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A DELAYED CHEMOSTAT MODEL WITH IMPULSIVE PERTURBATION ON THE NUTRIENT CONCENTRATION

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Abstract: A delayed chemostat model with impulsive perturbation on the nutrient concentration and a general nutrient uptake function is considered. The nutrient conversion process involves time delay. Using the discrete dynamical system determined by the stroboscopic map, we obtain the exact microorganism-eradication periodic solution of the model and prove the microorganism-eradication periodic solution is globally attractive, provided that the amount of impulsive substrate is less than some critical value. When the amount of impulsive substrate is larger than some critical value, the system is permanent. Computer simulations illustrate the results.

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Key Words: chemostat model, delay, impulsive input, permanence

1. Introduction

A chemostat is a very important laboratory apparatus used to culture microorganisms. In the literatures [1-3], the authors are assumed that sterile medium enters the chemostat at a constant rate; the volume with the chemostat is held constant by allowing excess medium (and microbes) to flow out through a siphon. Zhou et al [4] introduce and study a competitive system with Beddington CDeAngelis type functional response in periodic pulsed chemostat conditions. They assumed that the nutrient is the limiting substrate, which is pulled in periodically. A competitive chemostat model with impulsive effect was established and discussed.

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Recently, many papers studied chemostat models with impulsive effect [5,6]. The study of impulsive differential equations (IDE) are found in a almost every domain of applied sciences and have been studied in many investigations [7,8]. Funasaki and Kot [9] studied a predator-prey model in a chemostat with predator, prey, and periodically pulsed substrate. They have investigated the existence and stability of the periodic solutions of the impulsive subsystem with substrate and prey. Further, using numerical simulation method, they showed that impulsive invasion cause complex dynamics of system.

In general, time delays in dynamical systems can have a considerable influence on the qualitative behavior of these systems. Some non-stationary phenomena, such as periodic fluctuations and instabilities, can be explained by incorporating time delays in model systems (see [2], [10], [11] and references therein). Chemostat models with time delays have received much attention since delays can often cause some complicated dynamical behaviors. However, delayed chemostat models with impulsive perturbation have seldom been studied by authors. In this paper, we incorporate a delay in a chemostat-type model with impulsive nutrient supplied. The discrete delay is included since it is assumed that the concentration of the microorganism at time t depends on the concentration of nutrient at time $t-\omega$, that is the time delay ω denotes the time necessary to complete the nutrient conversion process. A general function is also used to describe the nutrient uptake.

The organization of this paper is as follows. In next section, we investigate a delayed chemostat model with impulsive perturbation on the nutrient concentration and introduce some definitions, notations and lemmas which will be useful in subsequent sections. In Section 3, using the discrete dynamical system determined by the stroboscopic map, we obtain the sufficient condition for the global attractivity of the microorganism-eradication periodic solution. The sufficient condition for the permanence of the system is also obtained in Section 4. In the final section, the numerical simulation verifies our analysis. We point out some future research directions.

2. Model Formulation and Preliminary Results

Smith and Waltman [1] describe a chemostat and formulate various mathematical chemostat models. The specific growth rate of bacteria saturates at sufficiently high-substrate concentration. Equations of the basic model take the form [12]

$$\begin{aligned} S'(t) &= Q(S^0 - S(t)) - \frac{1}{\delta} p(S(t))x(t), \\ x'(t) &= -Qx(t) + p(S(t))x(t), \end{aligned} \tag{2.1}$$

where the state variables $S(t)$ and $x(t)$ denote the concentrations of the limiting substrate and the microorganism at time t, respectively; S^0 denotes the input concentration of the limiting substrate per unit of time; δ is the yield of the microorganism per unit mass of substrate; Q is the dilution rate of the chemostat. The function $p(S)$ denotes the microbial growth rate.

