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BOUNDED SOLUTIONS OF THE NONLINEAR DIFFERENTIAL SYSTEMS

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Abstract: In this paper some bounded solutions of the nonlinear differential system $x_i^{,} = \sum_{j=1}^{n} f_{ij}(t,x)x_j,\ i = 1,2,\dots,n$, where $f_{ij}(t,x) \in C_0(D \equiv \langle t_0,\infty) \times R^n, R)$, for all $i,j = 1,2,\dots,n$ are investigated using transformation in generalized cylindrical coordinates.

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1. Generalized Cylindrical Coordinates

In this work we investigate bounded of solutions of the nonlinear differential system

$$x_i^{,} = \sum_{j=1}^{n} f_{ij}(t,x)x_j, \quad i = 1,2,\dots,n, \tag{1}$$

where $f_{ij}(t,x) \in C_0(D \equiv \langle t_0,\infty) \times R^n, R)$, for all $i,j = 1,2,\dots,n$. We define the generalized cylindrical coordinates

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$$\begin{aligned} x_1 &= r\cos u_1;\\ x_k &= r\left(\prod_{i=1}^{k-1}\sin u_i\right)\cos u_k,\ \ k=2,3,\dots,n-2;\\ x_{n-1} &= r\left(\prod_{i=1}^{n-2}\sin u_i\right);\\ x_n &= rv, \end{aligned} \tag{2}$$

where $u_i(t), r(t), v(t) \in C_1(\langle t_0,\infty), R), i = 1,2,\dots,n-2$. We assume that the arbitrary solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))$ of system (1) exists on the interval $\langle t_0,\infty)$, where $r(t) \geq 0$ is its polar function and $u_i(t)$ are its angle functions, $i = 1,2,\dots,n-2$.

Let $x(t)$ be the arbitrary nontrivial solution of system (1), hence $r(t) > 0$ for all $t \in \langle t_0,\infty)$. Then considering (2) for all $t \in \langle t_0,\infty)$,

$$\sum_{i=1}^{n} x_i^2(t) = r^2(t)\big(1+v^2(t)\big) \tag{3}$$

holds. Consequently the boundary of the solutions of system (1) depends only on the functions $r(t), v(t)$. The oscillatory properties of the solutions of system (1) depend on the conduct of the angle functions $u_i(t), i = 1,2,\dots,n-2, t \in \langle t_0,\infty)$.

Let $\emptyset \neq D^0 \subset D$ be an open set. We denote $g_i(t,u) = \sum_{j=1}^{n} f_{ij}(t,u)u_j$, $i = 1,2,\dots,n$. We suppose that exist and are continuous partial derivatives $\frac{\partial g_i(t,u_1,\dots,u_n)}{\partial u_j}$; $i,j = 1,2,\dots,n$, on $D^0 \subset D$. Then by (see [1]) one and only one integral curve of system (1) passes through every point $(t_0, u_1^0, u_2^0, \dots, u_n^0) \in D^0$.

Derivating each of equation (2) we get (for brevity we shall use $C_i(t)$ and $S_i(t)$ instead of $\cos u_i(t)$ and $\sin u_i(t)$)

$$\begin{aligned} a)\quad & x_1'(t) = r'(t)C_1(t) - r(t)u_1'(t)S_1(t);\\ b)\quad & x_k'(t) = r'(t)C_k(t)\prod_{i=1}^{k-1}S_i(t) + \sum_{i=1}^{k-1} r(t)u_i'(t)C_i(t)C_k(t)\prod_{j=1,j\neq i}^{k-1}S_j(t)\\ & -r(t)u_k'(t)S_k(t)\prod_{j=1}^{k-1}S_j(t),\ \text{where k} = 2,3,\dots,\text{n}-2;\\ c)\quad & x_{n-1}'(t) = r'(t)\prod_{i=1}^{n-2}S_i(t) + \sum_{i=1}^{n-2} r(t)u_i'(t)C_i(t)\prod_{j=1,j\neq i}^{n-2}S_j(t);\\ d)\quad & x_n'(t) = r'(t)v(t) + r(t)v'(t). \end{aligned} \tag{4}$$

