradiction shows that $x(t)$ must be bounded so there exists a constant $L > 0$ such that $x(t) \le L$ for $t \ge T_1$. It follows from (2.3) that

$$z(t) \ge -LM_2 \int_{T_1}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds \ge -LN_1 c_1 > -\infty,$$

which contradicts the fact that $\lim_{t\to\infty} z(t) = -\infty$. Therefore have $z'(t) \ge 0$ for $t \ge T_1$. Now, integrating (2.4) for $t \ge T_1$, we obtain

$$\infty > r(T_1)z'(T_1) \ge r(T_1)z'(T_1) - r(t)z'(t) \ge kM_1 \int_{T_1}^{t} x^\alpha(s-\delta)ds$$

and therefore $x^\alpha(t) \in L^1([t_1,\infty))$. Then, by Lemma 2.1 and (B_5), we have

$$X^\alpha(t) = (x(t) + c(t)h(x(t-\tau)))^\alpha \le 2^{\alpha-1}\Big(x^\alpha(t) + N_2^\alpha c_2^\alpha x^\alpha(t-\tau)\Big),$$

so

$$X^\alpha(t) \in L^1([t_1,\infty)). \tag{2.6}$$

On the other hand, from equation (2.3), we obtain

$$X'(t) \quad = z'(t) + \tfrac{1}{r(t)} \int_{t-\delta+\sigma}^{t} q(s)g(x(s-\sigma))ds \geq 0$$

so that $X(t)$ is nondecreasing. But then $X^{\alpha}(t) \geq X^{\alpha}(t_1)$ for $t \geq T_1$ implies that $X^{\alpha}(t) \notin L^1([T_1, \infty))$ contradicting (2.6). Therefore, every solution of equation (1.1) is oscillatory. □

Next, we establish an oscillation result for equation (1.2).

Theorem 2.3. *Assume that there is a constant c_3 such that $0 \leq c(t) \leq c_3 < \frac{1}{N_2}$. If*

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds < \infty, \tag{2.7}$$

then any solution $x(t)$ of equation (1.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (1.2), say $x(t) > 0$ for $t \geq t_1 \geq t_0 + \delta > 0$. Choose $T \geq t_1$ such that

$$\int_{T}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds \leq \frac{N_2 c_3}{M_2}. \tag{2.8}$$

Define

$$w(t) = x(t) - c(t)h(x(t-\tau)) - \int_{T}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)g(x(\xi-\sigma))d\xi ds. \tag{2.9}$$

Then analogous to (2.4) in the proof of Theorem 2.2, we obtain

$$(r(t)w'(t))' \leq -kM_1 x^{\alpha}(t-\delta) \leq 0 \tag{2.10}$$

for $t \geq T$, and conclude that $r(t)w'(t)$ is eventually nonincreasing. Therefore, $w'(t) < 0$ or $w'(t) \geq 0$ for all $t \geq T_1 \geq T$.

Assume that $w'(t) < 0$ for $t \geq T_1$; then $\lim_{t\to\infty} w(t) = -\infty$. We claim that $x(t)$ is bounded from above. If it is not the case, there exists a number $T_2 \geq T_1 + \tau$ such that

$$w(T_2) < 0 \quad \text{and} \quad \max_{T_1 \leq t \leq T_2 - \tau} x(t) = x(T_2 - \tau) \tag{2.11}$$

and we have

$$0 > w(T_2) = x(T_2) - c(T_2)h(x(T_2-\tau)) - \int_{T}^{T_2} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)g(x(\xi-\sigma))d\xi ds$$

$$\geq \Big[-N_2c_3 - M_2\int_T^{T_2}\frac{1}{r(s)}\int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\Big]x(T_2-\tau) \geq 0.$$

This contradiction shows that $x(t)$ must be bounded from above, so there exists a constant $L > 0$ such that $x(t) \leq L$ for $t \geq T_1$. It follows from (2.9) and (2.8) that

$$w(t) \geq -L\Big[N_2c_3 + M_2\int_T^{t}\frac{1}{r(s)}\int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\Big] > -\infty$$

which contradicts the fact that $\lim_{t\to\infty} w(t) = -\infty$. Hence $w'(t)$ must be eventually positive.