In (2.1), the microbial continuous culture has been considered by authors. But it is natural to consider chemostat models with periodic perturbations owing to periodic behaviors (for example, food supply, harvesting, mating habits and so on). This is an interesting issue on mathematical and laboratory experiment. But the research on the chemostat model with impulsive perturbations is not too much [5,9]. In [5], Sun and Chen investigate a Monod type chemostat model with impulsive perturbation of substrate

$$\begin{aligned} & S'(t) = -QS(t) - \frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))}, \quad t \neq kT, \\ & x'(t) = x(t)\left(\frac{\mu_m S(t)x(t)}{\delta(K_m + S(t))} - Q\right), \quad t \neq kT, \\ & S(kT^+) = S(kT) + \tau S^0, \quad x(kT^+) = x(kT), \quad n = 1, 2, \ldots, \\ & S(0^+) \geq 0, \quad x(0^+) \geq 0, \end{aligned} \tag{2.2}$$

where the first and second equations hold between pulses, the third equation describes the actual pulsing. $T = \tau/Q$ is the period of the pulsing; τS^0 is the amount of limiting substrate pulsed each T; QS^0 units of substrate are added, on average, per unit of time. Using Floquet theory and small amplitude perturbation method, the stability of the microorganism-eradication periodic solution and the permanence of the system (2.2) are discussed in detail.

The following system is a model for the behavior of the limiting substrate and the microorganism when the time delay and impulsive perturbation of substrate are taken into account:

$$\begin{aligned} & S'(t) = -QS(t) - \frac{1}{\delta} p(S(t))x(t), \quad t \neq kT, \\ & x'(t) = -Qx(t) + e^{-Q\omega} p(S(t-\omega))x(t-\omega), \quad t \neq kT, \\ & S(kT^+) = S(kT) + \tau S^0, \quad x(kT^+) = x(kT), \quad n = 1, 2, \ldots. \end{aligned} \tag{2.3}$$

As discussed in [13], $e^{-Q\omega}x(t-\omega)$ represents the concentration of the microorganism that consume nutrient ω units of time prior to time t and that survive in the chemostat the ω units of time necessary to complete the nutrient conversion process. We assume that the microbial growth rate $p(S)$ in (2.3) satisfies:

(H) $p(S):\ R_+ \mapsto R_+$ is continuously differentiable with $p'(S) > 0,\ p(0) = 0$ and a finite limit as $S \to \infty$, that is, there exists a positive constant $\alpha > 0$ such that $\lim_{S\to\infty} p(S) = \alpha$.

We also assume that system (2.3) with following initial conditions

$$\begin{array}{l} (S(t), x(t)) = (\psi_1(t),\ \psi_2(t)) \in C_2^+ \text{ and } t \in [-\omega,\ 0],\ \psi_i(0) > 0,\ i = 1, 2, \\ \psi_2(0) = \int_{-\omega}^{0} e^{Q\theta} p(\psi_1(\theta))\psi_2(\theta)d\theta, \end{array} \tag{2.4}$$

where $C_2^+ \triangleq C([-\omega,\ 0],\ R_+^2)$ and $R_+^2 \triangleq \{(x_1,\ x_2) |\ x_i \geq 0,\ i = 1, 2\}$.

Before stating and proving our main results, we give the following definitions, notations and lemmas which will be useful in the following sections.

Let $J \subset R$ and N be the set of nonnegative integers. We introduce the following space of functions:

(i) $PC(J,\ R) \triangleq \{u : J \mapsto R \mid u$ is continuous for $t \in J,\ t \neq \tau_k$, continuous from the left for $t \in J$, and has discontinuities of the first kind at the point $\tau_k \in J, k \in N\}$;

(ii) $PC^1(J,\ R) \triangleq \{u \in PC(J,\ R) \mid u$ is continuously differential for $t \in J,\ t \neq \tau_k;\ u'(\tau_k^+)$ and $u'(\tau_k^-)$ exist, $k \in N\}$.

Definition 2.1. A function $z:\ [-\omega,\ \infty) \mapsto R_+^2$ is said to be a solution of system (2.3) on $[-\omega,\ \infty)$ if:

(i) $z(t)$ is continuous on $(0,\ T]$ and each interval $(kT,\ (k+1)T],\ k \in N$;

(ii) for any $kT,\ k \in N$, $z(kT^+)$ and $z(kT^-)$ exist and $z(kT^-) = z(kT)$;

(iii) $z(t)$ satisfies the two first equations of system (2.3) for almost everywhere in $[0,\ \infty)/\{kT\}$ and satisfies the third equation for every $t = kT,\ k \in N$.