Multiplying equation (4a) by expression $C_1(t)$, multiplying equation (4b) successively by $C_k(t)\prod_{i=1}^{k-1} S_i(t)$ for $k = 2,3,\ldots,n-2$, multiplying equation (4c) by $\prod_{i=1}^{n-2} S_i(t)$, and after the addition of the obtained equations we get

$$r\text{'}(t) = x\text{'}_1(t)C_1(t) + \sum_{i=2}^{n-2} x\text{'}_i(t)C_i(t)\prod_{j=1}^{i-1} S_j(t) + x\text{'}_{n-1}(t)\prod_{i=1}^{n-2} S_i(t). \tag{5}$$

Let $n > 3$. Multiplying (see [1])equation (4a) by expression $(-S_1(t))$, multiplying the first equation in (4b) by $C_1(t)C_2(t)$ for $k = 2$, multiplying the equations in (4b) successively by $C_1(t)C_k(t)\prod_{i=2}^{k-1} S_i(t)$ for $k = 3,4,\ldots,n-2$ multiplying equation (4c) by $C_1(t)\prod_{i=2}^{n-2} S_i(t)$ and after the addition of the obtained equations we have

$$\begin{aligned} r(t)u\text{'}_1(t) &= -x\text{'}_1(t)S_1(t) + x\text{'}_2(t)C_1(t)C_2(t) \\ &\quad + \sum_{i=3}^{n-2} x\text{'}_i(t)C_1(t)C_i(t)\prod_{j=2}^{i-1} S_j(t) + x\text{'}_{n-1}(t)C_1(t)\prod_{i=2}^{n-2} S_i(t). \end{aligned} \tag{6}$$

We apply the method listed above for equation (4b) and (4c). Let $n > 4$. Multiplying equation (4b) by expression $(-S_2(t))$ for $k = 2$, multiplying the equations (4b) by $C_2(t)C_3(t)$ for $k = 3$, multiplying (see [3]) equations (4b) successively by $C_2(t)C_k(t)\prod_{i=3}^{k-1} S_i(t)$ for $k = 4,5,\ldots,n-2$, multiplying equation (4c) by $C_2(t)\prod_{i=3}^{n-2} S_i(t)$ and after the addition of the obtained equations we have

$$\begin{aligned} r(t)u\text{'}_2(t)S_1(t) &= -x\text{'}_2(t)S_2(t) + x\text{'}_3(t)C_2(t)C_3(t) \\ &\quad + \sum_{i=4}^{n-2} x\text{'}_i(t)C_2(t)C_i(t)\prod_{j=3}^{i-1} S_j(t) + x\text{'}_{n-1}(t)C_2(t)\prod_{i=3}^{n-2} S_i(t). \end{aligned} \tag{7}$$

2. Bounded Solutions

Theorem 1. *Let $(x_1(t), x_2(t),\ldots,x_n(t))$ be a non-trivial solution of system (1). Allowing that for coefficients $f_{ij}(t,x)$, $i,j = 1,2,\ldots,n$ of differential system (1) the statement $f_{ii}(t,x) = a(t,x)$; $f_{ij}(t,x) = -f_{ji}(t,x)$, for $i \neq j, i \neq n, j \neq n$; $f_{in}(t,x) = f_{ni}(t,x)$, for $i \neq n$; are valid on the domain $D^0 \subset D$. Then functions $r(t), v(t), u_i(t)$, $i = 1,2,\ldots,n-2$, satisfy the following (We denote $\sum_{i=i_0}^{i_1}(.) = 0$, $\prod_{i=i_0}^{i_1}(.) = 1$, if $i_0 > i_1$.)*

$$a)\quad \frac{r\text{'}(t)}{r(t)} = a(t,x) + \Big(f_{1n}(t,x)C_1(t) + \sum_{i=2}^{n-2} f_{in}(t,x)C_i(t)\prod_{r=1}^{i-1} S_r(t)$$

