BOUNDED SOLUTIONS OF THE NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract: In this paper some bounded solutions of the nonlinear differential system
\[ x_i = \sum_{j=1}^{n} f_{ij}(t, x)x_j, \quad i = 1, 2, \ldots, n, \]
where \( f_{ij}(t, x) \in C_0(D \equiv (t_0, \infty) \times R^n, R) \), for all \( i, j = 1, 2, \ldots, n \) are investigated using transformation in generalized cylindrical coordinates.

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1. Generalized Cylindrical Coordinates

In this work we investigate bounded of solutions of the nonlinear differential system
\[ x_i = \sum_{j=1}^{n} f_{ij}(t, x)x_j, \quad i = 1, 2, \ldots, n, \quad (1) \]
where \( f_{ij}(t, x) \in C_0(D \equiv (t_0, \infty) \times R^n, R) \), for all \( i, j = 1, 2, \ldots, n \). We define the generalized cylindrical coordinates

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where \( u_i(t), r(t), v(t) \in C_1((t_0, \infty), R), i = 1, 2, \ldots, n - 2 \). We assume that the arbitrary solution \( x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1) exists on the interval \((t_0, \infty)\), where \( r(t) \geq 0 \) is its polar function and \( u_i(t) \) are its angle functions, \( i = 1, 2, \ldots, n - 2 \).

Let \( x(t) \) be the arbitrary nontrivial solution of system (1), hence \( r(t) > 0 \) for all \( t \in (t_0, \infty) \). Then considering (2) for all \( t \in (t_0, \infty) \),

\[
\sum_{i=1}^{n} x_i^2(t) = r^2(t)(1 + v^2(t))
\]

holds. Consequently the boundary of the solutions of system (1) depends only on the functions \( r(t), v(t) \). The oscillatory properties of the solutions of system (1) depend on the conduct of the angle functions \( u_i(t), i = 1, 2, \ldots, n - 2, t \in (t_0, \infty) \).

Let \( \emptyset \neq D^0 \subset D \) be an open set. We denote \( g_i(t, u) = \sum_{j=1}^{n} f_{ij}(t, u)u_j, i = 1, 2, \ldots, n \). We suppose that exist and are continuous partial derivatives \( \frac{\partial g_i(t, u)}{\partial u_j}, i, j = 1, 2, \ldots, n, \) on \( D^0 \subset D \). Then by (see [1]) one and only one integral curve of system (1) passes through every point \((t_0, u_0^1, u_0^2, \ldots, u_0^n) \in D^0 \).

Derivating each of equation (2) we get (for brevity we shall use \( C_i(t) \) and \( S_i(t) \) instead of \( \cos u_i(t) \) and \( \sin u_i(t) \))

\[
a) \quad x_1'(t) = r'(t)C_1(t) - r(t)u_1'(t)S_1(t);
\]

\[
b) \quad x_k'(t) = r'(t)C_k(t) \prod_{i=1}^{k-1} S_i(t) + \sum_{i=1}^{k-1} r(t)u_i'(t)C_i(t)C_k(t) \prod_{j=1, j \neq i}^{k-1} S_j(t)
\]

\[
- r(t)u_k'(t)S_k(t) \prod_{j=1}^{k-1} S_j(t), \quad \text{where } k = 2, 3, \ldots, n - 2;
\]

\[
c) \quad x_{n-1}'(t) = r'(t) \prod_{i=1}^{n-2} S_i(t) + \sum_{i=1}^{n-2} r(t)u_i'(t)C_i(t) \prod_{j=1, j \neq i}^{n-2} S_j(t);
\]

\[
d) \quad x_n'(t) = r'(t)v(t) + r(t)v'(t).
\]
Multiplying equation (4a) by expression $C_1(t)$, multiplying equation (4b) successively by $C_k(t)$ for $k = 2, 3, \ldots, n - 2$, multiplying equation (4c) by $\prod_{i=1}^{n-2} S_i(t)$, and after the addition of the obtained equations we get

$$r(t) = a(t, x) + \sum_{i=2}^{n-2} \sum_{j=1}^{i-1} \prod_{r=1}^{i} S_r(t) + x'_{n-1}(t) \prod_{i=1}^{n-2} S_i(t). \quad (5)$$

Let $n > 3$. Multiplying (see [1]) equation (4a) by expression $(-S_1(t))$, multiplying the first equation in (4b) by $C_1(t)C_2(t)$ for $k = 2$, multiplying the equations in (4b) successively by $C_1(t)C_k(t)\prod_{i=1}^{k-1} S_i(t)$ for $k = 3, 4, \ldots, n - 2$ multiplying equation (4c) by $C_1(t)\prod_{i=2}^{n-2} S_i(t)$ and after the addition of the obtained equations we have

$$r(t)u_1(t) = -x_1(t)S_1(t) + x_2(t)C_1(t)C_2(t) + \sum_{i=3}^{n-2} \sum_{j=2}^{i-1} \prod_{r=1}^{i} S_r(t) + x'_{n-1}(t)C_1(t)\prod_{i=2}^{n-2} S_i(t). \quad (6)$$

