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ON WEAK* STATISTICAL CONVERGENCE OF SEQUENCE OF FUNCTIONALS

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Abstract: The main object of this paper is to introduce the concept of weak* statistical convergence of sequence of functionals in a normed space. It is shown that in a reflexive space, weak* statistically convergent sequences of functionals are the same as weakly statistically convergent sequences of functionals.

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1. Introduction

The idea of statistical convergence was given by Zygmund [23] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was formally introduced by Steinhaus [20] and Fast [5] and later reintroduced by Schoenberg [19]. Although, statistical convergence was introduced over nearly the last fifty years, it has become an active area of research in recent years. This concept has been applied in various areas such as number theory [4], measure theory [14], trigonometric series [23], summability theory [6], locally convex spaces [11], in the study of strong integral summability [2], turnpike theory [12, 13, 16], and Banach spaces [3].

If K is a subset of the positive integers $\mathbb{N}$, then K_n denotes the set $\{k \in K : k \leq n\}$ and $|K_n|$ denotes the number of elements in K_n. The natural density of K (see [15, chapter 11]) is given by $\delta(K) = \lim_{n\to\infty} n^{-1}|K_n|$. A sequence (x_k) of (real or complex) numbers is said to be statistically convergent to some

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number L, if for every $\epsilon > 0$, the set $K_\epsilon = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\}$ has natural density zero; in this case, we write st-$\lim_k x_k = L$.

For real-valued sequences, statistically convergent sequences often satisfy statistical analogs of the usual attributes of convergent sequences. For instance, statistically convergent sequences are statistically bounded, a sequence is statistically convergent if and only if it is statistically Cauchy, and there are statistical analogs of the lim sup, lim inf, and so forth, see [5, 7, 8, 9, 18].

We recall (see [7]) that if $x = (x_k)$ is a sequence such that x_k satisfies property P for all k except a set of natural density zero, then we say that $x = (x_k)$ satisfies P for "almost all k", and we abbreviate this by "a.a.k."

The following concept is due to Fridy [7]. A sequence (x_k) is said to be statistically Cauchy if for each $\epsilon > 0$, there exists a number $N(= N(\epsilon))$ such that $|x_k - x_N| < \epsilon$, for a.a. k, that is, $\delta(\{k \in \mathbb{N} : |x_k - x_N| \geq \epsilon\}) = 0$.

Fridy [7] proved that a number sequence is statistically convergent if and only if it is statistically Cauchy. It was shown by Kolk [10] that this result remains true in case the entries of the sequences come from a Banach space instead of being scalars.

A number sequence $x = (x_k)$ is statistically bounded [9] if there is a number B such that $\delta\{k : |x_k| > B\} = 0$, that is, $|x_k| \leq B$, for a.a.k.

Maddox [11] extended the concept of statistical convergence to sequences with values in arbitrary locally convex Hausdorff topological vector spaces. The statistical convergence in Banach spaces was studied by Kolk [10].

Quite recently, Connor et al. [3] have introduced a new concept of weak statistical convergence and have characterized Banach spaces with separable duals via weak statistical convergence. Pehlivan and Karaev [17] have also used the idea of weak statistical convergence in strengthening a result of Gokhberg and Krein on compact operators. Bhardwaj and Bala [1] have introduced a new concept of weak statistically Cauchy sequence in a normed space and it has been shown that in a reflexive space, weak statistically Cauchy sequences are the same as weakly statistically convergent sequences.

We now introduce the concept of weak* statistical convergence of sequence of functionals as follows.

Definition 1.1. Let X be a normed linear space. Denote the continuous dual of X by $X^{'}$. The sequence (f_k) of bounded linear functionals on X is weak* statistically convergent to $f \in X^{'}$ provided that, the sequence $(f_k(x) - f(x))$ is statistically convergent to 0 for each $x \in X$. In this case, we write w*-st-lim $f_k = f$ and f is called the weak* statistical limit of (f_k).

In this paper, we show that weak* statistical convergence is a generalization of the usual notion of weak* convergence. It is shown that in a reflexive space, weak* statistically convergent sequences are the same as weakly statistically convergent sequences.