An integration of inequality (2.10) for $t \geq T_1$ shows that $x^\alpha(t) \in L^1([T_1,\infty))$. Define

$$y(t) = x(t) - c(t)h(x(t-\tau)), \quad \text{for} \quad t \geq T_1. \tag{2.12}$$

From (2.9), we see that

$$y'(t) = w'(t) + \frac{1}{r(t)}\int_{t-\delta+\sigma}^{t} q(s)g(x(s-\sigma))ds \geq 0,$$

that is, $y(t)$ is a nondecreasing function, so

$$\lim_{t\to\infty} y(t) = \mu \quad \text{where} \quad -\infty < \mu \leq \infty.$$

This implies that $\lim_{t\to\infty} y^\alpha(t) = \mu^\alpha$ where $-\infty < \mu^\alpha \leq \infty$. Since $x^\alpha(t) \in L^1([T_1,\infty))$ and $x(t) > y(t)$, we have $\mu^\alpha \in (-\infty,\infty)$.

If $0 < \mu^\alpha < \infty$, then for $0 < \epsilon < \mu^\alpha$, there exist a $T_3 \geq T_2$ such that

$$x^\alpha(t) \geq y^\alpha(t) > \mu^\alpha - \epsilon \quad \text{for} \quad t \geq T_2.$$

But this implies that $x^\alpha(t) \notin L^1([t_1,\infty))$, which is a contradiction.

If $-\infty < \mu^\alpha < 0$, then for any $0 < \epsilon < -\mu^\alpha$, there exist a $T_3 \geq T_2$ such that

$$y^\alpha(t) < \mu^\alpha + \epsilon < 0 \quad \text{for} \quad t \geq T_3.$$

From (2.12), we have

$$x^\alpha(t-\tau) > -(\mu^\alpha + \epsilon).$$

But this implies that $x^\alpha(t-\tau) \notin L^1([T_3,\infty))$, which contradicts the fact that $x^\alpha(t) \in L^1([T_3,\infty))$.

Finally, if $\mu = 0$, then we claim that $x(t)$ is bounded from above. If this is not the case, then there exists an increasing sequence $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \max_{t_1\leq t\leq t_n} x(t) = x(t_n), \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = \infty.$$

Then, we see that

$$y(t_n) \geq (1 - c_3 N_2) x(t_n)$$

and taking the limit as $n \to \infty$, we obtain

$$0 \geq (1 - c_3 N_2) \lim_{n \to \infty} x(t_n) = \infty,$$

which is a contradiction. Hence, $x(t)$ is bounded from above.

We then have

$$y(t) \geq x(t) - c_3 N_2 x(t - \tau).$$

Hence,

$$\begin{aligned} 0 &\geq \limsup_{t \to \infty} \Big[x(t) - c_3 N_2 x(t - \tau) \Big] \\ &\geq \limsup_{t \to \infty} x(t) + \liminf_{t \to \infty} \Big[- c_3 N_2 x(t - \tau) \Big] \\ &\geq \limsup_{t \to \infty} x(t) - c_3 N_2 \limsup_{t \to \infty} x(t - \tau) \\ &\geq (1 - c_3 N_2) \limsup_{t \to \infty} x(t). \end{aligned}$$

Therefore, $\lim_{t \to \infty} x(t) = 0$, and this completes the proof of the theorem. □

3. Forced Equations

Next we consider equations (1.1) and (1.2) with forcing terms

$$\Big[r(t)[x(t) + c(t)h(x(t - \tau))]' \Big]' + p(t) f(x(t - \delta)) - q(t) g(x(t - \sigma)) = e(t), \quad (3.1)$$

$$\Big[r(t)[x(t) - c(t)h(x(t - \tau))]' \Big]' + p(t) f(x(t - \delta)) - q(t) g(x(t - \sigma)) = e(t), \quad (3.2)$$

for $t > 0$, where $e(t) \in C([0, \infty); \mathbb{R})$. We will assume that there is a function $E \in C^1([0, \infty); \mathbb{R})$ such that

(B_8) $r(t)E'(t) \in C^1([0, \infty); \mathbb{R})$, $\Big[r(t)E'(t) \Big]' = e(t)$, and $\lim_{t \to \infty} E(t) = 0$.

