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WHEN A FINITE SUBSET $S \subset X \subset \mathbb{P}^n$ COMPUTES THE X-RANK OF SOME $P \in \mathbb{P}^n$?

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Abstract: Let $X \subset \mathbb{P}^n$ be an integral and projective variety. For any $P \in \mathbb{P}^n$ the X-rank of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$; any such S with minimal cardinality is said to compute the X-rank of P. Fix $S \subset X$. Here we give onditions on X and S which imply the existence of $P \in \mathbb{P}^n$ such that S compute the X-rank of P.

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Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. The integer $r_X(P)$ is called the X-rank of P. For any $P \in \mathbb{P}^n$ let $\mathcal{S}(X,P)$ denote the set of all $S \subset X$ computing $r_X(P)$, i.e. such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Here we continue the study of these sets $\mathcal{S}(X,P)$ started (to the best of our knowledge) in [1]. Let $\gamma(X)$ be the maximal integer t such that every subset $S \subset X$ with $\sharp(S) = t$ is linearly independent. Let $\beta(X)$ be the maximal integer t such that every zero-dimensional subscheme $E \subset X$ of degree $\leq t$ is linearly independent. Obviously $\beta(X) \leq \gamma(X) \leq n+1$ and both inequalities are equalities equality if X is a rational normal curve. For every integer $s > 0$ set $\mathcal{B}(X,s) := \{S \subset X : \sharp(S) = s, \dim(\langle S \rangle) \leq s-2\}$. Obviously $\mathcal{B}(X,s) \neq \emptyset$ if

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and only if $s > \gamma(X)$. It is easy to check that $\mathcal{B}(X, s)$ is a constructible subset of the Hilbert scheme of X.

For any finite subset $S \subset \mathbb{P}^n$ set $\ll S \gg := \{P \in \langle S \rangle : P \notin \langle S' \rangle \ \forall S' \subsetneq S\}$. Notice that $\ll S \gg = \emptyset$ if and only if either $S = \emptyset$ or S is linearly dependent. Assume $s := \sharp(S) > 0$ and S linearly independent. Obviously $P \in \ll S \gg$ if and only if $P \in \langle S \rangle$ and P is not contained in any of the s hyperplanes $\langle S' \rangle$ of S with $S' \subset S$ and $\sharp(S') = s - 1$. Let $\gamma(X)_2$ be the maximal integer $s \geq \gamma(X) + 1$ such that every $S \in \mathcal{B}(X, s)$ contains $A \in \mathcal{B}(X, \gamma(X) + 1)$. Every $S \in \mathcal{B}(X, \gamma(X) + 1)$ satisfies $h^1(\mathcal{I}_S(1)) = 1$ and every proper subset of it is linearly independent. Let $\gamma(X)'$ be the maximal integer $t \geq \gamma(X) + 1$ such that every $S \subset \mathcal{B}(X, t)$ satisfies $h^1(\mathcal{I}_S(1)) = 1$. Let $\gamma(X)''$ be the maximal integer $t \geq \gamma(X) + 1$ such that every $S \subset \mathcal{B}(X, t)$ satisfies $h^1(\mathcal{I}_S(1)) = 1$ and it is of the form $S = A \sqcup B$ with $B \in \mathcal{B}(X, \gamma(X) + 1)$ and $\sharp(A) = t - \gamma(X) - 1$; notice that in any such decomposition the linear space $\langle B \rangle$ is unique (it is a minimal subspace containing a linearly dependent subset of S and the assumption " $h^1(\mathcal{I}_S(1)) = 1$ " gives its uniqueness); thus in any such decomposition both A and B are unique. Here we prove the following result, the first one being of the type " for all $P \in \ll S \gg$ ", while the second one being of the type " for a general $P \in \ll S \gg$ ".

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety. Fix an integer s such that $2s - 1 \leq \gamma(X)''$. Let $S \subset X$ be any subset such that $\sharp(S) = s$ and $B \cap S = \emptyset$ for all $B \in \mathcal{B}(X, \gamma(X) + 1)$. Then $S \in \mathcal{S}(X, P)$ for every $P \in \ll S \gg$.*

Theorem 2. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Fix an integer s such that $1 \leq s \leq (n + 1)/2$. Fix any $S \subset X$ such that $\sharp(S) = s$ and S is linearly independent. Then $S \in \mathcal{S}(X, P)$ for a general $P \in \langle S \rangle$.*

We recall the following lemma ([1], Lemma 1).

