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POINTS OF A PROJECTIVE SPACE WITH A PRESCRIBED NUMBER OF SUBSETS COMPUTING THEIR X-RANK

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^n$ be an integral non-degenerate subvariety. For any $P \in \mathbb{P}^n$ the X-rank of P is the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. Let $\mathcal{S}(X,P)$ be the set of all $S \subset X$ computing the X-rank of P. Here we construct smooth curves $X \subset \mathbb{P}^n$ and $P \in \mathbb{P}^n$ such that $\sharp(\mathcal{S}(X,P))$ is a prescribed integer.

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Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. The integer $r_X(P)$ is called the X-rank of P ([4]). Let $\mathcal{S}(X,P)$ the set of all $S \subset X$ computing $r_X(P)$, i.e. the set of all subsets $S \subset X$ such that $\sharp(S) = r_X(P)$ and $P \in \langle S \rangle$. Notice that any $S \in \mathcal{S}(X,P)$ is linearly independent and $P \notin \langle S' \rangle$ for any $S' \subsetneq S$. For every integer $t \geq 1$ let $\sigma_k(X)$ denote the closure in $\mathbb{P}^n$ of all $(k-1)$-dimensional linear spaces spanned by t points of Y. Set $\sigma_0(X) := \emptyset$. For any $P \in \mathbb{P}^n$ the border X-rank $b_X(P)$ is the minimal integer $t \geq 1$ such that $P \in \sigma_t(X)$, i.e. the only integer $t \geq 1$ such that $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$.

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If $\sigma_{k-1}(X) \neq \mathbb{P}^n$, then a general $P \in \sigma_k(X)$ has X-rank k. An important problem is to find conditions on X and P such that $\sharp(\mathcal{S}(X,P)) = 1$. Very seldom $\sharp(\mathcal{S}(X,P)) = 1$ for a general $P \in \mathbb{P}^n$ (see [2] for a full classification when X is smooth and $\dim(X) \leq 3$). Now we fix an integer $k \geq 2$ such that $k(\dim(X)+1) \leq n$. There is an easy to check condition which implies that $\sharp(S(X,P)) = 1$ for a general $P \in \sigma_k(X)$: it is sufficient to assume that X is not weakly $(k-1)$-defective ([3], Proposition 1.5), i.e. it is sufficient to assume that for a general $S \subset X$ such that $\sharp(S) = k$ the points of S are isolated component of the contact locus of a general hyperplane of $\mathbb{P}^n$ tangent to X at each point of X. No curve is weakly defective ([1] or [3], Remark 1.2). Here we prove the following results.

Theorem 1. *Fix integers b, r such that $r > b \geq 2$ and a smooth and connected projective curve C. Set $g := p_a(C)$ and $d_0 := 2g+4b+4r-1$. Fix an integer $d \geq d_0$ and $R \in \mathrm{Pic}^d(C)$. Then there exist a non-degenerate curve $X \subset \mathbb{P}^n$ such that $X \cong C$ and $P \in \mathbb{P}^n$ such that $r_X(P) = r$, $b_X(P) = b$ and $\sharp(\mathcal{S}(X,P)) = 1$. Moreover, we may assume that the given isomorphism $X \to C$ maps R onto $\mathcal{O}_X(1)$ and that $n = d-g-1$.*

Theorem 2. *Fix integers $x \geq 2$ and $r \geq 2$, $x \geq 2$. Fix a smooth and connected projective curve C. Set $g := p_a(C)$ and $d_1 := 2g+(x+1)r+2$. Fix an integer $d \geq d_1$ and $R \in \mathrm{Pic}^d(C)$. Then there exist a non-degenerate curve $X \subset \mathbb{P}^n$ and $P \in \mathbb{P}^n$ such that $X \cong C$, $r_X(P) = r$ and $\sharp(\mathcal{S}(X,P)) = x$. Moreover, we may assume that the given isomorphism $X \to C$ maps R onto $\mathcal{O}_X(1)$ and that $n = d-g-x+1$.*

Proof of Theorem 1. Since $d \geq d_0 \geq 2g+1$, $h^1(C,R) = 0$, $h^0(C,R) = d+1-g$ and R is very ample. We identify C with its embedding $C \hookrightarrow \mathbb{P}^{d-g}$ induced by the complete linear system $|R|$. This identification gives an equality $R = \mathcal{O}_C(1)$. Fix $O \in C$. Since $d-2r \geq 2g-1$, for any $W \subset C$ such that $\deg(W) \leq r$ we have $h^1(C, R(-W)) = 0$, i.e. $\dim(\langle W \rangle) = \deg(W)-1$, i.e. W is linearly independent (seen as a zero-dimensional subscheme of $\mathbb{P}^{d-g}$). For any $Q \in \mathbb{P}^{d-g}$ let $\ell_Q : \mathbb{P}^{d-g} \setminus \{Q\} \to \mathbb{P}^{d-g-1}$ denote the linear projection from Q. Fix $O \in C$ and take any $A \in \langle bO \langle \setminus \langle (b-1)O \rangle$. Since $2\deg(W) \leq 2r$ In particular bO is the only subscheme of C with degree $\leq k$ and whose linear span contains A. We get $b_C(A) = b$ and $r_C(A) \geq 2r+1-b$. Fix a finite set $E \subset C \setminus \{O\}$ such that $\sharp(S) = r$. Thus $\deg(E \cup bO) = r+b$ and $\dim(\langle E \cup bO \rangle) = r+b-1$. Fix a general $Q \in \langle E \cup bO \rangle$; Since $\dim(\langle E \cup bO \rangle) = r+b-1$, we have $Q \notin \langle W' \rangle$ for any $W' \subsetneqq E \cup bO$. Since $d-2(r+b) \geq 2g-1$, it is easy to check that if Z is a zero-dimensional subscheme of C such that $\deg(Z) \leq 2(r+b)$ and $Q \in \langle Z \rangle$, then $bO \cup S \subseteq Z$. Hence $b_C(Q) = r+b$. We have $\langle \{Q\} \cup bO \rangle \cap \langle \{Q\} \cup S \rangle = \{Q\}$.