Theorem 3.1. *Assume that there are constants c_4 and c_5 such that $0 \le c_4 \le c(t) \le c_5$ and*

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds < \infty. \tag{3.3}$$

Then every solution $x(t)$ of equation (3.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of equation (3.1), say $x(t) > 0$ for all $t \ge T_1 \ge t_0 + \delta > 0$. Choose $T \ge T_1$ so that

$$\int_{T}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds \le \frac{N_1 c_4}{M_2}. \tag{3.4}$$

Define

$$Z(t) = z(t) - E(t), \tag{3.5}$$

where $z(t)$ is given in (2.3). Then, analogous to (2.4), we obtain

$$(r(t)Z'(t))' \le -kM_1 x^{\alpha}(t-\delta) < 0 \tag{3.6}$$

for $t \ge T$. Thus, $r(t)Z'(t)$ is eventually nonincreasing so there exists $T_1 \ge T + \delta$ such that $Z'(t) < 0$ or $Z'(t) \ge 0$ for $t \ge T_1$.

First assume that $Z'(t) < 0$ for $t \ge T_1$. Then $\lim_{t\to\infty} Z(t) = -\infty$. If $x(t)$ is unbounded from above, then there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ satisfying $t_1 \ge T_1$,

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{t\to\infty} Z(t_n) = -\infty, \quad \lim_{n\to\infty} E(t_n) = 0,$$

$$\max_{T_1 \le t \le t_n - \tau} x(t) = x(t_n - \tau), \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = \infty.$$

Then, we have

$$\begin{aligned} Z(t_n) &= x(t_n) + c(t_n)h(x(t_n - \tau)) - \int_{T_1}^{t_n} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) g(x(\xi - \sigma)) d\xi ds \\ &\quad - E(t_n) \\ &\ge \Big[N_1 c_4 - M_2 \int_{T_1}^{t_n} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds\Big] x(t_n - \tau) - E(t_n) \\ &\ge \Big[N_1 c_4 - M_2 \int_{T_1}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds\Big] x(t_n - \tau) - E(t_n), \end{aligned}$$

and taking the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} Z(t_n) \geq \Big[N_1 c_4 - M_2 \int_{T_1}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\Big] \lim_{n\to\infty} x(t_n - \tau) = \infty,$$

which contradicts the fact that $\lim_{t\to\infty} Z(t) = -\infty$. Hence, $x(t)$ is bounded from above, so there exists a constant $L > 0$ such that $x(t) \leq L$ for $t \geq T_1$. From (3.5) we have

$$Z(t) \geq -LM_2 \int_{T}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds - E(t) > -\infty$$

contradicting the fact that $\lim_{t\to\infty} Z(t) = -\infty$.

Now assume that $Z'(t) \geq 0$ for $t \geq T_1$. Integrating (3.6) for $t \geq T_1$, we obtain

$$\infty > r(T_1)Z'(T_1) \geq r(T_1)Z'(T_1) - r(t)Z'(t) \geq kM_1 \int_{T_1}^{t} x^{\alpha}(s-\delta)ds$$

which implies that $x^{\alpha}(t) \in L^1([T_1, \infty))$ Let $X(t) = x(t) + c(t)h(x(t-\tau))$. Then by Lemma 2.1, we have

$$X^{\alpha}(t) \leq 2^{\alpha-1}\Big(x^{\alpha}(t) + N_2^{\alpha} c_5^{\alpha} x^{\alpha}(t-\tau)\Big)$$

so $X^{\alpha}(t) \in L^1([T_1, \infty))$. If we set

$$\phi(t) = X(t) - E(t) \quad \text{for} \quad t \geq T_1, \tag{3.7}$$

then from (3.6), we obtain

$$\phi'(t) = Z'(t) + \frac{1}{r(t)} \int_{t-\delta+\sigma}^{t} q(s)g(x(s-\sigma))ds \geq 0$$

so that $\phi(t)$ is nondecreasing. Therefore,

$$\lim_{t\to\infty} \phi(t) = \lim_{t\to\infty} X(t) = \mu,$$

where $0 \leq \mu \leq \infty$.

