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SPECTRAL FINITE DIFFERENCE ANALYSIS OF NATURAL CONVECTION IN A MULTIPLY-CONNECTED REGION WITH SUDDEN EXPANSION

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Abstract: Steady-state two-dimensional laminar natural convection in an infinite doubly-connected region with sudden expansion is analyzed numerically, using a spectral finite difference scheme. The region is assumed to be one with a nearly circular cylinder inside. A boundary-fitted conformal map is generated analytically through a shift factor together with a multiplication factor. In addition, transformation of a variable is adopted to support the condition for an artificially generated boundary (not univalent with zero measure).

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1. Introduction

Analysis of natural convection heat transfer is one element of heat and fluid flow problems, e.g. heat transfer to liquid metals from cylinders [1], to air from cylinders [2]. Recently-developed spectral finite difference schemes [3] are very effective to analytical or numerical treatment especially of heat and/or fluid flow problems in two-dimensional or in axisymmetric solutions, but not restricted to them. Mathematical introduction of multiply-connectedness is required as shown in [4] (doubly-connected), [5] (triply-connected), and [6] (quadruply-connected). The spectral finite difference scheme has the following property: mathematically exact spatial spectral decomposition, high spatial resolution,

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high speed computation, and accepting non-uniform grid spacing.

2. Analysis

2.1. General

Consider two-dimensional laminar natural convection enclosed in horizontal parallel walls of infinite extension over a nearly circular cylinder with sudden expansion. Under a Boussinesq approximation neglecting dissipation terms, the vorticity transport equation and the energy equation can be expressed by

$$J\frac{\partial \zeta}{\partial t} + \frac{\partial(\zeta,\psi)}{\partial(\alpha,\beta)} = \frac{1}{\sqrt{Gr}}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\zeta + \frac{\partial(T,y)}{\partial(\alpha,\beta)}, \tag{1}$$

$$J\frac{\partial T}{\partial t} + \frac{\partial(T,\psi)}{\partial(\alpha,\beta)} = \frac{1}{Pr\sqrt{Gr}}\left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)T, \tag{2}$$

respectively, where (x, y): dimensionless Cartesian coordinate based on the reference length L, y: vertically upward, (α, β): boundary-fitted conformal mapping coordinate, $J = \dfrac{\partial(x,y)}{\partial(\alpha,\beta)}$, ζ: vorticity, ψ: stream function, Gr: Grashof number, Pr: Prandtl number, T: dimensionless temperature $\equiv$ (local temperature$-$ $T_L)/(T_H - T_L)$, T_H: uniform cylinder surface temperature ($> T_L$), T_L: uniform wall temperature. The cylinder surface is assumed to be given by $\alpha = \alpha_0 (< 0)$ such that $e^{\alpha_0} \ll 1$, and the walls are assumed to be given by $\alpha = 0$, although a part of coordinate surface $\alpha = 0$ may correspond to the interior region. The relation between vorticity and a stream function is given by

$$J\zeta + \left(\frac{\partial^2}{\partial\alpha^2} + \frac{\partial^2}{\partial\beta^2}\right)\psi = 0. \tag{3}$$

2.2. Conformal Mapping System

Given a configuration of boundaries, the corresponding conformal mapping system is not determined uniquely. Thus consider the following:

$$z(w) = \cosh^{-1}\left(\frac{2w - k - 1}{k - 1}\right) - \frac{1}{k}\cosh^{-1}\left\{\frac{(k+1)w - 2k}{(k-1)w}\right\}, \tag{4}$$

$$w = -i\frac{\xi - 1}{\xi + 1}a + b, \xi \equiv e^{\alpha + i\beta}, \tag{5}$$

$$z \equiv x + iy, a > 0, b : \text{real}, k > 1, |\beta| \leq \pi, \alpha_0 \leq \alpha \leq 0,$$

