CHARACTERISATION OF PASTING SYSTEM USING ARRAY TOKEN PETRI NETS

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Abstract: Pasting system has been introduced in [3]. We characterize this pasting system through array token Petri nets [4, 1]. In addition we study some properties of array token Petri nets.

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1. Petri Nets and Example

Definition 1.1. A Petri net structure [5] is a four tuple $C = (P, T, I, O)$ where $P = \{p_1, p_2, \ldots, p_n\}$ is a finite set of places, $n > 0$, $T = \{t_1, t_2, \ldots, t_m\}$ is a finite set of transitions $m > 0$, $P \cap T = \emptyset$, $I : T \rightarrow P^\infty$ is the input function from transitions to bags of places and $O : T \rightarrow P^\infty$ is the output function from transitions to bags of places.

Definition 1.2. A Petri net marking is an assignment of tokens to the places of a Petri net. The tokens are used to define the execution of a Petri net. The number and position of tokens may change during the execution of a Petri net. The marking can be defined as an $n$-vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ where $\mu_i$ = no of tokens in $p_i$, $i = 1, 2, \ldots, n$. We can also write $\mu(p_i) = \mu_i$. 
Definition 1.3. A Petri net $C$ with initial marking $\mu$ is called a market Petri net. A marked Petri net $M = (C, \mu)$ can also be written as $M = (P, T, I, O, \mu)$.

When a transition is fired one token is removed from its input place and one token is placed in each of its output place. For example when $t_1$ is fired in the following figure one token from place $A$ is removed and one token is placed in both $B$ & $C$ which are the output places of $t_1$.

2. Triangular Pasting System

Triangular pasting system is introduced in [3]. Let $A, B, C, D$ be the four types of isosceles right angled triangle with dimensions $1/\sqrt{2}, 1/\sqrt{2}, 1$ unit. The tile $A$ has $a_1, a_3$ of equal length and $a_2$ is of dimension 1. For the tiles $B, C$ and $D$ we give similar labels.

The set of all edge labels is called an edge set, denoted by $E$. An edge pasting rule is a pair $(x, y)$ where $x, y \in E$. The pasting rules are as follows:

1. Tile $A$ can be pasted with $B$ by the rules
   $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$, with tile $C$ by the rule $\{(a_3, c_1)\}$ and with tile $D$ by the rule $\{(a_1, d_3)\}$.

2. Tile $B$ can be pasted with $A$ by the rules
   $\{(b_1, a_1), (b_2, a_2), (b_3, a_3)\}$, with tile $C$ by the rule $\{(b_1, c_3)\}$ and with tile $D$ by the rule $\{(b_3, d_1)\}$.
3. Tile $C$ can be pasted with $D$ by the rules
\{(c_1, d_1), (c_2, d_2), (c_3, d_3)\}, with tile $A$ by the rule \{(c_1, a_3)\} and with tile $B$ by the rule \{(c_3, b_1)\}.

4. Tile $D$ can be pasted with $C$ by the rules \{(d_1, c_1), (d_2, c_2), (d_3, c_3)\}, with tile $A$ by the rule \{(d_3, a_1)\} and with tile $B$ by the rule \{(d_1, b_3)\}.

**Definition 2.1.** A $k$-tabled triangular pasting system ($k$–TPS) is a $(k + 4)$-tuple $S = (\Sigma, \Sigma', E, T_1, T_2, \ldots, T_k, t_0)$ where $\Sigma$ is a finite non-empty set of isosceles right angled triangular tiles $A$, $B$, $C$ and $D$. $\Sigma'$ consists of tiles called completion tiles denoted by $A'$, $B'$, $C'$ and $D'$ which completes each generation when used such that $\Sigma \cup \Sigma' = \phi$. $E$ is the set of edge labels of the tiles in $\Sigma \cup \Sigma'$. $T_1, T_2, \ldots, T_{k-1}$ are finite set of pasting rules called the intermediate pasting rules and $T_k$ is a finite set of pasting rules called the final pasting rules. $t_0$ is the axiom, which is a finite tiling of tiles in $\Sigma \cup \Sigma'$.

A tiling pattern $t_{i+1}$ on the square grid is generated from a pattern $t_i$ in $k$ stages:

1. In the first $k - 1$ stages the tables $T_1, T_2, \ldots, T_{k-1}$ are used sequentially. The rules of the tables in each stage are applied in parallel to the boundary edges of the pattern obtained in the previous stage.