If $0 < \mu \leq \infty$, then $X(t)$ is bounded from below and so $X^{\alpha}(t)$ can not belong to $L^1([T_1, \infty))$ which is a contradiction. If $\mu = 0$, then since $x(t) \leq X(t)$, we have that $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the theorem. □

Our final result is an oscillation criteria for equation (3.2).

Theorem 3.2. *Assume that there is a constant c_6 such that $0 \le c(t) \le c_6 < \frac{1}{N_2}$ and*

$$\int_{t_0}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds < \infty. \tag{3.8}$$

Then every solution $x(t)$ of equation (3.2) is either oscillatory or satisfies $\lim\limits_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a nonoscillatory solution of (3.2), say $x(t) > 0$ for $t \ge T \ge t_0$. Choose $T_1 \ge T$ such that

$$\int_{T_1}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds < \frac{1 - N_2 c_6}{M_2} \tag{3.9}$$

and set

$$W(t) = w(t) - E(t) \tag{3.10}$$

where $w(t)$ is defined in (2.9). Then, as before,

$$(r(t)W'(t))' \le -kM_1 x^{\alpha}(t-\delta) < 0 \quad \text{for} \quad t \ge T_2 \tag{3.11}$$

for some $T_2 \ge T_1$, and we have two cases to consider.

If $W'(t) < 0$ for $t \ge T_3 \ge T_2$, then $\lim_{t\to\infty} W(t) = -\infty$. If $x(t)$ is not bounded from above, then there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_1 \ge T_3$,

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{t\to\infty} W(t_n) = -\infty, \quad \lim_{n\to\infty} E(t_n) = 0,$$

$$\max_{T_3 \le t \le t_n} x(t) = x(t_n), \quad \text{and} \quad \lim_{n\to\infty} x(t_n) = \infty.$$

Now

$$\begin{aligned}
W(t_n) &= x(t_n) - c(t_n)h(x(t_n - \tau)) - \int_{T_3}^{t_n} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) g(x(\xi - \sigma)) d\xi ds \\
&\quad - E(t_n) \\
&\ge x(t_n) - N_2 c_6 x(t_n - \tau) - M_2 \int_{T_3}^{t_n} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) x(\xi - \sigma) d\xi ds - E(t_n) \\
&\ge \Big[1 - N_2 c_6 - M_2 \int_{T_3}^{t_n} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds\Big] x(t_n) - E(t_n) \\
&\ge \Big[1 - N_2 c_6 - M_2 \int_{T_3}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi) d\xi ds\Big] x(t_n) - E(t_n),
\end{aligned}$$

so

$$\lim_{n\to\infty} W(t_n) \ge \Big[1 - N_2c_6 - M_2 \int_{T_3}^{\infty} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\Big] \lim_{n\to\infty} x(t_n) = \infty,$$

which is a contradiction. Therefore $x(t)$ must be bounded from above and so there exists a constant $L > 0$ such that $x(t) \le L$ for $t \ge T_3$. It follows from (3.10) that

$$\begin{aligned} W(t) &\ge -c(t)h(x(t-\tau)) - \int_{t_0}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)g(x(\xi-\sigma))d\xi ds - E(t) \\ &\ge -L\Big[N_2c_6 + M_2 \int_{T_3}^{t} \frac{1}{r(s)} \int_{s-\delta+\sigma}^{s} q(\xi)d\xi ds\Big] - E(t) > -\infty, \end{aligned}$$

which contradicts the fact that $\lim_{t\to\infty} W(t) = -\infty$.