$$\cosh^{-1}(q) \equiv \ln\left(q + \sqrt{q-1}\sqrt{q+1}\right),$$

where $\sqrt{\ }$ and ln() stand for a principal value. The reference length L is so chosen that the wider width of the parallel walls is π, and that the narrower one is $\pi(1-1/k)$. The location of the center of the cylinder, z^*, is given by $z^* = z(ai+b)$, which can be assigned to any point except $\left\{p_0 + (\pi - \pi/k)\, i \,\middle|\, 0 < p_0 \leq 1/\sqrt{k}\right\}$. The configuration corresponding to the coordinate surface $\alpha = 0$ is shown in Figure 1, where points A, B (B'), C, D, E, F, A' correspond to $\beta = -\pi, -2\tan^{-1}(b/a), \beta_1[\equiv 2\tan^{-1}\{(w_1 - b)/a\}], \beta_0[\equiv 2\tan^{-1}\left\{(1/\sqrt{k} - b)/a\right\}]$, $2\tan^{-1}\{(1-b)/a\}, 2\tan^{-1}\{(k-b)/a\}, \pi$ respectively. For β_1, $z(1) = z(w_1)$, $w_1 < 1/\sqrt{k}$. The interval C to E ($\beta_1 \leq \beta \leq 2\tan^{-1}\frac{1-b}{a}$) does not correspond to the physical boundary, and so it constitutes an apparent boundary, where auxiliary conditions are required.

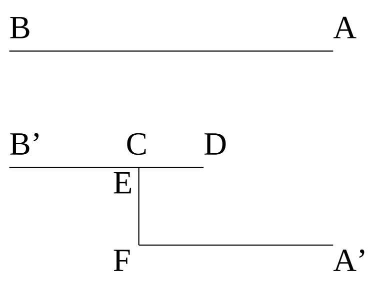

Figure 1: Schematic configuration for $\alpha = 0$

2.3. Variable Transformation of β

Let ω be a monotonously increasing function of β such that $-\pi \leq \omega \leq \pi$. Then

$$\frac{\partial^2}{\partial \beta^2} = \left(\frac{d\omega}{d\beta}\right)^2 \frac{\partial^2}{\partial \omega^2} + \frac{d^2\omega}{d\beta^2}\frac{\partial}{\partial \omega}.$$

Taking $1/z'(1) = 0$ into account, the following applies:
For $\beta(\beta_1 \leq \beta \leq \beta_0, \omega_1 \leq \omega \leq 0, \omega_1 \equiv -\sqrt{\beta_0 - \beta_1}\ )$

$$\omega - \omega_1 = \sqrt{\beta - \beta_1}. \tag{6}$$

For β $(\beta_0 < \beta \leq 2\tan^{-1}\dfrac{1-b}{a}, 0 < \omega \leq -\omega_1\ )$

$$z\left\{a\tan\frac{\beta(\omega)}{2}+b\right\} = z\left\{a\tan\frac{\beta(-\omega)}{2}+b\right\},$$

$$a\tan\frac{\beta(\omega)}{2}+b > \frac{1}{\sqrt{k}}. \tag{7}$$

For β $(-\pi < \beta < \beta_1, -\pi < \omega < \omega_1)$

$$\tan\frac{\beta}{2}-\tan\frac{\beta_1}{2} = \tan\frac{\omega}{2}-\tan\frac{\omega_1}{2}. \tag{8}$$

For β $(2\tan^{-1}\dfrac{1-b}{a} < \beta \leq \pi)$

$$\tan\frac{\beta}{2}-\frac{1-b}{a} = \tan\frac{\omega}{2}+\tan\frac{\omega_1}{2}. \tag{9}$$

Thus Eqs.(1),(2), and (3) become

$$\begin{aligned} & J\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{\partial\zeta}{\partial t}+\left(\frac{d\omega}{d\beta}\right)^{-3}\frac{\partial(\zeta,\psi)}{\partial(\alpha,\omega)} \\ = \ & \frac{1}{\sqrt{Gr}}\left\{\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{\partial^2}{\partial\alpha^2}+\left(\frac{d\omega}{d\beta}\right)^{-2}\frac{\partial^2}{\partial\omega^2}+\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{d^2\omega}{d\beta^2}\frac{\partial}{\partial\omega}\right\}\zeta \\ & +\left(\frac{d\omega}{d\beta}\right)^{-3}\frac{\partial(T,y)}{\partial(\alpha,\omega)}, \end{aligned} \tag{10}$$

