THE LERAY-LIONS EQUATION AND ITS DOUBLE OBSTACLE PROBLEM WITH VARIABLE EXPONENT

Tong Yuxia\(^{1,\S}\), Gu Jiantao\(^{2}\), Hou Xiuli\(^{3}\)

\(^{1,2}\)College of Science
Hebei United University
Tangshan, Hebei, 063009, P.R. CHINA

\(^{3}\)Qian’an College
Hebei United University
Qian’an, Hebei, 064400, P.R. CHINA

Abstract: The relation between the solutions to the Leray-Lions equation and the solutions to its double obstacle problem with variable exponent has been studied. The Caccioppoli estimate of the solution to double obstacle problem has been obtain, which is important in studying the integrablity of the solutions.

AMS Subject Classification: 31C45, 35J60

Key Words: double obstacle problem, Leray-Lions equation, variable exponent

1. Introduction

Partial differential equations with nonlinearities involving nonconstant exponents have attracted an increasing amount of attention in recent years. The development, mainly by [1], of a theory modeling the behavior of electrorheological fluids, an important class of non-Newtonian fluids, seems to have boosted a still far from completed effort to study and understand nonlinear PDEs involving variable exponents. This, in turn, gave rise to a revival of the interest in Lebesgue and Sobolev spaces with variable exponent.

Received: January 27, 2011
© 2011 Academic Publications, Ltd.

\(^{\S}\)Correspondence author
There has been new interesting development in the study of the Leray-Lions equation, largely pertaining to applications in quasiformal analysis and nonlinear elasticity, that is

$$\text{div} A(x, \nabla u) = 0$$  \hspace{1cm} (1.1)$$

where the operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions

$$|A(x, \xi)| \leq \alpha |\xi|^{p(x)-1}, \quad \langle A(x, \xi), \xi \rangle \geq \beta |\xi|^p,$$

$$\langle A(x, \xi) - A(x, \zeta), \xi - \zeta \rangle \geq 0.$$  \hspace{1cm} (1.2)$$

for almost every $x \in \Omega$ and all $\xi, \zeta \in \mathbb{R}^n$. Here $\alpha, \beta > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.1).

In this paper, we consider the Leray-Lions equation with potential applications to the modeling of combustion, thermal explosions, nonlinear heat generation, gravitational equilibrium of polytropic stars, glaciology, non-Newtonian fluids, and the flow through porous media. Many of properties of the Leray-Lions equation have already been analyzed for constant exponents of nonlinearity (cf. [2]-[5], and the references therein) but it seems to be more realistic to assume the exponent to be variable.

The aim of the present paper is to study the Leray-Lions equation of type (1.1) with the more general growth conditions than (1.2), that is, we assume the operator $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following assumptions

$$\langle A(x, \xi), \xi \rangle \geq \alpha |\xi|^{p(x)};$$  \hspace{1cm} (1.3)$$

$$|A(x, \xi)| \leq \beta |\xi|^{p(x)-1};$$  \hspace{1cm} (1.4)$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$, where $\alpha, \beta$ is a positive constant;

$$\langle A(x, \xi) - A(x, \zeta), \xi - \zeta \rangle > 0;$$  \hspace{1cm} (1.5)$$

for almost every $x \in \Omega$ and for every $\xi, \zeta \in \mathbb{R}^n$, with $\xi \neq \zeta$.