Denote $R_+ = [0,\ \infty)$, $\Omega = \text{int } R_+^2$. Let $f = (f_1,\ f_2)^T$ the map defined by the right hand of the two first equations of system (2.3).

Let $V:\ R_+ \times R_+^2 \to R_+$, then $V \in V_0$ if

(i) V is continuous in $(kT,\ (k+1)T] \times R_+^2$ and for each $z \in R_+^2,\ k \in N$, $\lim_{(t,\ y)\to(kT^+,\ z)} V(t,\ y) = V(kT^+,\ z)$ exists;

(ii) V is locally Lipschitzian in z.

Definition 2.2 Let $V \in V_0$, then for $(t,\ z) \in (kT,\ (k+1)T] \times R_+^2$, the upper derivative of $V(t,\ z)$ with respect to the system (2.3) is defined as

$$D^+V(t,\ z) = \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\ z+hf(t,\ z)) - V(t,\ z)].$$

Obviously, the smoothness properties of function f guarantee the global existence and uniqueness of solutions of system (2.3) (see Ref. [14 for details). Therefore, the following Lemmas are obvious.

Lemma 2.1. *Suppose that $z(t)$ is a solution of system (2.3) with initial conditions (2.4). Then $z(t) \geq 0$.*

Lemma 2.2. (Lakhsmikantham et al [14]) *Let $V : R_+ \times R_+^2 \mapsto R_+$ and $V \in V_0$. Assume that*

$$\begin{array}{l} D^+V(t,\ z(t)) \leq g(t,\ V(t,\ z(t))), \quad t \neq kT, \\ V(t,\ z(t^+)) \leq \varphi_n(V(t,\ z(t))), \quad t = kT, \end{array} \tag{2.5}$$

where $g : R_+ \times R_+ \mapsto R$ is continuous in $(kT,\ (k+1)T] \times R_+$ and for each $v \in R_+,\ n \in N,\ \lim_{(t,\ y)\to(kT^+,\ v)} g(t,\ y) = g(kT^+,\ v)$ exists; $\varphi_n : R_+ \to R_+$ is nondecreasing. Let $R(t) = R(t,\ 0,\ u_0)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{array}{l} u'(t) = g(t,\ u(t)), \quad t \neq kT, \\ u(t^+) = \varphi_n(u(t)), \quad t = kT, \\ u(0^+) = u_0, \end{array} \tag{2.6}$$

existing on $[0,\ \infty)$. Then $V(0^+,\ z_0) \leq u_0$ implies that $V(t,\ z(t)) \leq R(t),\ t \geq 0$, where $z(t) = z(t,\ 0,\ z_0)$ is any solution of (2.3) existing on $[0,\ \infty)$.

Remark 1. If we have some smoothness conditions of g to guarantee the existence and uniqueness of solutions for (2.5), then $R(t)$ is exactly the unique solution of (2.5).

Remark 2. In Lemma 2.2, if the directions of the inequalities in (2.5) are reversed, that is,

$$\begin{array}{l} D^+V(t,\ z(t)) \geq g(t,\ V(t,\ z(t))), \quad t \neq kT, \\ V(t,\ z(t^+)) \geq \varphi_n(V(t,\ z(t))), \quad t = kT, \end{array}$$

then $V(t,\ z(t)) \geq r(t),\ t \geq 0$, where $r(t)$ is the minimal solution of (2.6) on $[0,\ \infty)$.