$$\begin{aligned} & J\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{\partial T}{\partial t}+\left(\frac{d\omega}{d\beta}\right)^{-3}\frac{\partial(T,\psi)}{\partial(\alpha,\omega)} \\ = \ & \frac{1}{Pr\sqrt{Gr}}\left\{\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{\partial^2}{\partial\alpha^2}+\left(\frac{d\omega}{d\beta}\right)^{-2}\frac{\partial^2}{\partial\omega^2}+\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{d^2\omega}{d\beta^2}\frac{\partial}{\partial\omega}\right\}T, \end{aligned} \tag{11}$$

$$\begin{aligned} & J\left(\frac{d\omega}{d\beta}\right)^{-4}\zeta+\left\{\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{\partial^2}{\partial\alpha^2}+\left(\frac{d\omega}{d\beta}\right)^{-2}\frac{\partial^2}{\partial\omega^2}\right. \\ & \left.+\left(\frac{d\omega}{d\beta}\right)^{-4}\frac{d^2\omega}{d\beta^2}\frac{\partial}{\partial\omega}\right\}\psi = 0. \end{aligned} \tag{12}$$

Consequently the coefficients

$$\left(\frac{d\omega}{d\beta}\right)^{-4}, \left(\frac{d\omega}{d\beta}\right)^{-3}, \left(\frac{d\omega}{d\beta}\right)^{-2}, \left(\frac{d\omega}{d\beta}\right)^{-4} \frac{d^2\omega}{d\beta^2}$$

are piecewise continuous in each interval $[-\pi, \omega_1]$, $[\omega_1, 0]$, $[0, -\omega_1]$, $[-\omega_1, \pi]$. At $\omega = 0, \omega = \pm\omega_1$, both the limits of the above coefficients (from the right and from the left) exist, and at $\omega = -\pi$ the limits from the right exist, and at $\omega = \pi$ the limits from the left exist.

2.4. Boundary Conditions at Physical Boundaries

On the surface of the walls, C_1, specified by $\alpha = 0$, and $-\omega_1 \leq |\omega| \leq \pi$, no slip flow conditions for dynamical conditions and an isothermal condition of its surface temperature are assumed:

$$T(\alpha = 0, \beta) = 0. \tag{13}$$

Without loss of generality for a finite number of the cylinder

$$\psi(\alpha = 0, \beta) = 0, \tag{14}$$

$$\frac{\partial}{\partial \alpha}\psi(\alpha = 0, \beta) = 0. \tag{15}$$

On the surface of the cylinder, C_0, specified by $\alpha = \alpha_0, |\omega| \leq \pi$, no slip flow condition and an isothermal condition of its surface higher temperature are assumed:

$$\psi(\alpha = \alpha_0, \beta) = \psi_0(\text{ constant to be determined }), \tag{16}$$

$$\frac{\partial}{\partial \alpha}\psi(\alpha = \alpha_0, \beta) = 0, \tag{17}$$

$$T(\alpha = \alpha_0, \beta) = 1. \tag{18}$$

2.5. Auxiliary Conditions at $\alpha = 0$ and $|\omega| \leq -\omega_1$

Necessary conditions are given by the condition that any scalar quantity and its gradient are continuous along the line $(-\omega_1 \geq |\omega| \geq 0)$, which becomes, for any scalar quantity ϕ, e.g. ψ, ζ , and T,

$$\frac{1}{2}\left\{\phi(\alpha = 0, \omega) - \phi(\alpha = 0, -\omega)\right\} = 0, \tag{19}$$

$$
\begin{aligned}
&\frac{\partial}{\partial\alpha}\frac{1}{2}\left\{\phi(\alpha=0,\omega)+\phi(\alpha=0,-\omega)\right\} \\
= &\quad \frac{\sqrt{J(\alpha=0,\omega)}-\sqrt{J(\alpha=0,-\omega)}}{\sqrt{J(\alpha=0,\omega)}+\sqrt{J(\alpha=0,-\omega)}} \\
&\quad \times\frac{\partial}{\partial\alpha}\frac{1}{2}\left\{\phi(\alpha=0,\omega)-\phi(\alpha=0,-\omega)\right\}. \qquad (20)
\end{aligned}
$$

Throughout Eqs.(19) and (20), the second arguments of J and ϕ should be regarded as the transformed variable ω from β.