Concerning the exponent $p(\cdot)$ appearing in (1.3) and (1.4), it is a measurable function $p(\cdot) : \Omega \to \mathbb{R}^n$ such that

$$p(\cdot) \in W^{1,\infty}(\Omega) \quad \text{and} \quad 1 < \inf_{x \in \Omega} p(x) \leq \sup_{x \in \Omega} p(x) \leq n.$$  \hspace{1cm} (1.6)$$

The following is a brief overview of the state of the art concerning Lebesgue spaces with variable exponent, and Sobolev spaces modeled upon them. Given a measurable function $p(\cdot) : \Omega \to [1, +\infty)$, we will use the following notation throughout the paper:

$$p_- = \inf_{x \in \Omega} p(x) \quad \text{and} \quad p_+ = \sup_{x \in \Omega} p(x).$$  \hspace{1cm} (1.7)$$
Let
\[ L^p(x)(\Omega) = \left\{ u \mid u \text{ is a measurable real valued function on } \Omega, \int_{\Omega} |u|^p(x) \, dx < \infty \right\}, \]
\[ W^{1,p}(x)(\Omega) = \left\{ u \in L^p(x)(\Omega) \mid |\nabla u| \in L^p(x)(\Omega) \right\}. \]

We can introduce norms on \( L^p(x)(\Omega) \) and \( W^{1,p}(x)(\Omega) \), respectively, as
\[ \|u\|_{p(x)} = \inf \left\{ \lambda > 0 \mid \int_{\Omega} \frac{|u(x)|}{\lambda} \, dx \leq 1 \right\}, u \in L^p(x)(\Omega) \]
and
\[ \|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)} \]
then \((L^p(x)(\Omega), \| \cdot \|_{p(x)}) \) and \((W^{1,p}(x)(\Omega), \| \cdot \|_{p(x)}) \) are both separable and reflexive Banach spaces.

Denote by \( W^{1,p}_0(\Omega) \) the closure of \( C^\infty_0(\Omega) \) in \( W^{1,p}(x)(\Omega) \); we know that \( |\nabla u|_{p(x)} \) is an equivalent norm on \( W^{1,p}_0(\Omega) \).

In this work we do not study variable exponent spaces themselves, but rather related differential equations. For our purposes, the most important facts about the variable exponent Lebesgue spaces are the following. If \( E \) is a measurable set with a finite measure, \( p \) and \( q \) are variable exponents satisfying \( q(x) \leq p(x) \) for almost every \( x \in E \), then \( L^{p(x)}(E) \) embeds continuously into \( L^{q(x)}(E) \) and the norm of the embedding cannot exceed \( 1 + \|E\| \). We use also a variable exponent version of Hölder’s inequality
\[ \int_{\Omega} fg \, dx \leq C \|f\|_{p(x)} \|g\|_{p'(x)} \quad (1.8) \]
where the constant \( C \) depends only on \( p^- \) and \( p^+ \). Here \( 1/p(x) + 1/p'(x) = 1 \) for every \( x \). For all these facts we refer the reader to [9].

2. The Relation between the Solutions to (1.1) and the Solutions to its Double Obstacle Problem

By the similar definitions as the solution, supersolution (or subsolution) to quasilinear elliptic equation, we can give the definitions of the solution, supersolution (or subsolution) to (1.1).
**Definition 2.1.** If a function \( u \in W^{1,p}_{{\text{loc}}} (\Omega) \) satisfies
\[
\int_\Omega (A(x, \nabla u), \nabla \varphi) \, dx = 0,
\] (2.1)
for any \( \varphi \in W^{1,p}_0 (\Omega) \), then we say that \( u \) is a solution to (1.1). If for any \( 0 \leq \varphi \in W^{1,p}_0 (\Omega) \), we have
\[
\int_\Omega (A(x, \nabla u), \nabla \varphi) \, dx \geq 0 (\leq 0),
\] (2.2)
then we say that \( u \) is a supersolution (subsolution) to (1.1).

We can see that if \( u \) is a subsolution to (1.1), then for \( 0 \geq \varphi \in W^{1,p}_0 (\Omega) \), we have
\[
\int_\Omega (A(x, \nabla u), \nabla \varphi) \, dx \geq 0.
\] (2.3)

A function \( u \) is a subsolution in \( \Omega \) if \( -u \) is a supersolution in \( \Omega \), and a solution in \( \Omega \) if it is both a supersolution and a subsolution in \( \Omega \).

According to the above definitions, we can get the following theorem.