Lemma 2.3. *There exists a constant $M > 0$ such that $S(t) \leq M$ and $x(t) \leq M$ for each positive solution $(S(t),\ x(t)$ of system (2.3) with sufficiently large t.*

Proof. Suppose $(S(t),\ x(t))$ is any solution of system (2.3). Let $W(t) = \delta S(t) + e^{Q\omega}x(t) + \int_{t-\omega}^{t} p(s(\theta))x(\theta)d\theta$. By a simple computation, we are easy to obtain that there exists a constant $\beta > 0$ such that

$$D^+W(t,\ z(t)) \geq -QW(t) + \beta, \quad t \in (kT,\ (k+1)T],$$

and

$$W(kT^+) = W(kT) + \delta\tau S^0.$$

According to Theorem 1.4.1 in [14], we derive

$$W(t) < W(0)e^{-Qt} + \int_0^t \beta e^{-Q(t-s)} ds + \Sigma_{0<kT<t} \delta\tau S^0 e^{-Q(t-kT)}$$
$$\to \frac{\beta}{Q} + \frac{\delta\tau S^0 e^{QT}}{e^{QT}-1} \quad \text{as} \quad t \to \infty.$$

Let $M \triangleq \frac{\beta}{Q} + \frac{\delta\tau S^0 e^{QT}}{e^{QT}-1}$. In view of the definition of $W(t)$ we can easily see that each positive solution of system (2.3) is uniformly ultimately bounded. □

Lemma 2.4 (see Halany [15]) *Let t_0 be a real number and τ_0 be a nonnegative number. If $u : [t_0 - \tau_0, \infty) \to [0, \infty)$ satisfies*

$$m'(t) \le -\rho m(t) + \rho_0 \left[\sup_{t-\tau_0 \le s \le t} m(s) \right] \quad \text{for} \quad t \ge t_0,$$

and if $\rho > \rho_0 > 0$, then there exist positive numbers μ and γ such that

$$m(t) \le \mu e^{-\gamma(t-t_0)} \quad \text{for} \quad t \ge t_0.$$

Lemma 2.5. *Consider the following system*

$$\begin{aligned} &U'(t) = -QU(t) - \Gamma, \quad t \ne kT, \\ &U(t^+) = U(t) + \tau S^0, \quad t = kT. \end{aligned} \tag{2.7}$$

Then system (2.7) has a unique $T-$periodic solution given by

$$\tilde{U}(t) = (U^* + \frac{\Gamma}{Q})e^{-Q(t-kT)} - \frac{\Gamma}{Q}, \qquad kT < t \le (k+1)T, \quad k \in N,$$

which is globally asymptotically stable, where $U^* = \frac{\tau S^0 - \frac{\Gamma}{Q}(1-e^{-QT})}{1-e^{-QT}}$.

By using the discrete dynamical system determined by the stroboscopic map, we can easily obtain the result of Lemma 2.5. The proof of Lemma 2.5 is similar to the proof of Lemma 2.1 in the literature [7].

3. Global Attractivity of the Microorganism-Eradication Periodic Solution

From the Lemma 2.5, we can see that system (2.3) has a microorganism-eradication periodic solution $(\tilde{S}(t),\ 0)$, where

$$\tilde{S}(t) = S^* e^{-Q(t-kT)} \quad \text{and} \quad S^* = \frac{\tau S^0}{1-e^{-QT}}, \quad kT < t \le (k+1)T,\ k \in N.$$

In this section, we establish the global attractivity condition for the microorganism-eradication periodic solution $(\tilde{S}(t),\ 0)$ of system (2.3).

Theorem 3.1. *The microorganism-eradication periodic solution* $(\tilde{S}(t),\ 0)$ *of system (2.3) is globally attractive provided that*

$$\tau < \tau_* \triangleq \frac{(1-e^{-QT})p^{-1}(Qe^{Q\omega})}{S^0}.$$

Proof. Since $\tau < \tau_*$, we can choose $\epsilon > 0$ sufficiently small such that

$$Q - e^{-Q\omega} p\left(\frac{\tau S^0}{1-e^{-QT}} + \epsilon\right) > 0. \tag{3.1}$$

It follows from the first equation of (2.3) that $S'(t) \le -QS(t)$. From Lemmas 2.2, 2.3 and 2.5, we have that for the given $\epsilon > 0$ there exists an integer $k_1 > 0$ such that

$$S(t) < \tilde{S}(t) + \epsilon \le \frac{\tau S^0}{1-e^{-QT}} + \epsilon \triangleq S^M, \quad kT < t \le (k+1)T,\ k > k_1. \tag{3.2}$$