2. And in the $k^{th}$ stage, the pasting rules of $T_k$ are applied parallel to the boundary edges of the pattern obtained in the $(k - 1)^{th}$ stage deriving $t_{i+1}$.

The set of all patterns derived from the axiom of the pasting system is denoted by $T(S)$

$$T(S) = \{t_j : t_0 \Rightarrow t_j/j \geq 0\}$$

The family of all patterns generated by the system is $f(k – TPS)$.

**Example 2.1.** A 2–TPS generating a sequence of right angled isosceles triangles $S = (\Sigma, \Sigma', E, T_1, T_2, t_0)$, where

$$\Sigma = \left\{ \begin{array}{c}
\frac{b_2}{b_3} \frac{b_1}{c_1} \frac{c_3}{c_2} \\
\end{array} \right\},$$

$$\Sigma' = \left\{ \begin{array}{c}
\frac{a_1'}{a_2'} \frac{a_3'}{d_1'} \\
\end{array} \right\}.$$
$t_0 = \begin{array}{c}
\begin{array}{c}
D
\end{array}
\end{array}$, $E = \{b_1, b_2, b_3, c_1, c_2, c_3, a'_1, a'_2, a'_3, d'_1, d'_2, d'_3\}$

$T_1 = \{(a'_3, b_3), (d'_1, c_1)\}$, $T_2 = \{(b_1, a'_1), (b_2, a'_2), (c_2, d'_2), (c_3, d'_3), (a'_1, d'_3)\}$.

The first three members of $T(S)$ are shown below

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure}
\caption{3. Parallel Array Token Petri Nets}
\end{figure}

Parallel Array Token Petri nets are introduced in [1]. In this paper we characterize Triangular Pasting system using Parallel Array Token Petri nets (PATPN).

Let $V$ be the alphabet containing the four types of isosceles right angled triangle with sides being $1/\sqrt{2}, 1/\sqrt{2}$ and 1 unit.

$$V = \left\{ \triangle, \bigtriangleup, \square, \bigcirc \right\}.$$ 

The catention rules between these triangles which is assigned to the transitions of the Petri net are as follows

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure}
\caption{3. Parallel Array Token Petri Nets}
\end{figure}
$D \checkmark_1 C$ catenate $C$ to the left of $D$,
$C \checkmark_2 D$ catenate $D$ to the right of $D$,
$B \checkmark_3 A$ catenate $A$ in the upward direction of $B$,
$A \checkmark_4 B$ catenate $B$ in the downward direction of $A$,
$A \checkmark_5 B$ catenate $B$ in the lu (left up) direction of $A$,
$C \checkmark_6 D$ catenate $D$ in the ld (left down) direction of $C$,
$A \checkmark_7 B$ catenate $B$ in the ru (right up) direction of $A$,
$D \checkmark_8 C$ catenate $C$ in the rd (right down) direction of $D$.

The transitions of the Petri net are attached to rules of the form $\theta_1 \otimes \theta_2$ where $\theta_1$ and $\theta_2$ are elements of $V$ and $\#$ is any one of the catenation rules we have defined above. When the transition fires the array in the input place gets catenated according to the transition rule and the new array is placed in the output place. The rules of catenation are applied to all the triangles of the array in the input place with the same label. The following example explains the parallel catenation.

**Example 3.1.** If the input place of a transition has the array
\[
\begin{array}{c}
A \\
A
\end{array}
\]
as token and the transition is attached to the rule
$A \checkmark_4 B$ then after firing the output places of the transition will have the array
\[
\begin{array}{c}
A \\
B \\
A \\
B
\end{array}
\]
as token.

**Definition 3.1.** A PATPN is a 5-tuple $N = (\Sigma, C, \sigma, S, F)$ where $\Sigma$ is a finite nonempty subset of $V$. $C$ is a Petri net. $\sigma$ is a mapping which attaches rules to the various transitions of the Petri net in the form $\sigma(t_i) = \theta_1 \otimes \theta_2$. $S$ is the starting place where an array is placed as a token. $F$ is a subset of the set of places of the Petri net where the final array is stored after all the firing of the various possible transitions of the Petri net.