**Theorem 2.1.** A function \( u \in W^{1,p}_{{\text{loc}}} (\Omega) \) is a solution to (1.1) if and only if \( u \) is both supersolution and subsolution to (1.1).

**Proof.** The sufficiency is obvious, we only prove the necessity. For any \( \varphi \in W^{1,p}_0 (\Omega) \), let
\[
\varphi_1 = \varphi^+ \geq 0, \quad \varphi_2 = \varphi^- \leq 0.
\]
By definition 2.1, for \( u \) is a supersolution to (1.1), it holds that
\[
\int_\Omega (A(x, \nabla u), \nabla \varphi_1) \, dx \geq 0, \quad \text{(2.4)}
\]
and for \( u \) is a subsolution to (1.1), we have
\[
\int_\Omega (A(x, \nabla u), \nabla \varphi_2) \, dx \geq 0 \quad \text{(2.5)}
\]
So
\[
0 \leq \int_\Omega (A(x, \nabla u), \nabla \varphi_1) \, dx + \int_\Omega (A(x, \nabla u), \nabla \varphi_2) \, dx
= \int_\Omega (A(x, \nabla u), \nabla \varphi_1 + \nabla \varphi_2) \, dx
= \int_\Omega (A(x, \nabla u), \nabla \varphi) \, dx.
\] (2.6)
Using $-\varphi$ in place of $\varphi$, we also can get
\[
\int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle dx \leq 0.
\]  \hspace{1cm} (2.7)

Thus
\[
\int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle dx = 0.
\]  \hspace{1cm} (2.8)

Therefore $u$ is a solution to (1.1).

The next we will introduce the double obstacle problem to (1.1), which definition is according to the similar definition as the obstacle problem of quasilinear elliptic equation. We refer the reader to [7] for some details of the obstacle problem of quasilinear elliptic equation.

Supposed that $\Omega$ is a bounded domain. $\psi_1, \psi_2$ are any functions in $\Omega$ with values in the extended reals $[-\infty, \infty]$, and $\theta \in W^{1,p(x)}(\Omega)$ is a function which gives the boundary values. Let
\[
K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega) = \left\{ v \in W^{1,p(x)}(\Omega) : \psi_1 \leq v \leq \psi_2 \text{ a.e., } v - \theta \in W^{1,p(x)}_0(\Omega) \right\}.
\]

The problem is to find a function in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ such that for any $v \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ we have
\[
\int_{\Omega} \langle A(x, \nabla u), \nabla (v - u) \rangle dx \geq 0.
\]  \hspace{1cm} (2.9)

**Definition 2.2.** A function $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is called a solution to the double obstacle problem of Leray-Lions equation (1.1) with obstacles $\psi_1, \psi_2$ and boundary values $\theta$ or a solution to the double obstacle problem of Leray-Lions equation (1.1) in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ if $u$ satisfies (2.9) for any $v \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$.

If $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a solution to the double obstacle problem of (1.1), then for any constant $\varepsilon > 0$, we have $u - \varepsilon \in K^{\theta-\varepsilon,p(x)}_{\psi_1-\varepsilon,\psi_2-\varepsilon}(\Omega)$ is a solution to the double obstacle problem of (1.1).