We apply the method listed above for equation (4b) and (4c). Let $n > 4$. Multiplying equation (4b) by expression $(-S_2(t))$ for $k = 2$, multiplying the equations (4b) by $C_2(t)C_3(t)$ for $k = 3$, multiplying (see [3]) equations (4b) successively by $C_2(t)C_k(t)\prod_{i=3}^{k-1} S_i(t)$ for $k = 4, 5, \ldots, n - 2$, multiplying equation (4c) by $C_2(t)\prod_{i=3}^{n-2} S_i(t)$ and after the addition of the obtained equations we have

$$r(t)u_2(t)S_1(t) = -x_2(t)S_2(t) + x_3(t)C_2(t)C_3(t) + \sum_{i=4}^{n-2} \sum_{j=3}^{i-1} \prod_{r=1}^{i} S_r(t) + x'_{n-1}(t)C_2(t)\prod_{i=3}^{n-2} S_i(t). \quad (7)$$

2. Bounded Solutions

Theorem 1. Let $(x_1(t), x_2(t), \ldots, x_n(t))$ be a non-trivial solution of system (1). Allowing that for coefficients $f_{ij}(t, x)$, $i, j = 1, 2, \ldots, n$ of differential system (1) the statement $f_{ii}(t, x) = a(t, x)$; $f_{ij}(t, x) = -f_{ji}(t, x)$, for $i \neq j, i \neq n, j \neq n$; $f_{in}(t, x) = f_{ni}(t, x)$, for $i \neq n$; are valid on the domain $D^0 \subset D$. Then functions $r(t), v(t), u_i(t)$, $i = 1, 2, \ldots, n - 2$, satisfy the following (We denote $\sum_{i=i_0}^{i_1}(\cdot) = 0$, $\prod_{i=i_0}^{i_1}(\cdot) = 1$, if $i_0 > i_1$.)

a) \[
r'/(r(t) = a(t, x) + (f_{1n}(t, x)C_1(t) + \sum_{i=2}^{n-2} f_{in}(t, x)C_i(t) \prod_{r=1}^{i-1} S_r(t) \prod_{i=1}^{n-2} S_i(t). \quad (8)
\]
\[ + f_{n-1,n}(t, x) \prod_{r=1}^{n-2} S_r(t) v(t); \]

\[ b) \ u_1(t) = - \left( f_{12}(t, x) C_2(t) + \sum_{j=3}^{n-2} f_{1j}(t, x) C_j(t) \prod_{r=2}^{j-1} S_r(t) \right) v(t); \]

\[ + f_{1,n-1}(t, x) \prod_{r=2}^{n-2} S_r(t) + (f_{1n}(t, x) S_n(t) + f_{2n}(t, x) C_1(t) C_2(t) \]

\[ + \sum_{i=3}^{n-1} f_{1i}(t, x) C_i(t) \prod_{r=2}^{i-1} S_r(t) + f_{n-1,n}(t, x) C_1(t) \prod_{r=2}^{n-2} S_r(t) \right) v(t); (8) \]

\[ c) \ u_k(t) \prod_{r=1}^{n-2} S_r(t) = - \left( \sum_{j=k+1}^{n-2} f_{kj}(t, x) C_j(t) \prod_{r=1}^{j-1} S_r(t) + f_{kn}(t, x) \times \prod_{r=1, r \neq k}^{n-2} S_r(t) \right) + \left( f_{k1}(t, x) S_k(t) - f_{k+1,1}(t, x) C_k(t) C_{k+1}(t) \right) \]

\[ - \sum_{i=k+2}^{n-2} f_{i1}(t, x) C_k(t) C_i(t) \prod_{r=2, r \neq k}^{i-1} S_r(t) - f_{n-1,1}(t, x) C_k(t) \]

\[ \times \prod_{r=1, r \neq k}^{n-2} S_r(t) \right) C_1(t) + \sum_{j=2}^{k-1} \left( f_{kj}(t, x) S_k(t) - f_{k+1,j}(t, x) C_k(t) C_{k+1}(t) \right) \]

\[ - \sum_{i=k+2}^{n-2} f_{ij}(t, x) C_k(t) C_i(t) \prod_{r=k+1}^{i-1} S_r(t) - f_{n-1,j}(t, x) C_k(t) \]

\[ \times \prod_{r=k+1}^{n-2} S_r(t) \right) C_j(t) \prod_{r=1}^{j-1} S_r(t) + \left( f_{kn}(t, x) S_k(t) - f_{k+1,n}(t, x) C_k(t) \right. \]

\[ \times C_{k+1}(t) - \sum_{i=k+2}^{n-2} f_{in}(t, x) C_k(t) C_i(t) \prod_{r=k+1}^{i-1} S_r(t) - f_{n-1,n}(t, x) C_k(t) \]

\[ \times \prod_{r=k+1}^{n-2} S_r(t) \right) v(t), \text{ where } k = 2, 3, \ldots, n - 2; \]

\[ d) \ v'(t) = (1 - v^2(t)) \left( f_{1n}(t, x) C_1(t) + \sum_{j=2}^{n-2} f_{jn}(t, x) C_j(t) \prod_{r=1}^{j-1} S_r(t) \right. \]
\[ + f_{n-1,n}(t, x) \prod_{r=1}^{n-2} S_r(t) \].