**Definition 3.2.** The language generated by the PATPN is the set of all arrays stored in the final places and is denoted by $\ell(N)$.

**Example 3.2.** $N = (\Sigma, C, \sigma, S, F)$ where $\Sigma = \{A, A_1, A_2, B, B_1, B_2\}$, $C$ is a Petri net with $P = \{p_1, p_2, p_3, p_4, p_5, p_6, p_7\}$,
$T = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}$. $I$, $O$ and $\sigma(t_i)$ are as given in the diagram, $F = S$.

The sequence of firing $t_1t_2t_3t_4t_5t_6t_7$ gives a triangle with a hole.
The sequence of firing $(t_1t_2t_3t_4t_5t_6t_7)^2$ gives the triangle.
4. Characterisation of Pasting Systems

In this section we characterize triangular pasting systems described in Section 2 by \( PATPN \). In other words given a \( PATPN \) we discuss how to construct a \( k-TPS \) and vice versa. Some pasting rules equivalent to the catenation rules are given below:

\( D \overset{1}{\rightarrow} C \) is equivalent to the pasting rule \((c_2, d_2)\),
\( B \overset{3}{\rightarrow} A \) is equivalent to the pasting rule \((b_2, a_2)\),
\( A \overset{5}{\rightarrow} D \) is equivalent to the pasting rule \((a_1, d_3)\),
\( C \overset{6}{\rightarrow} D \) is equivalent to the pasting rule \((c_1, d_1)\),
\( A \overset{7}{\rightarrow} B \) is equivalent to the pasting rule \((a_3, b_3)\),
\( D \overset{8}{\rightarrow} A \) is equivalent to the pasting rule \((d_3, a_1)\)
Algorithm. Given a PATPN generating a family of patterns over $V$ to construct a $k$–TPS which generates the same family.

**Input:** A PATPN generating a family of patterns over $V$.

**Output:** A $k$–TPS generating the same family.

**Begin**

1. **Step 1.** The first member of the family is taken as $t_0$. This array may have two types of triangles i) which get catenated ii) which do not get catenated. Put all the first type of triangles in $\Sigma'$. Put all the second type of triangles in $\Sigma$.

2. **Step 2.** One sequence of firing of all the transitions in the Petri net develops the pattern $P_{n+1}$ from the pattern $P_n$ in the start symbol. This development comes in $k$ stages in the pasting system.

   - **Stage 1.** The triangles that gets catenated through firing of transitions, to the triangles of $\Sigma'$ in $t_0$, are put in $\Sigma$. These firing rules are put as pasting rules in $T_1$.

   - **Stage 2.** In this stage the triangles coming through stage 1 are catenated to more triangles through firing of transitions. Put these firing rules as pasting rules in $T_2$. If we are able to generate the next member of the family, then the new triangles are put in $\Sigma'$ in which case we have a $2$–TPS equivalent to the given PATPN. If not these triangles are put in $\Sigma$ and we move to the next stage.

   - **Stage 3.** In this stage the triangles coming through stage 2 are catenated to more triangles through firing of transitions. Put these firing rules as pasting rules in $T_3$. If we are able to generate the next member of the family, then the new triangles are put in $\Sigma'$ in which case we have a $3$–TPS equivalent to the given PATPN. If not these triangles are put in $\Sigma$ and we move to the next stage.
We repeat this process so that at some stage, say $k$, we generate the next member of the family. Then we have $k$ tables and always in the last stage all the triangles pasted are from $\Sigma'$. Thus we have a $k-TPS$.

End.

Let us consider the $PATPN$ of Example 3.2 to construct the corresponding $k-TPS$.

**Step 1.** The first member of the family is taken as $t_0$ which has four triangles $A, B, A_1, A_2$. Out of these four only $A_1, A_2$ get catenated. Hence put them in $\Sigma'$. $A, B$ are put in $\Sigma$.

**Step 2. Stage 1.** The triangles $A_1$ and $A_2$ of $t_0$ get catenated to $B_1$ and $B_2$, so put them in $\Sigma$. These firing rules are put as pasting rules of $T_1$. Hence $T_1 = \{(a_{13}, b_{23}),(a_{23}, b_{13})\}$.