A function $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a solution of the obstacle problem if and only if it is a quasiminimizer of the Dirichlet energy integral among the functions in $K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$. This is shown in the usual way. Explicitly, if $u \in K^{\theta,p(x)}_{\psi_1,\psi_2}(\Omega)$ is a
solution of the obstacle problem, then
\[
\alpha \int_\Omega |\nabla u|^{p(x)} \, dx \leq \int_\Omega \langle A(x, \nabla u), \nabla u \rangle \, dx \\
\leq \int_\Omega \langle A(x, \nabla u), \nabla v \rangle \, dx \\
\leq \int_\Omega |A(x, \nabla u)||\nabla v| \, dx \\
\leq \beta \int_\Omega |\nabla u|^{p(x)-1}|\nabla v| \, dx \\
\leq \beta \int_\Omega \left( \frac{\varepsilon(p(x) - 1)}{p(x)} |\nabla u|^{p(x)} + \frac{\varepsilon^{1-p(x)}}{p(x)} |\nabla v|^{p(x)} \right) \, dx \\
\leq \beta \frac{\varepsilon(p^+ - 1)}{p^-} \int_\Omega |\nabla u|^{p(x)} \, dx + \beta \frac{\varepsilon^{1-p^+}}{p^-} \int_\Omega |\nabla v|^{p(x)} \, dx, 
\]
for every \( v \in K_{\psi_1, \psi_2}^{\theta,p(x)}(\Omega) \), where we have used Young’s inequality. Moving all the terms with \( u \) to the left hand side, and let \( \varepsilon \) satisfies that \( \beta \frac{\varepsilon(p^+ - 1)}{p^-} = \frac{\alpha}{2} \), we see that
\[
\int_\Omega |\nabla u|^{p(x)} \, dx \leq C(\alpha, \beta, p^-, p^+) \int_\Omega |\nabla v|^{p(x)} \, dx 
\]
(2.11)
So it is a quasiminimizer of the Dirichlet energy integral among the functions in \( K_{\psi_1, \psi_2}^{\theta,p(x)}(\Omega) \).

We have some relations between the solution to quasilinear elliptic equation and the solution to obstacle problem in PDE. As to the Leray-Lions equation and its double obstacle problem with variable exponent, we also have some relations between the solution to nonhomogeneous Leray-Lions equation and the solution to obstacle problem of it. We have the following two theorems.

**Theorem 2.2.** If \( u \) is a supersolution to (1.1), then \( u \) is a solution to the obstacle problem of (1.1) in \( K_{\psi_1, \psi_2}^{u,p(x)}(\Omega) \).

*Proof.* If \( u \) is a supersolution to (1.1) in \( \Omega \) then for any \( v \in K_{u,\psi_2}^{u,p(x)}(\Omega) \), we have \( v - u \geq 0, \ v - u \in W_0^{1,p(x)}(\Omega) \). Thus let
\[
\varphi = v - u, 
\]
(2.15)
we have
\[
0 \leq \int_\Omega \langle A(x, \nabla u), \nabla \varphi \rangle \, dx = \int_\Omega \langle A(x, \nabla u), \nabla v - \nabla u \rangle \, dx. 
\]
(2.16)
So $u$ is a solution to the obstacle problem of (1.1) in $K^{u,p(x)}_{u,\psi_2}(\Omega)$.

**Theorem 2.3.** A function $u \in W^{1,p(x)}(\Omega)$ is a solution to (1.1) if only if $u$ is a solution to the obstacle problem of (1.1) in $K^{\theta,p(x)}_{-\infty,\infty}(\Omega)$ with $\theta$ satisfying $u - \theta \in W^{1,p(x)}_0(\Omega)$.

**Proof.** If $u$ is a solution to the obstacle problem of (1.1) in $K^{\theta,p(x)}_{-\infty,\infty}(\Omega)$, then for any $\theta \in W^{1,p(x)}_0(\Omega)$, let $v = u + \varphi$, then

$$v - \theta = u - \theta + \varphi \in W^{1,p(x)}_0(\Omega).$$

Then we have $v \in K^{\theta,p(x)}_{-\infty,\infty}(\Omega)$. So we can obtain

$$\int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle dx = \int_{\Omega} \langle A(x, \nabla u), \nabla v - \nabla u \rangle dx \geq 0. \quad (2.17)$$

By using $-\varphi$ in place of $\varphi$, we also can get

$$\int_{\Omega} \langle A(x, \nabla u), \nabla (-\varphi) \rangle dx = \int_{\Omega} \langle A(x, \nabla u), \nabla v - \nabla u \rangle dx \geq 0. \quad (2.18)$$

So

$$\int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle dx = 0. \quad (2.19)$$

Thus $u$ is a solution to (1.1) in $\Omega$.