**Proof.** According to the first assumption of Theorem 1 it follows that \( r(t) \neq 0 \) for all \( t \in (-t_0, \infty) \). If we put into (5) expression (1) instead of \( x_i(t) \), \( i = 1, 2, \ldots, n \), and expression (2) instead of \( x_i(t) \), \( i = 1, 2, \ldots, n \), after arrangement we get

\[
\frac{r'(t)}{r(t)} = (f_{11}(t, x)C_1(t) + \sum_{j=2}^{n-2} f_{1j}(t, x)C_j(t) \prod_{r=1}^{j-1} S_r(t) \\
+ f_{1,n-1}(t, x) \prod_{r=1}^{n-2} S_r(t) + f_{1n}(t, x)v(t))C_1(t) + \sum_{i=2}^{n-2} (f_{i1}(t, x)C_1(t) \\
+ \sum_{j=2}^{n-2} f_{ij}(t, x)C_j(t) \prod_{r=1}^{j-1} S_r(t) + f_{i,n-1}(t, x) \prod_{r=1}^{n-2} S_r(t) \\
+ f_{in}(t, x)v(t))C_i(t) \prod_{r=1}^{i-1} S_r(t) + (f_{n-1,1}(t, x)C_1(t) \\
+ \sum_{j=2}^{n-2} f_{n-1,j}(t, x)C_j(t) \prod_{r=1}^{j-1} S_r(t) + f_{n-1,n-1}(t, x) \prod_{r=1}^{n-2} S_r(t) \\
+ f_{n-1,n}(t, x)v(t)) \prod_{r=1}^{n-2} S_r(t).
\]

From this equation with the assumptions of Theorem 1 (a) follows. We can prove relations (b), (c), (d) analogously. Theorem 1 is proved. \( \Box \)

**Definition 2.** Let \( i \) be a fixed integer, \( 1 \leq i \leq n \). We say that the solution \( (x_1(t), x_2(t), \ldots, x_n(t)) \) of system (1) is \( i \)-bounded, if for all \( t \geq a \) it holds \( |x_i(t)| \leq K \), where \( K > 0 \) is a constant. System (1) is \( i \)-bounded, if all its solutions are \( i \)-bounded. Otherwise, the considered solution of system (1) is \( i \)-unbounded.

**Theorem 3.** Let coefficients \( f_{ij}(t, x), i, j = 1, 2, \ldots, n \), of (1) satisfy the conditions of Theorem 1 on the domain \( D^0 \subset D \). If

\[
\sup_{X \in \mathbb{R}^n} \int_{t_0}^{\infty} \left( |a(t, X)| + \sum_{i=1}^{n-1} |f_{in}(t, X)| \right) dt < +\infty, \ t_0 \geq a
\]
then there exist two different nontrivial solutions of system (1), which are n-bounded.

Proof. Let the assumptions of Theorem 1.2 be fulfilled. From equation (9d) it follows that \( v_1(t) = -1, \ v_2(t) = 1, \ t \in \langle t_0, \infty \rangle \) are its solutions. Integrating equation (9a) from \( t_0 \) to \( t, \ a \leq t_0 \leq t \), for \( v(t) = 1 \) it follows

\[
\left| \int_{t_0}^{t} \frac{r(s)}{r(s)} ds \right| \leq \int_{t_0}^{t} \left( |a(s, x(s))| + |f_{1n}(s, x(s))C_1(s)\right.
\]

\[
+ \sum_{i=2}^{n-2} f_{in}(s, x(s))C_i(s) \prod_{r=1}^{i-1} S_r(s) + f_{n-1,n}(s, x(s)) \prod_{r=1}^{n-2} S_r(s) \left| \right) ds
\]

\[
\leq \int_{t_0}^{t} \left( |a(s, x(s))| + \sum_{i=1}^{n-1} |f_{in}(s, x(s))| \right) ds.
\]

Let \( t \to \infty \). Then for function \( r(t) \)

\[
\left| \int_{t_0}^{\infty} \frac{r'(t)}{r(t)} dt \right| \leq \int_{t_0}^{\infty} \left( |a(t, x(t))| + \sum_{i=1}^{n-1} |f_{in}(t, x(t))| \right) dt
\]

\[
\leq \sup_{X \in \mathbb{R}^n} \int_{t_0}^{\infty} \left( |a(t, X)| + \sum_{i=1}^{n-1} |f_{in}(t, X)| \right) dt
\]

holds. Then according to the assumption of Theorem 1 it follows that function \( r(t) \) is bounded on the interval \( < a, \infty \). If now we use the relations (2) and (3), we have for \( v(t) = 1 \) the solution \((x_1(t), x_2(t), \ldots, x_n(t))\) of system (1), which is n-bound. Analogously we could find the solution of system (1) in the case of \( v(t) = -1 \). Thus Theorem 3 is proved.

References