If $W'(t) \ge 0$ for $t \ge T_3$, then from (3), we can obtain that $x^{\alpha}(t) \in L^1([t_1,\infty))$. Let

$$X_1(t) = x(t) - c(t)h(x(t-\tau) \quad \text{and} \quad \psi(t) = X_1(t) - E(t).$$

Then,

$$\begin{aligned} \psi'(t) &= X_1'(t) - E'(t) \\ &= W'(t) + \frac{1}{r(t)} \int_{t-\delta+\sigma}^{t} q(s)g(x(s-\sigma))ds \ge 0 \end{aligned}$$

for $t \ge T_1$, so $\psi(t)$ is nondecreasing. Hence,

$$\lim_{t\to\infty} \psi(t) = \lim_{t\to\infty} X_1(t) = \lim_{t\to\infty} \Big[x(t) - c(t)x(t-\tau)\Big] = \mu$$

where $\mu \in (-\infty,\infty]$. This implies that

$$\lim_{t\to\infty} \psi^{\alpha}(t) = \mu^{\alpha}$$

where $\mu^{\alpha} \in (-\infty,\infty]$. Since $x(t) \ge X_1(t)$ and $x^{\alpha}(t) \in L^1([t_1,\infty))$, we have $\mu^{\alpha} \in (-\infty,\infty)$. As in the proof of Theorem 2.3, we obtain a contradiction if $\mu \ne 0$, and if $\mu = 0$, we find that $\lim_{t\to\infty} x(t) = 0$. Thus, the conclusion of the theorem holds. □

4. Examples

In this section, we present some examples to illustrate the results obtained in the previous section.

Example 4.1. Consider the equation

$$\Big[e^{-t}\Big(x(t)+\frac{5}{2}\frac{x(t-\pi)(1+x^2(t-\pi))}{2+x^2(t-\pi)}\Big)'\Big]' + 8e^{2t}x^3(t-2\pi)(1+x^2(t-2\pi)) \\ - 4e^{-2(t+\pi)}\frac{x^3(t-\pi)}{1+x^2(t-\pi)} = 0, \quad t \geq 0. \quad (4.1)$$

Here we have $\tau = \pi$, $\delta = 2\pi, \sigma = \pi$, $r(t) = e^{-t}$, $c(t) = \frac{5}{2}$, $p(t) = 8e^{2t}$, $q(t) = 4e^{-2(t+\pi)}$, $h(u) = \frac{u(1+u^2)}{2+u^2}$, $f(u) = u^3(1+u^2)$, and $g(u) = \frac{u^3}{1+u^2}$. We see that $\int_0^\infty \frac{1}{r(t)}dt = \int_0^\infty e^t dt = \infty$; and for $u \neq 0$ we have $\frac{f(u)}{u^3} > 1 = M_1$, $\frac{g(u)}{u} < 1 = M_2$, $\frac{g(u)}{f(u)} < 1 = M$, and $N_1 = \frac{1}{2} \leq \frac{h(u)}{u} \leq 1 = N_2$. Also, $p(t) - Mq(t-\delta+\sigma) = 8e^{2t} - 4e^{-2t} \geq 4 > 0$ and

$$\int_0^\infty \frac{1}{r(s)}\int_{s-\delta+\sigma}^s q(\xi)d\xi ds = \frac{2(e^{2\pi}-1)}{e^{2\pi}} < \infty.$$

Therefore, by Theorem 2.2, all solutions of equation (4.1) are oscillatory.