2.6. Multiply-Connectedness

Multiply-connectedness gives rise to

$$\oint_{C_0}\frac{\partial p}{\partial\beta}d\beta=0, \qquad (21)$$

where p stands for pressure. Since the cylinder is fixed and isothermal (from the assumption), combining ∇p from the Navier-Stokes equation (not explicitly given) and Eq.(21) gives

$$\oint_{C_0}\frac{\partial\zeta}{\partial\alpha}d\beta=\oint_{C_0}\frac{\partial\zeta}{\partial\alpha}\left(\frac{d\omega}{d\beta}\right)^{-1}d\omega=0. \qquad (22)$$

2.7. Spectral Decomposition

Spectral decomposition of variables is based on Fourier series:

$$
\begin{aligned}
\begin{bmatrix}\psi(\alpha,\beta,t)\\ \zeta(\alpha,\beta,t)\\ T(\alpha,\beta,t)\end{bmatrix} &= \sum_{n=1}^{\infty}\begin{bmatrix}\psi_{sn}(\alpha,t)\\ \zeta_{sn}(\alpha,t)\\ T_{sn}(\alpha,t)\end{bmatrix}\sin n\omega \\
&\quad +\sum_{n=0}^{\infty}\begin{bmatrix}\psi_{cn}(\alpha,t)\\ \zeta_{cn}(\alpha,t)\\ t_{cn}(\alpha,t)\end{bmatrix}\cos n\omega, \qquad (23)
\end{aligned}
$$

where $-\pi\leq\omega\leq\pi$.

2.8. Discretization and Time Integration

Numerical integration schemes are as follows: the system of Eqs.(10)-(12) with respect to ω are decomposed into the corresponding Fourier components of ω, discretized in time and space using a finite difference scheme with respect to α, together with a mixed type of boundary conditions at $\alpha = 0$ [7]. Although any non-uniform grid spacing in α can be accepted, the following may work for the n-th grid point $\alpha_n (0 \leq n \leq M+1;\ M$: suitably chosen integer, $M+1$ for $\alpha = 0$)

$$\alpha_n = \alpha_0 + h \left\{ \frac{\sinh \gamma (n-1)}{\sinh \gamma} + 1 \right\}, \tag{24}$$

$$h \equiv -\alpha_0 \bigg/ \left(\frac{\sinh \gamma M}{\sinh \gamma} + 1 \right), \tag{25}$$

where γ is a real parameter (> 0), and the limit $\gamma \to 0$ corresponds to a uniform grid spacing in α. The larger the value of γ, the finer the relative grid spacing near the surface of the cylinder at $\alpha = \alpha_0$. First, steady-state pure heat conduction solution temperature corresponding to no flow would be sought through the same spectral finite difference scheme by iteration under a semi-implicit scheme, supplemented with a diagonal dominant form. Then, as the initial thermal field the said pure heat conduction field is adopted, and an initial stationary flow field is assumed, which is integrated semi-implicitly with respect to time to get a steady-state solution, applying a diagonal dominant form. Dimensionless total force $\boldsymbol{F}$ acting on the stationary surface C_0 (excluding stationary buoyancy force) is given by

$$\begin{aligned} \boldsymbol{F} &= \frac{i}{\sqrt{Gr}} \oint_{C_0} \zeta \frac{dz}{dw} \frac{dw}{d(\alpha + i\beta)} d\beta - \frac{i}{\sqrt{Gr}} \oint_{C_0} \frac{\partial \zeta}{\partial \alpha} z\,(w)\, d\beta \\ &= \frac{i}{\sqrt{Gr}} \oint_{C_0} \zeta \frac{dz}{dw} \frac{dw}{d(\alpha + i\beta)} \left(\frac{d\omega}{d\beta} \right)^{-1} d\omega \\ &\quad - \frac{i}{\sqrt{Gr}} \oint_{C_0} z\,(w) \frac{\partial \zeta}{\partial \alpha} \left(\frac{d\omega}{d\beta} \right)^{-1} d\omega, \end{aligned} \tag{26}$$