The following well-known lemmas are required for establishing the results of this paper.

Lemma 1.2. (see [19]) *If* st-lim $x_k = l$ *and* $g(x)$*, defined for all real* x*, is continuous at* $x = l$*, then* st-lim $g(x_k) = g(l)$*.*

Lemma 1.3. (see [18]) *A number sequence* (x_k) *is statistically convergent to* l *if and only if there exists such a set* $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ *that* $\delta(K) = 1$ *and* $\lim_{n\to\infty} x_{k_n} = l$*.*

Lemma 1.4. (see [18]) *If* st-lim $x_k = l$ *and* st-lim $y_k = m$ *and* α *is a real number, then*

(i) st-lim$(x_k + y_k) = l + m$,

(ii) st-lim$(\alpha x_k) = \alpha l$.

Lemma 1.5. (see [21]) *A number sequence* (x_k) *is statistically bounded if and only if there exists such a set* $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ *that* $\delta(K) = 1$ *and* (x_{k_n}) *is bounded.*

Lemma 1.6. (see [22]) *Let* $x_k \leq y_k$*, for a.a.k. If* st-lim x_k *and* st-lim y_k *exist, then:*

$$\mathrm{st} - \lim x_k \leq \mathrm{st} - \lim y_k.$$

2. Main Results

Our first result shows that weak* statistical convergence is a generalization of the usual notion of weak* convergence.

Theorem 2.1. *Let* (f_k) *be a weak* convergent sequence in a normed space* X*, and* w^*-lim $f_k = f$*. Then* (f_k) *is weak* statistically convergent to* f*. The converse is not generally true.*

Proof. If w^*-lim $f_k = f$, then $(f_k(x))$ is convergent to $f(x)$, for all $x \in X$ which implies w^*-st-lim $f_k = f$. To show that the converse is not true, we give the following example.

Example 2.2. Let $f_k \in \ell_1^{'}$ be defined by

$$f_k(x) = \begin{cases} x_1, & \text{if } k = m^2, \\ x_k, & \text{if } k \neq m^2. \end{cases}$$

For $k \neq m^2$ and for every $x \in \ell_1$, we have $f_k(x) \to 0$ as $k \to \infty$. Hence, st-lim $f_k(x) = 0$, by Lemma 1.3, which in turn implies that w^*-st-lim $f_k = 0$.

For $k = m^2$, consider a sequence $x \in \ell_1$ defined by $x = (1, 0, 0, 0, \ldots)$. Clearly $f_k(x) = x_1 = 1 \nrightarrow 0$, as $k \to \infty$. Hence, (f_k) is not weak* convergent.

Theorem 2.3. *Let X be a normed space. If a sequence (f_k) in $X^{'}$ is weakly statistically convergent to $f \in X^{'}$, then it is weak* statistically convergent.*

Proof. Let w-st-lim $f_k = f$. Then for every $g \in X^{''}$ and $\epsilon > 0$, we have $\delta(\{k : |g(f_k) - g(f)| \geq \epsilon\}) = 0$. Let $x \in X$ and $g = Cx$, where $C : X \to X^{''}$ is the canonical mapping. We have $g(f_k) = f_k(x)$ and $g(f) = f(x)$ for every $x \in X$. Thus we have $\delta(\{k : |f_k(x) - f(x)| \geq \epsilon\}) = 0$ for every $x \in X$ and $\epsilon > 0$. This means that w^*-st-lim $f_k = f$.

Theorem 2.4. *Let X be a reflexive normed space. If a sequence (f_k) in $X^{'}$ is weak* statistically convergent to $f \in X^{'}$, then it is weakly statistically convergent.*

Proof. Let X be a reflexive normed space and w^*-st-lim $f_k = f$. Thus we have $\delta(\{k : |f_k(x) - f(x)| \geq \epsilon\}) = 0$ for each $x \in X$ and $\epsilon > 0$. To show that w-st-lim $f_k = f$ in $X^{'}$, let $g \in X^{''}$. Then $g = Cx$ for some $x \in X$, where $C : X \to X^{''}$ is the canonical mapping. Since $g(f_k) = f_k(x)$ and $g(f) = f(x)$, we have

$$\delta(\{k : |g(f_k) - g(f)| \geq \epsilon\}) = 0$$

for each $\epsilon > 0$. Since this is true for every $g \in X^{''}$, hence w-st-lim $f_k = f$.