Further, from the second equation of system (2.3), we know that (3.2) implies

$$x'(t) \le -Qx(t) + e^{-Q\omega} p(S^M) x(t-\omega), \quad t > kT + \omega,\ k > k_1.$$

Consider the following comparison differential system:

$$y'(t) = -Qy(t) + e^{-Q\omega} p(S^M) y(t-\omega), \quad t > kT + \omega,\ k > k_1. \tag{3.3}$$

From (3.1), we have $Q > e^{-Q\omega} p(S^M)$. According to Lemma 2.4 we have $\lim_{t\to\infty} y(t) = 0$.

Let $(S(t),\ x(t))$ be the solution of system (2.3) with initial conditions (2.4) and $S(0^+) = \bar{S},\ x(0^+) = \bar{x}$, $y(t)$ be the solution of system (3.3) with initial value $y(0^+) = \bar{x}$. In view of the comparison theorem, we have $\limsup_{t\to\infty} x(t) \leq \limsup_{t\to\infty} y(t) = 0$. Incorporating into the positivity of $x(t)$, we know that $\lim_{t\to\infty} x(t) = 0$. Therefore, for any $\epsilon_1 > 0$ $(0 < \epsilon_1 < \frac{\tau S^0 \delta Q}{\alpha(1-e^{-QT})})$, there exists an integer $k_2 > k_1$ (where $k_2T > k_1T + \omega$) such that $x(t) < \epsilon_1$ for all $t > k_2T$.

For the first equation of system (2.3) and hypothesis (H), we have

$$S'(t) > -QS(t) - \frac{1}{\delta}\alpha\epsilon_1, \quad \text{for} \quad t > k_2T.$$

From Lemma 2.2 and Lemma 2.5, there exists an integer $k_3(> k_2)$ such that $S(t) \geq z(t)$ and $z(t) \to \tilde{S}(t)$ as $t > k_3T$, where $z(t)$ is the solution of

$$\begin{aligned} &z'(t) = -Qz(t) - \frac{\epsilon_1\alpha}{\delta}, \quad t \neq kT, \\ &z(t^+) = z(t) + \tau S^0, \quad t = kT. \end{aligned}$$

where

$$\tilde{z}(t) = (z^* + \frac{\epsilon_1\alpha}{\delta Q})e^{-Q(t-kT)} - \frac{\epsilon_1\alpha}{\delta Q}, \quad z^* = \frac{\tau S^0 - \frac{\epsilon_1\alpha}{\delta Q}(1-e^{-QT})}{1-e^{-QT}}$$

Therefore, we have $\tilde{z}(t) - \epsilon_1 < S(t) < \tilde{S}(t) + \epsilon$ for t large enough. Let $\epsilon,\ \epsilon_1 \to 0$, we have $z(t) \to \tilde{S}(t)$. Hence $S(t) \to \tilde{S}(t)$ as $t \to \infty$. □

According to Theorem 3.1 we can easily obtain the following results.

Corollary 3.1. *The the microorganism-eradication periodic solution* $(\tilde{S}(t),\ 0)$ *is globally attractive provided that* $\tau S^0 \geq p^{-1}(Qe^{Q\omega})$.

Corollary 3.2. *Assume that* $\tau S^0 < p^{-1}(Qe^{Q\omega})$. *Then the infection-free periodic solution* $(\tilde{S}(t),\ 0)$ *is globally attractive provided that* $T > T^*$, *where* $T^* = -\frac{1}{Q}\ln\left(1 - \frac{\tau S^0}{p^{-1}(Qe^{Q\omega})}\right)$.

Theorem 3.1 determines the global attractivity of (2.3) in Ω for the case $\tau < \tau^*$. Its epidemiological implication is that the microorganism vanishes in time owing to the insufficient biomass of microorganism of input substrate. Corollary 3.2 implies that the microorganism will disappear if the period of pulsing is longer enough.

4. Permanence of System (2.3)

In this section, our main purpose is to establish the conditions for the permanence of system (2.3).