where $\boldsymbol{F}$ is based on $\rho U^2 L$, (ρ: density of the fluid, L: reference length, $U \equiv (\nu/L)\sqrt{Gr}$, ν: kinematic viscosity of the fluid), and on $C_0, w = a \tan(\beta/2) + b$. Mean Nusselt number Nu_m at the surface C_0 is given by

$$\begin{aligned} Nu_m &= - \oint_{C_0} \frac{\partial T}{\partial \alpha} d\beta \bigg/ \oint_{C_0} \sqrt{J} d\beta \\ &= - \oint_{C_0} \frac{\partial T}{\partial \alpha} \left(\frac{d\omega}{d\beta} \right)^{-1} d\omega \bigg/ \oint_{C_0} \sqrt{J} d\beta. \end{aligned} \tag{27}$$

For the estimate of the denominator

$$\oint_{C_0} \sqrt{J} d\beta \approx 4\pi a \mathrm{e}^{\alpha_0} |z'(ai + b)|. \tag{28}$$

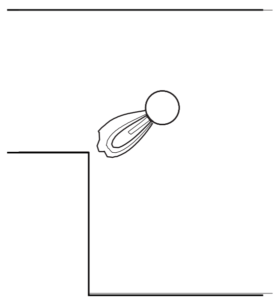

Figure 2: Streamlines at $Gr = 1$, $Pr = 0.7$, $k = 2$, $\delta\psi = 5 \times 10^{-12}$

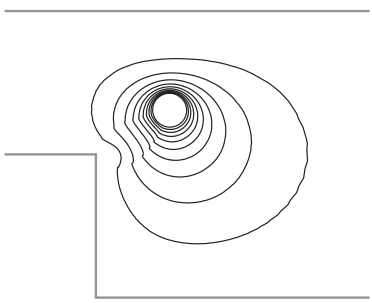

Figure 3: Isotherms at $Gr = 1$, $Pr = 0.7$, $k = 2$, $\delta T = 0.1$

3. Results and Discussions

3.1. Field Characteristics

Figures 2 and 3 show an example of steady-state streamlines and isotherms respectively at $Gr = 1, Pr = 0.7$ for $a = 1, b = 0, k = 2, \alpha_0 = -2$. $\delta\psi$ and δT stand for the difference of streamlines and isotherms respectively. Apart from the neighbourhood of C_0, $|\zeta| \ll 1$ and $|\nabla\psi| \ll 1$, so that streamlines faraway are sparse and not reproduced in these figures. Examples of global quantities are $\boldsymbol{F} = 5.3 \times 10^{-4} - 6.3 \times 10^{-5}i, Nu_m = 6.80$.

3.2. Characteristics with *Gr*

As long as Gr is small, steady-state thermal field is expected to be nearly independent of Gr if Pr is moderate as 0.7, which results in, if the object geometrical configuration is fixed under the same thermal boundary condition (Dirichlet condition), that the spatial dimensionless vorticity distribution would be nearly proportional to $\sqrt{Gr}$, and $\boldsymbol{F}$ would be nearly independent of Gr, which is the case, whereas Nu_m increases slightly with Gr, e.g. $Nu_m = 6.17$ at $Gr = 0.1$, $Nu_m = 6.80$ at $Gr = 1$, $Nu_m = 7.17$ at $Gr = 2$.

3.3. Number of Spectral Components Used

The maximum degree of Fourier series used in Figures 2-3 is 15 for both sine and cosine components.

4. Conclusions

Introduction of multiply-connectedness leads to a reasonable analysis of natural convection for parallel walls with sudden expansion, using a spectral finite difference scheme with a suitably chosen conformal mapping function, supplemented with variable transformation.

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