Conversely, if $u$ is a solution to (1.1) in $\Omega$, then for any $v \in K^{\theta,p(x)}_{-\infty,\infty}(\Omega)$, we have $v - u \in W^{1,p(x)}_0(\Omega)$. Now let $\varphi = v - u$, then we have

$$0 = \int_{\Omega} \langle A(x, \nabla u), \nabla \varphi \rangle dx = \int_{\Omega} \langle A(x, \nabla u), \nabla v - \nabla u \rangle dx. \quad (2.20)$$

Thus

$$0 \leq \int_{\Omega} \langle A(x, \nabla u), \nabla v - \nabla u \rangle dx. \quad (2.21)$$

So the theorem 2.3 is proved.
Next we consider a Caccioppoli type estimate of the solutions to obstacle problem taking into account its importance in studying the integrability of the solutions.

**Theorem 3.1.** Assume $u$ is a solution to the obstacle problem $K_{\psi_1, \psi_2}^{\theta, p(x)}(\Omega)$ with positive obstacle $\psi_1$, non-positive obstacle $\psi_2$. Let $B \subset \Omega$, $R > 0$ and $\eta \in C_0^\infty(B)$ with $0 \leq \eta \leq 1$. Then

$$
\int_\Omega |\nabla u|^p(x) \eta^p_B + |\nabla \eta|_p^p dx \leq C \int_\Omega u^p(x) |\nabla \eta|_p^p dx.
$$

(3.1)

Here the constant $C$ depends only on $p^-$ and $p^+$.

**Proof.** Let $v = u - sgn(u)\varepsilon_0 \phi$, where $\phi = u_+\eta^p_B$,

$$
sgn(u) = \begin{cases} 
-1, & \text{if } u(x) < 0 \\
1, & \text{if } u(x) \geq 0 
\end{cases}
$$

$\varepsilon_0 > 0$ is a constant which is small enough to ensure that $v$ is between the positive obstacle $\psi_1$ and non-positive obstacle $\psi_2$. It is clearly that $v$ has the same boundary values as $u$. Since

$$
|\nabla \phi| \leq \left| \eta^p_B \nabla u_+ + u_+ p^+_B \eta^{p^+_B} - \nabla \eta \right| \leq |\nabla u| + C|u_+|
$$

(3.2)

we observe that $|\nabla \phi| \in L^{p^+}(\Omega)$, and $|\phi| \in L^{p^+}(\Omega)$, thus, it is enough to show that $\phi \in W^{1, p^+}(\Omega)$.

Since $v = u - sgn(u)\varepsilon_0 \phi \in K_{\psi_1, \psi_2}^{\theta, p(x)}(\Omega)$, we find

$$
0 \leq \int_\Omega \langle A(x, \nabla u), \nabla(-sgn(u)\varepsilon_0 \phi) \rangle dx \\
= - \int_\Omega \langle A(x, \nabla u), sgn(u)\varepsilon_0 \eta^p_B \nabla u_+ + sgn(u)\varepsilon_0 u_+ p^+_B \eta^{p^+_B} - \nabla \eta \rangle dx \quad (3.3) \\
\leq -\alpha \varepsilon_0 \int_\Omega |\nabla u_+|^p(x) \eta^p_B + \beta \varepsilon_0 \int_\Omega |\nabla \eta|_p^p - u_+ p^+_B \eta^{p^+_B - 1} |\nabla \eta|_p^p dx
$$