Theorem 4.1. *System (2.3) is permanent if $\tau > \tau^*$, where $\tau^* \triangleq e^{QT}\tau_* = \frac{(e^{QT}-1)p^{-1}(Qe^{Q\omega})}{S^0}$.*

Proof. Note that the second equation of (2.3) can be rewritten as

$$x'(t) = -Qx(t) + e^{-Q\omega}p(S(t))x(t) - e^{-Q\omega}\frac{d}{dt}\int_{t-\omega}^{t} p(S(\theta))x(\theta)d\theta. \tag{4.1}$$

Let us consider any positive solution $(S(t),\ x(t))$ of system (2.3) with initial values $S_0 > 0$ and $x_0 > 0$. According to this solution, we define

$$V(t) = x(t) + e^{-Q\omega}\int_{t-\omega}^{t} p(S(\theta))x(\theta)d\theta.$$

According to (4.1), we calculate the derivative of V along the solutions of (2.3)

$$V'(t) = (e^{-Q\omega}p(S(t)) - Q)x(t). \tag{4.2}$$

Since $\tau > \tau^*$, we can choose sufficiently small x^* and ξ such that

$$e^{-Q\omega}p\left(\frac{\tau S^0 - \frac{x^*\alpha}{\delta Q}(e^{QT}-1)}{e^{QT}-1} - \xi\right) - Q > 0. \tag{4.3}$$

We claim that for any $t_0 > 0$, it is impossible that $x(t) < x^* < \frac{\tau S^0 \delta Q}{\alpha(1-e^{-QT})}$ for all $t \geq t_0$. Suppose that the claim is not valid. Then there is a $t_0 > 0$ such that $x(t) < x^*$ for all $t \geq t_0$. It follows from the first equation of (2.3), that for $t \geq t_0$,

$$\begin{aligned} S'(t) &= -QS(t) - \frac{1}{\delta}p(S(t))x(t) \\ &\geq -QS(t) - \frac{x^*\alpha}{\delta}. \end{aligned}$$

Consider the following comparison impulsive system for $t \geq t_0$,

$$\begin{aligned} &u'(t) = -Qu(t) - \frac{x^*\alpha}{\delta}, \quad t \neq kT, \\ &u(t^+) = u(t) + \tau S^0, \quad t = kT, \\ &u(0^+) = S_0. \end{aligned} \tag{4.4}$$

In view of Lemma 2.5, we obtain that

$$\tilde{u}(t) = \left(u^* + \frac{x^*\alpha}{\delta Q}\right) e^{-Q(t-kT)} - \frac{x^*\alpha}{\delta Q}, \quad kT < t \le (k+1)T$$

is the globally asymptotically stable positive periodic solution of (4.4), where

$$u^* = \frac{\tau S^0 - \frac{x^*\alpha}{\delta Q}(1 - e^{-QT})}{1 - e^{-QT}}.$$

By comparison theorem in impulsive differential equation, we know that, there exists t_1 $(> t_0 + \omega)$ such that the following inequality holds true for $t \ge t_1$

$$\begin{aligned} S(t) &> \tilde{u}(t) - \xi > u^* e^{-QT} - \frac{x^*\alpha}{\delta Q}(1 - e^{-QT}) - \xi \\ &= \frac{\tau S^0 - \frac{x^*\alpha}{\delta Q}(e^{QT} - 1)}{e^{QT} - 1} - \xi \triangleq \sigma \quad \text{for} \quad t \ge t_1. \end{aligned} \tag{4.5}$$

It follows from (4.3) that

$$e^{-Q\omega} p(\sigma) - Q > 0.$$

Moreover,

$$V'(t) > (e^{-Q\omega} p(\sigma) - Q)x(t), \quad \text{for} \quad t \ge t_1. \tag{4.6}$$

Denote

$$x_l \triangleq \min_{t \in [t_1,\ t_1+\omega]} x(t).$$

We will show that $x(t) \ge x_l$ for all $t \ge t_1 > t_0$. Suppose the contrary. Then there is a $T_0 \ge 0$ such that $x(t) \ge x_l$ for $t_1 \le t \le t_1+\omega+T_0$ and $x(t_1+\omega+T_0) = x_l$ and $x'(t_1 + \omega + T_0) \le 0$.