$$+f_{n-1,n}(t,x)\prod_{r=1}^{n-2}S_r(t)\big)v(t);$$

$$b)\quad u_1'(t)=-\big(f_{12}(t,x)C_2(t)+\sum_{j=3}^{n-2}f_{1j}(t,x)C_j(t)\prod_{r=2}^{j-1}S_r(t)$$

$$+f_{1,n-1}(t,x)\prod_{r=2}^{n-2}S_r(t)\big)+\big(f_{1n}(t,x)S_1(t)+f_{2n}(t,x)C_1(t)C_2(t)$$

$$+\sum_{i=3}^{n-2}f_{in}(t,x)C_1(t)C_i(t)\prod_{r=2}^{i-1}S_r(t)+f_{n-1,n}(t,x)C_1(t)\prod_{r=2}^{n-2}S_r(t)\big)v(t);\quad(8)$$

$$c)\quad u_k'(t)\prod_{r=1}^{k-1}S_r(t)=-\big(\sum_{j=k+1}^{n-2}f_{kj}(t,x)C_j(t)\prod_{r=1,r\neq k}^{j-1}S_r(t)+f_{kn}(t,x)$$

$$\times\prod_{r=1,r\neq k}^{n-2}S_r(t)\big)+\big(f_{k1}(t,x)S_k(t)-f_{k+1,1}(t,x)C_k(t)C_{k+1}(t)$$

$$-\sum_{i=k+2}^{n-2}f_{i1}(t,x)C_k(t)C_i(t)\prod_{r=k+1}^{i-1}S_r(t)-f_{n-1,1}(t,x)C_k(t)$$

$$\times\prod_{r=k+1}^{n-2}S_r(t)\big)C_1(t)+\sum_{j=2}^{k-1}\big(f_{kj}(t,x)S_k(t)-f_{k+1,j}(t,x)C_k(t)C_{k+1}(t)$$

$$-\sum_{i=k+2}^{n-2}f_{ij}(t,x)C_k(t)C_i(t)\prod_{r=k+1}^{i-1}S_r(t)-f_{n-1,j}(t,x)C_k(t)$$

$$\times\prod_{r=k+1}^{n-2}S_r(t)\big)C_j(t)\prod_{r=1}^{j-1}S_r(t)+\big(f_{kn}(t,x)S_k(t)-f_{k+1,n}(t,x)C_k(t)$$

$$\times C_{k+1}(t)-\sum_{i=k+2}^{n-2}f_{in}(t,x)C_k(t)C_i(t)\prod_{r=k+1}^{i-1}S_r(t)-f_{n-1,n}(t,x)C_k(t)$$

$$\times\prod_{r=k+1}^{n-2}S_r(t)\big)v(t),\ where\ k=2,3,\dots,n-2;$$

$$d)\quad v'(t)=\big(1-v^2(t)\big)\big(f_{1n}(t,x)C_1(t)+\sum_{j=2}^{n-2}f_{jn}(t,x)C_j(t)\prod_{r=1}^{j-1}S_r(t)$$

$$+f_{n-1,n}(t,x)\prod_{r=1}^{n-2}S_r(t)\big).$$

Proof. According to the first assumption of Theorem 1 it follows that $r(t) \neq 0$ for all $t \in< t_0, \infty)$. If we put into (5) expression (1) instead of $x_i^{\text{'}}(t)$, $i = 1, 2, \dots, n$, and expression (2) instead of $x_i(t)$, $i = 1, 2, \dots, n$, after arrangement we get