Lemma 1. *Fix any $P \in \mathbb{P}^n$ and two zero-dimensional subschemes A, B of $\mathbb{P}^n$ such that $A \neq B$, $P \in \langle A \rangle$, $P \in \langle B \rangle$, $P \notin \langle A' \rangle$ for any $A' \subsetneq A$ and $P \notin \langle B' \rangle$ for any $B' \subsetneq B$. Then $h^1(\mathbb{P}^n, \mathcal{I}_{A \cup B}(1)) > 0$.*

Proof of Theorem 1. If $\ll S \gg = \emptyset$, then there is nothing to prove. Thus we may assume $\ll S \gg \neq \emptyset$. Fix $P \in \ll S \gg$ and assume $S \notin \mathcal{S}(X, P)$, i.e. assume $r_X(P) \leq s - 1$. Fix S_P computing $r_X(P)$ and set $S' := S \cup S_P$. Lemma 1 gives $h^1(\mathcal{I}_{S'}(1)) > 0$. Since $\sharp(S') \leq 2s - 1 \leq \gamma(X)''$, we have $S' = A \sqcup B$ with $\sharp(A) = \sharp(S') - \gamma(X) - 1$, $B \in \mathcal{B}(X, \gamma(X) + 1)$. Since $S \cap B = \emptyset$, we have $B \subseteq S_P$. Since S_P computes $r_X(P)$, it is linearly independent. Since $B \subseteq S_P$, we get a contradiction. □

Proof of Theorem 2. Since the case $s \leq 2$ is obvious, we may assume $s \geq 3$. There is a positive integer r such that $r_X(P) = r$ for a non-empty open subset W of $\ll S \gg$. Since $W \subset \langle S \rangle$, then $r \leq s$. Thus it is sufficient to prove $r = s$. Assume $r \leq s - 1$. Fix a general $P \in \ll S \gg$ and $A \subset X$ computing $r_X(P)$. Let $\ell : \mathbb{P}^n \setminus \langle S \rangle \to \mathbb{P}^{n-s}$ denote the linear projection from $\langle S \rangle$. Let C denote the closure of $\ell(X \setminus \langle S \rangle \cap X)$ in $\mathbb{P}^{n-s}$. First assume $\langle A \rangle \cap \langle S \rangle = \{P\}$. Since $\dim(\langle S \rangle) = s - 1$ and $\dim(X) = 1$, we get $r = s - 1$ and that a general $(s-1)$-dimensional secant space of X meets $\langle S \rangle$ at a unique point. Hence the rational map $\ell | X \setminus \langle S \rangle \cap X$ induces a birational map from X onto C and a general union of $s - 1$ points of C is linearly dependent, contradicting the generalized trisecant lemma (here we use that $n - s \geq s - 1$). Now assume $\rho := \dim(\langle S \rangle \cap \langle A \rangle) \geq 1$. Take any $B \subset A$ such that $\sharp(B) = r - \rho$. We get that $\langle S \rangle \cap \langle B \rangle \neq \emptyset$. Let W be a non-empty open subset of $\ll S \gg$ such that $r_X(P) = r$ and $\dim(\langle S \rangle \cap \langle A \rangle) = \rho$ (for at least one choice of $S_P \in \mathcal{S}(X, P)$). Since $\cup_{P \in W}(\langle S_P \rangle \cap \langle S \rangle)$ covers a non-empty open subset of $\langle S \rangle$, we get that the family of all such sets S_P, $P \in W$, has dimension at least $s - 1 - \rho$. Thus $r \geq s - 1 - \rho$. Taking a quasi-finite morphism $u : W' \to W$ with W' integral and $\dim(W') = s - 1 - \rho$ we may select for each $Q \in W'$ a subset B_Q of S_P such that $\sharp(B_Q) = s - 1 - \rho$ and the algebraic family $\{B_Q\}_{Q \in W'}$ of subsets of X has dimension $s - 1 - \rho$. Thus for a general $B \subset X$ such that $\sharp(B) = s - 1 - \rho$ there is at least one $Q \in W'$ such that $B = B_Q$. The linear projection ℓ from $\langle S \rangle$ shows that a general subset of C with cardinality $s - 1 - \rho$ is linearly dependent, contradiction. □

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764