From the second equation of system (2.3), we have

$$\begin{aligned} x'(t_1 + \omega + T_0) &> -Qx_l + e^{Q\omega} p(S(t_1 + T_0))x(t_1 + T_0) \\ &> (-Q + e^{-Q\omega} p(\sigma))x_l > 0 \end{aligned}$$

which leads to a contradiction. Therefore

$$x(t) \ge x_l \quad \text{for all} \quad t \ge t_1. \tag{4.7}$$

As a consequence, (4.6) and (4.7) lead to

$$V'(t) > (e^{-Q\omega} p(\sigma) - Q)x_l > 0 \quad \text{for} \quad t \ge t_1$$

which implies that $V(t) \to \infty$ as $t \to \infty$. This contradicts $V(t) \leq M(1 + \alpha\omega e^{-Q\omega})$. Hence, the claim is proved.

By the claim, we are left to consider two cases. First, $x(t) \geq x^*$ for t large enough. Second, $x(t)$ oscillates about x^* for t large enough. Define $q = \min\left\{\frac{x^*}{2},\ x^* e^{-Q\omega}\right\}$.

In the following, we hope to show that $x(t) \geq q$ for t large enough. The conclusion is evident in the first case. For the second case, let $t^* > 0$ and $\gamma > 0$ satisfy

$$x(t^*) = x(t^* + \gamma) = x^*,$$

and

$$x(t) < x^* \quad \text{for} \quad t^* < t < t^* + \gamma, \tag{4.8}$$

where t^* is sufficiently large such that

$$S(t) > \sigma \quad \text{for} \quad t^* < t < t^* + \gamma. \tag{4.9}$$

Moreover, $x(t)$ is uniformly continuous since the positive solutions of (2.3) are ultimately bounded and $x(t)$ is not effected by impulses. Hence, there exists a η $(0 < \eta < \omega$, and η is independent of the choice of $t^*)$ such that $x(t) > \frac{x^*}{2}$ for $t^* \leq t \leq t^* + \eta$.

If $\gamma \leq \eta$, there is nothing to prove. Let us consider the case $\eta < \gamma$. There two subcases to consider:

(i) If $\eta < \gamma \leq \omega$, it follows from (4.7), (4.8) and (4.9) that $x'(t) \geq -Qx(t)$ for $t^* + \eta \leq t \leq t^* + \gamma$. Hence $x(t) \geq x^* e^{-Q\omega}$ for $t^* + \eta \leq t \leq t^* + \gamma$.

(ii) If $\gamma > \omega$, by (4.7), we can easily obtain $x(t) \geq q$ for $t \in [t^*, t^* + \omega]$. Then, proceeding exactly as the proof for above claim, we see that $x(t) \geq q$ for $t \in [t^* + \omega, t^* + \gamma]$. Since this kind of interval $[t^*, t^* + \gamma]$ is chosen in an arbitrary way (we only need t^* to be large), we conclude that $x(t) \geq q$ for t large enough in the second case.

In view of our above discussions, the choices of q is independent of the positive solution, and we have proved that any positive solution of (2.3) satisfies $x(t) \geq q$ for t large enough.

From system (2.3) and Lemma 2.5, we have

$$\begin{aligned} &S'(t) > -QS(t) - \frac{\alpha M}{\delta}, \quad t \neq kT \\ &S(t^+) = S(t) + \tau S^0, \quad t = kT. \end{aligned}$$

According to Lemma 2.5, we are easy to show that there exists a positive constant r such that $S(t) \geq r$ for t large enough.