Example 4.2. Consider the equation

$$\Big[e^{-t}\Big(x(t)-\frac{1}{e}\frac{x(t-\pi)(1+|x(t-\pi)|)}{2+|x(t-\pi)|}\Big)'\Big]' + \Big(1+\frac{1}{e}\Big)x^5(t-2\pi)(1+|x(t-2\pi)|) \\ - e^{-2t-2\pi}\frac{x^5(t-\pi)}{(1+|x(t-\pi)|^4)} = 0, \quad t \geq 0. \quad (4.2)$$

We have $\tau = \pi$, $\delta = 2\pi$, $\sigma = \pi$, $r(t) = e^{-t}$, $c(t) = \frac{1}{e}$, $p(t) = 1+\frac{1}{e}$, $q(t) = e^{-2t-2\pi}$, $h(u) = \frac{u(1+|u|)}{2+|u|}$, $f(u) = u^5(1+|u|)$, and $g(u) = \frac{u^5}{1+|u|^4}$. Clearly, $\int_0^\infty \frac{1}{r(t)}dt = \int_0^\infty e^t dt = \infty$; for $u \neq 0$, $\frac{f(u)}{u^5} > 1 = M_1$, $\frac{g(u)}{u} < 1 = M_2$, $\frac{g(u)}{f(u)} < 1 = M$, and $N_1 = \frac{1}{2} \leq \frac{h(u)}{u} \leq 1 = N_2$. In addition, $p(t) - Mq(t-\delta+\sigma) = 1+\frac{1}{e}-e^{-2t} \geq \frac{1}{e} > 0$ and

$$\int_0^\infty \frac{1}{r(s)}\int_{s-\delta+\sigma}^s q(\xi)d\xi ds = \frac{1}{2}(1-e^{-2\pi}) < \infty.$$

Therefore, according to Theorem 2.3, any solution $x(t)$ of equation (4.2) is either oscillatory or satisfies $\lim_{t\to\infty} x(t) = 0$.

Remark 4.3. It appears that no existing results applied to equations (4.1) and (4.2) will yield these conclusions.

References

[1] R.P. Agarwal, M. Bohner, W.T. Li, *Nonoscillation and Oscillation: Theory of Functional Differential Equations*, Marcel Dekker, New York (2004).

[2] R.P. Agarwal, S.R. Grace, D. O'Regan, *Oscillation Theory for Second Order Dynamic Equations*, Taylor and Francis, London (2003).

[3] L.H. Erbe, Q. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations,* Marcel Dekker, New York (1995).

[4] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, London (1934).

[5] J. Manojlovic, Y. Shoukaku, T. Tanigawa, N. Yoshida, Oscillation criteria for second order differential equations with positive and negative coefficients, *Appl. Math. Comput.*, **181** (2006), 853-863.

[6] J. Manojlovic, Y.Shoukaku, T. Tanigawa, N. Yoshida, Oscillatory behavior of second order neutral differential equations with positive and negative coefficients, *Appl. Appl. Math.* **3** (2008), 1-17.

[7] S. Padhi, Oscillation and asymptotic behaviour of solutions of second order neutral differential equations with positive and negative coefficients, *Fasc. Math.*, **38** (2007), 105-114.

[8] S. Padhi, Oscillation and asymptotic behaviour of solutions of second order homogeneous neutral differential equations with positive and negative coefficients, *Funct. Differ. Eqn.*, **14** (2007), 363-371.

[9] N. Parhi, S. Chand, Oscillation of second order neutral differential equations with positive and negative coefficients, *J. Ind. Math. Soc.*, **66** (1999), 227-235.

[10] A. Weng, J. Sun, Oscillation of second order delay differential equations, *Appl. Math. Comp.*, **198** (2008), 930-935.

[11] A. Yang, Z. Zhang, W. Ge, Existence of nonoscillatory solutions of second-order nonlinear neutral differential equations, *Indian J. Pure Appl. Math.*, **39** (2008), 227-235.

[12] M.K. Yildiz, B. Karpuz, O. Ocalan, Oscillation of nonlinear neutral delay differential equations of second-order with positive and negative coefficients, *Turk J. Math.*, **33** (2009), 341-350.

[13] Z. Zhang, A.Yang, C. Di, Existence of positive solutions of second-order nonlinear neutral differential equations with positive and negative terms, *J. Appl. Math. Comp.*, **25** (2007), 245-253.

[14] Y. Zhou, B.G. Zhang, Existence of nonoscillatory solutions of higher order neutral differential equations with positive and negative coefficients, *Appl. Math. Lett.*, **15** (2002), 867-874.