This implies that

$$
\int_\Omega |\nabla u_+|^p(x) \eta^p_B \leq \frac{\beta p^+_B}{\alpha} \int_\Omega |\nabla u_+|^{p(x)-1} u_+ \eta^{p^+_B - 1} |\nabla \eta|_p^p dx \quad (3.4)
$$
We denote the right hand side of (3.4) by $I$. By Young’s inequality we obtain

$$I \leq \frac{\beta p_B^+ \alpha}{\alpha} \int \frac{\varepsilon^{1-p(x)}}{p(x)} u_+^p(x) |\nabla \eta|^p x \eta_{p_B^+} - \eta_{p_B^+} \eta_{p_B^+} \langle dx
\leq \frac{\beta p_B^+ \alpha}{\alpha} \left( \int \frac{\varepsilon^{1-p(x)}}{p(x)} u_+^p(x) |\nabla \eta|^p x \eta_{p_B^+} - \eta_{p_B^+} \eta_{p_B^+} \langle dx + \varepsilon \int \nabla u_+^p(x) \eta_{p_B^+} dx \right),$$

(3.5)

where $\varepsilon$ satisfies $\frac{\beta p_B^+ \alpha}{\alpha} \varepsilon < 1$. For this and the fact that $\int \nabla u_+^p(x) \eta_{p_B^+} \langle < \infty$, which combining with the inequality (3.4), we obtain

$$\int \nabla u_+^p(x) \eta_{p_B^+} \langle \leq C \int u_+^p(x) |\nabla \eta|^p x.$$

(3.6)

The proof completed.

We can also prove a Caccioppoli type estimate for the negative parts of solutions to obstacle problem.

**Theorem 3.2.** Assume $u$ is a solution to the obstacle problem $K_{\theta,p(x)}(\Omega)$ with positive obstacle $\psi_1$, non-positive obstacle $\psi_2$. Let $B \subset \subset \Omega$, $R > 0$ and $\eta \in C_0^\infty(B)$ with $0 \leq \eta \leq 1$. Then

$$\int_B |\nabla u_-^p(x) \eta_{p_B^+} \langle \leq C \int_B |u_-^p(x) |\nabla \eta|^p x.$$

(3.7)

Here the constant $C$ depends only on $p_B^+$ and $p_B^+$.

**Proof.** Let $v = u + sgn(u)\varepsilon_0 u_- \eta_{p_B^+}$, we find that

$$0 \leq \int \langle A(x, \nabla u), \nabla (sgn(u)\varepsilon_0 u_- \eta_{p_B^+}) \rangle dx
= \int \langle A(x, \nabla u), sgn(u)\varepsilon_0 \eta_{p_B^+} \nabla u_- + sgn(u)u_- \eta_{p_B^+} \eta_{p_B^+} - \beta \varepsilon_0 \nabla \eta \rangle dx
\leq -\alpha \varepsilon_0 \int |\nabla u_-^p(x) \eta_{p_B^+} \langle - \beta \varepsilon_0 \int |\nabla u_-^p(x) |\nabla \eta|^p x.$$
This implies that
\[
\int_{\Omega} \left| \nabla u - |p(x)\eta p^B \right| \leq \frac{\beta p_B^+}{\alpha} \int_{\Omega} -\left| \nabla u - |p(x) - 1 u - \eta p^B - 1 |\right| \nabla \eta \right| dx \tag{3.9}
\]
\[
= \frac{\beta p_B^+}{\alpha} \int_{\Omega} -u_- |\nabla \eta| p_B^+ - 1 \frac{p_B^+}{p(x)} |\nabla u - |p(x) - 1 \eta p(x) \nabla \eta| p(x) \right| dx
\]

Notice that \( \int_{\Omega} |\nabla u - |p(x)\eta p^B < \infty \), then by the same methods as the proof in Theorem 3.1, we obtain
\[
\int_{\Omega} \left| \nabla u - |p(x)\eta p^B \right| \leq C \int_{\Omega} |u_+ - |p(x)| \nabla \eta| p(x) \right| dx \tag{3.10}
\]

Acknowledgments

The research supported by Natural Science Foundation of Hebei Province (A2010000910), and Scientific Research Fund of Zhejiang Provincial Education Department (Y201016044).

References


378