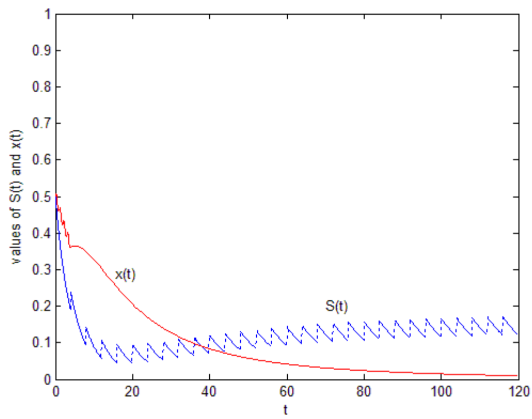

Figure 1: This figure shows that movement paths of S and x as functions of time t, $\tau_* = 2.094$. The microorganism will extinct. Parameters are $\mu_m = 0.2,\ K_m = 1,\ Q = 0.08,\ \delta = 0.4,\ \omega = 1,\ T = 4,\ S^0 = 0.1,\ \tau = 0.5$

Finally, let $\Omega_0 = \{(S,\ x)|\ r \le S \le M,\ q \le x \le M\}$. We know that Ω_0 is a bounded compact region in R_+^2 which has positive distance from coordinate hyperplanes. Ω_0 is a global attractor in Ω, and of course, every solution of system (2.3) with initial conditions (2.4) will eventually enter and remain in region Ω_0. Therefore, system (2.3) is permanent. $\square$

From Theorem 4.1, we also easily obtain the following result.

Corollary 4.1. *Assume that $\tau S^0 < p^{-1}(Qe^{Q\omega})$. Then system (2.3) is permanence provided that $T < T_*$, where $T_* = \frac{1}{Q}\ln\left[1 + \frac{\tau S^0}{p^{-1}(Qe^{Q\omega})}\right]$.*

5. Numerical Simulation and Summary

We have analyzed the delayed chemostat model with impulsive perturbation on the nutrient concentration. Two thresholds have been established, one for global attractivity of the microorganism-eradication periodic solution and one for permanence of the chemostat system.

From Theorem 3.1, we know that the microorganism-eradication periodic solution of impulsive system (2.3) is global attractive in Ω for the case $\tau < \tau_*$. Its epidemiological implication is that the microorganism is extinct. Theorem

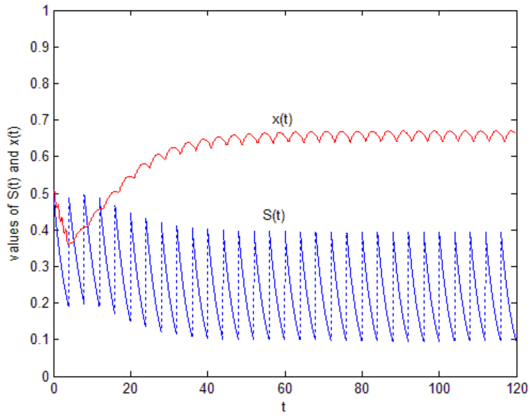

Figure 2: This figure shows that movement paths of S and x as functions of time t, $\tau^* = 2.8837$. The microorganism persists. Parameters are $\mu_m = 0.2,\ K_m = 1,\ Q = 0.08,\ \delta = 0.4,\ \omega = 1,\ T = 4,\ S^0 = 0.1,\ \tau = 3$

4.1 determines the permanence of (2.3) in Ω for the case $\tau > \tau^*$, thus the microorganism will persist. But for $\tau \in [\tau_*,\ \tau^*]$, the dynamical behaviors of model (2.3) have not been studied in this paper. In the following, we verify our main results by numerical simulation. Throughout this section, the microbial growth function $p(S) = \frac{\mu_m S}{K_m + S}$ is used. Let $\mu_m = 0.2,\ K_m = 1,\ Q = 0.08,\ \delta = 0.4,\ \omega = 1,\ T = 4,\ S^0 = 0.1$, then $\tau_* = 2.094$ and $\tau^* = 2.8837$. Figure 1 ($\tau = 0.5$) shows that the microorganism will extinct, whereas Figure 2 ($\tau = 3$) shows that the microorganism will persist.

It is well known that, as pointed out by Cushing [16], it is quite difficult to determine the stability of periodic solutions for delay systems. Moreover, delays in many models can destabilize an equilibrium and thus lead to periodic solutions by Hopf bifurcation and chaos [17]. The issues of the behavior of the model in this paper would be left as our future consideration.

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