**Stage 2.** The triangles $B_1$ and $B_2$ of the first stage get catenated to $A$. The second member of the family is not generated at this stage so the triangle $A$ is put in $\Sigma$. The corresponding rules are put in $T_2$. Hence $T_2 = \{(b_{12}, a_2),(b_{21}, a_1)\}$. We move on to the next stage.

**Stage 3.** The triangle $A$ of the second stage gets catenated to $B$. The second member of the family is not generated at this stage so the triangle $B$ is put in $\Sigma$. The corresponding rules are put in $T_3$. Hence $T_3 = \{(a_3, b_3)\}$. We move on to the next stage.

**Stage 4.** The triangle $B$ of the third stage gets catenated to $A_1, A_2$. At this stage the second member of the family is generated. So these triangles are put in $\Sigma'$. The corresponding rules are put in $T_4$. Hence $T_4 = \{(b_1, a_{11}),(b_2, a_{22})\}$.

Since four stages are needed to develop the next member the corresponding pasting system is $4-TPS$ and it is given below.

The $4-TPS$ $S = (\Sigma, \Sigma', E, T_1, T_2, T_3, T_4, t_0)$, where

\[
\Sigma = \{ \begin{array}{c} \triangle \quad \square \quad \bigtriangleup \\ \bigtriangledown \quad \bigtriangledown \end{array} \}, \quad \Sigma' = \{ \begin{array}{c} \triangle \quad \square \quad \bigtriangleup \\ \bigtriangledown \quad \bigtriangledown \end{array} \},
\]

$T_1 = \{(a_{23}, b_{13}),(a_{13}, b_{23})\}$, $T_2 = \{(b_{12}, a_2),(b_{21}, a_1)\}$,
$T_3 = \{(a_3, b_3)\}$, $T_4 = \{(b_1, a_{11}),(b_2, a_{22})\}$.

**Algorithm.** Given a $k-TPS$ algorithm to construct a $PATPN$.

**Input:** A $k-TPS$ generating a family of patterns over $V$.

**Output:** A $PATPN$ generating the same family.
Begin

Step 1. All the triangles of $\Sigma$ and $\Sigma'$ are put in a single set $\Sigma$.

Step 2. The array in $t_0$ is put as a token in the start place of the Petri net.

Step 3. For all pasting rules of $T_1, T_2, \ldots, T_k$ have a transition placed with the corresponding catenation rule attached. The firing of each transition should catenate the array in the input place with the needed triangle and deposit it in the output array.

Step 4. The last transition on firing should deposit the second pattern of the family in the start place, so that when the full sequence of transitions fire again we are able to develop the third pattern.

End.

Let us consider the $2-TPS$ of example 2.1. Put $A, B, C, D$ in $\Sigma$. The start place should have the array $\begin{array}{c} A \\ B \\ C \\ D \end{array}$ as a token. Since there are six pasting rules in $T_1$ and $T_2$ the Petri net should have six transitions with one input place and one output place with the corresponding catenation rule attached. The last transition should have $S$ as its output place. Thus we have the $PATPN$ $N = (\Sigma, C, \sigma, S, F)$ where $\Sigma = \{A, B, C, D\}$, $C$ is a Petri net with $P = \{p_1, p_2, p_3, p_4, p_5, p_6\}$, $T = \{t_1, t_2, t_3, t_4, t_5, t_6\}$. $I, O$ and $\sigma(t_i)$ are as given in the diagram. $F = S$.

This $ATPN$ generates the same family of triangles.
5. Properties of PATPN

1. Since \{A, B\} is a proper subset of the set \{A, B, C, D\}, the class of patterns generated by the PATPN with the set \{A, B\} as its alphabet can be generated by a similar PATPN with \{A, B, C, D\} as its alphabet. Hence the class of patterns generated by \{A, B\} is a proper subclass of the patterns generated by \{A, B, C, D\}.

2. The patterns generated using the catenation rules 1, 3, 4, 5, 6 (of Section 3) will develop to the left of the vertical axis. The patterns generated using the rules 2, 3, 4, 7, 8 (of Section 3) will develop to the right of the vertical axis. Hence there is symmetry with respect to Y-axis when these rules are used.

3. The patterns generated using the rules 1, 2, 3, 5, 7 (of Section 3) will develop above the horizontal axis. The patterns generated using the rules 1, 2, 4, 6, 8 (of Section 3) will develop below the horizontal axis. Hence there is symmetry with respect to X-axis when these rules are used.

References