$$\frac{r\text{'}(t)}{r(t)} = \big(f_{11}(t,x)C_1(t) + \sum_{j=2}^{n-2} f_{1j}(t,x)C_j(t)\prod_{r=1}^{j-1}S_r(t)$$
$$+ f_{1,n-1}(t,x)\prod_{r=1}^{n-2}S_r(t) + f_{1n}(t,x)v(t)\big)C_1(t) + \sum_{i=2}^{n-2}\big(f_{i1}(t,x)C_1(t)$$
$$+\sum_{j=2}^{n-2} f_{ij}(t,x)C_j(t)\prod_{r=1}^{j-1}S_r(t) + +f_{i,n-1}(t,x)\prod_{r=1}^{n-2}S_r(t)$$
$$+ f_{in}(t,x)v(t)\big)C_i(t)\prod_{r=1}^{i-1}S_r(t) + \big(f_{n-1,1}(t,x)C_1(t)$$
$$+\sum_{j=2}^{n-2} f_{n-1,j}(t,x)C_j(t)\prod_{r=1}^{j-1}S_r(t) + f_{n-1,n-1}(t,x)\prod_{r=1}^{n-2}S_r(t)$$
$$+ f_{n-1,n}(t,x)v(t)\big)\prod_{r=1}^{n-2}S_r(t).$$

From this equation with the assumptions of Theorem 1 (a) follows. We can prove relations (b), (c), (d) analogously. Theorem 1 is proved. □

Definition 2. Let i be a fixed integer, $1 \leq i \leq n$. We say that the solution $(x_1(t), x_2(t), \dots, x_n(t))$ of system (1) is i - bounded, if for all $t \geq a$ it holds $\mid x_i(t) \mid\leq K$, where $K > 0$ is a constant. System (1) is i -bounded, if all its solutions are i-bounded. Otherwise, the considered solution of system (1) is i - unbounded.

Theorem 3. *Let coefficients $f_{ij}(t,x)$, $i, j = 1, 2, \dots, n$, of (1) satisfy the conditions of Theorem 1 on the domain $D^0 \subset D$. If*

$$\sup_{X\in R^n}\int_{t_0}^{\infty}\left(\mid a(t,X)\mid + \sum_{i=1}^{n-1}\mid f_{in}(t,X)\mid\right)dt < +\infty,\ t_0 \geq a$$

then there exist two different nontrivial solutions of system (1), *which are n-bounded.*

Proof. Let the assumptions of Theorem 1 2 be fulfilled. From equation (9d) it follows that $v_1(t) = -1$, $v_2(t) = 1$, $t \in < t_0, \infty)$ are its solutions. Integrating equation (9a) from t_0 to t, $a \le t_0 \le t$, for $v(t) = 1$ it follows

$$\mid \int_{t_0}^{t} \frac{r'(s)}{r(s)} ds \mid \le \int_{t_0}^{t} \Big(\mid a(s,x(s)) \mid + \mid f_{1n}(s,x(s)) C_1(s) + \sum_{i=2}^{n-2} f_{in}(s,x(s)) C_i(s) \prod_{r=1}^{i-1} S_r(s) + f_{n-1,n}(s,x(s)) \prod_{r=1}^{n-2} S_r(s) \mid \Big) ds$$
$$\le \int_{t_0}^{t} \Big(\mid a(s,x(s)) \mid + \sum_{i=1}^{n-1} \mid f_{in}(s,x(s)) \mid \Big) ds.$$

Let $t \to \infty$. Then for function $r(t)$

$$\mid \int_{t_0}^{\infty} \frac{r'(t)}{r(t)} dt \mid \le \int_{t_0}^{\infty} \Big(\mid a(t,x(t)) \mid + \sum_{i=1}^{n-1} \mid f_{in}(t,x(t)) \mid \Big) dt$$
$$\le \sup_{X \in R^n} \int_{t_0}^{\infty} \Big(\mid a(t,X) \mid + \sum_{i=1}^{n-1} \mid f_{in}(t,X) \mid \Big) dt$$

holds. Then according to the assumption of Theorem 1 it follows that function $r(t)$ is bounded on the interval $< a, \infty)$. If now we use the relations (2) and (3), we have for $v(t) = 1$ the solution $(x_1(t), x_2(t), \dots, x_n(t))$ of system (1), which is n - bounded. Analogously we could find the solution of system (1) in the case of $v(t) = -1$. Thus Theorem 3 is proved. □

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