Our next result shows that in a Banach space, every weak* statistically convergent sequence is statistically bounded.

Lemma 2.5. *Let X be a Banach space. If a sequence (f_k) in $X^{'}$ is weak* statistically convergent to $f \in X^{'}$, then $(\|f_k\|)$ is statistically bounded.*

Proof. Let w^*-st-lim $f_k = f$. Then $(f_k(x))$ is a statistically convergent sequence for all $x \in X$ and hence is statistically bounded. So by Lemma 1.5, there exists a set $K = \{k_1 < k_2 < \ldots\} \subset \mathbb{N}$ such that $\delta(K) = 1$ and $(f_{k_n}(x))$ is bounded for every $x \in X$. Since X is complete, $(\|f_{k_n}\|)$ is bounded by the uniform boundedness theorem. Again, by Lemma 1.5, it follows that $(\|f_k\|)$ is statistically bounded.

Theorem 2.6. *In a Banach space X, we have* w*-st-lim $f_k = f$ *if and only if*

(i) The sequence $(\|f_k\|)$ *is statistically bounded.*

(ii) For every element x in a total subset M of X, we have st-lim $f_k(x) = f(x)$.

Proof. If w*-st-lim $f_k = f$, Then (i) follows from Lemma 2.5 and (ii) is trivial.

Conversely, suppose that (i) and (ii) hold. Consider any $x \in X$ and we will show that st-lim $f_k(x) = f(x)$. This will be done in two steps. First, it will be shown that this is true for all $x \in \operatorname{span} M$ and then for $x \in \overline{\operatorname{span} M}$.

To prove the first conclusion, let $x \in \operatorname{span} M$. Then $x = \sum_{i=1}^{n} \alpha_i x_i$ for $x_1, x_2, \ldots, x_n \in M$ and scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$. By hypothesis (ii), st-lim $f_k(x_i) = f(x_i)$ for all i, $1 \leq i \leq n$ and hence st-lim $f_k(x) = f(x)$, by Lemma 1.4. Thus the first conclusion is established.

For the second conclusion, suppose $y \in \overline{\operatorname{span} M}$. By hypothesis (i), there exists a constant $c > 0$ such that $\|f_k\| < c$, for a.a.k, and therefore, for any $x \in M \subset X$, we have $|f_k(x)| < c\|x\|$, for a.a.k which by Lemma 1.6 gives that st-lim $|f_k(x)| < c\|x\|$. Again, using Lemma 1.2, we have $|f(x)| < c\|x\|$ which implies $\|f\| < c$. Since $y \in \overline{\operatorname{span} M}$, for a given $\epsilon > 0$, there exists $x_j \in \operatorname{span} M (j = 1, 2, \ldots)$ such that $\|y - x_j\| < \frac{\epsilon}{3c}$ for all $j > n_0$.

Consider

$$\begin{aligned} |f_k(y) - f(y)| &\leq \|y - x_j\|\|f_k\| + |f_k(x_j) - f(x_j)| + \|x_j - y\|\|f\| \\ &< \frac{\epsilon}{3c}c + |f_k(x_j) - f(x_j)| + \frac{\epsilon}{3c}c, \quad \text{for a.a.}k, \quad \text{provided } j > n_0. \end{aligned}$$

Since $x_j \in \operatorname{span} M$, so by the first part of the proof, st-lim $f_k(x_j) = f(x_j)$ and hence $|f_k(x_j) - f(x_j)| < \frac{\epsilon}{3}$, for a.a.$k$. Hence $|f_k(y) - f(y)| < \epsilon$, for a.a.$k$, and so w*-st-lim $f_k = f$.

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