Thus $\ell_Q(\langle bO\rangle)$ and $\ell_Q(\langle S\rangle)$ are linear subspaces, respectively of dimension $b-1$ and $r-1$ and their intersection in a unique point, P. Call P this point. Since $b_Q(X) > b \geq 2$, then $Q \notin \sigma_2(X)$. Thus $\ell_Q|C$ induces an embedding onto a smooth curve $X \subset \mathbb{P}^{d-g-1}$. Hence $\ell_Q(bO)$ is a degree b linearly independent subscheme of X and $\ell_Q(S)$ is a degree r linearly independent. We get $b_X(P) \leq b$ and $r_X(P) \leq r$. Since $Q \notin \sigma_b(C)$, ℓ_Q is a morphism at each point of $\sigma_b(C)$. Thus $\ell_Q(\sigma_b(C))$ is an integral complete variety of dimension $2b-1$. Since $\ell_Q|C$ is injective and the secant varieties of non-degenerate curves have the expected dimension, we get $\ell_Q(\sigma_b(C)) = \sigma_b(X)$. Thus $b_X(P) = b$. Set $S := \ell_Q(E)$. Obviously $P \in \langle S\rangle$. Since $Q \notin \sigma_2(C)$, we have $\sharp(S) = r$. To conclude the proof it is sufficient to prove the non-existence of a set $B \subset X$ such that $\sharp(B) \leq r$, $B \neq S$ and $P \in \langle B\rangle$. Assume that B exists and write $B = \ell_Q(A)$ with $A \subset C$. Since $d - 2b - 2r \geq 2g - 1$, and C is linearly normal, for every every zero-dimensional scheme $W \subset C$ such that $\deg(W) \leq 2r$ and $Q \in \langle W\rangle$ we have $W \supseteq bO \cup E$. Since $P \in \langle A\rangle \cap \langle S\rangle$, $A \cap S \neq S$ and $P \notin \langle S'\rangle$ for any $S' \subsetneqq S$, we get $Q \in \langle B\cup E\rangle$. Thus $bO\cup E \subseteq B\cup E$. Since $b \geq 2$, we got a contradiction. □

Proof of Theorem 2. Since $d \geq d_0 \geq 2g+1$, $h^1(C, R) = 0$, $h^0(C, R) = d+1-g$ and R is very ample. We identify C with its embedding $C \hookrightarrow \mathbb{P}^{d-g}$ induced by the complete linear system $|R|$. This identification gives an equality $R = \mathcal{O}_C(1)$. For any linear subspace $V \subset \mathbb{P}^{d-g}$ let $\ell_V : \mathbb{P}^{d-g}\backslash V \to \mathbb{P}^a$, $a := d-g-\dim(V)-1$, denote the linear projection from V. Fix subsets $B_i \subset C$, $1 \leq i \leq x$, such that $\sharp(B_i) = r$ for all i and $B_i \cap B_j = \emptyset$ for all $i \neq j$. Fix $P \in \langle B_i\rangle$ such that $P_i \notin \langle B_i'\rangle$ for any $B_i' \subsetneqq B_i$. Since $d \geq 2g-1+xr$ we have $\dim(\langle \cup_{i=1}^x B_i\rangle) - xr - 1$. Hence $\langle\{P_1, \dots, P_x\}\rangle = x - 1$. Let $V \subset \langle\{P_1, \dots, P_x\}\rangle$ be a hyperplane such that $\{P_1, \dots, P_x\} \cap V = \emptyset$. Hence each $\ell_V(P_i)$ is defined and $\ell_V(P_i) = \ell_V(P_j)$ for all i, j. Set $P := \ell_V(P_1)$, $n := d-g-x+1$ and $X := \ell_V(C)$. Since $xr+2+2g-1 \leq d$, the only line tangent or secant to C and intersecting $\langle \cup_{i=1}^x B_i\rangle$ is a line spanned by two of the points of $\cup_{i=1}^x B_i$. For general $P_1, \dots, P_x$ no such line is contained in $\langle\{P_1, \dots, P_x\}\rangle$. Hence for general $P_1, \dots, P_x$ we may find V avoiding any such secant line. Thus $V \cap \sigma_2(C) = \emptyset$. Hence $\ell_V|C$ is an isomorphim. Since $\ell_V|C$ is injective, then $\sharp(\ell_V(B_i)) = r$ for all i and $\ell_V(B_i) \cap \ell_V(B_j)$ for all $i \neq j$. Since $P \in \langle \ell_V(B_i)\rangle$ for all i, to prove Theorem 2 it is sufficient to prove that for each $S \subset X$ such that $\sharp(S) \leq r$ and $P \in \langle S\rangle$, there is $i \in \{1, \dots, x\}$ such that $S = \ell_V(B_i)$. Fix any such set S and write $S = \ell_V(B)$ with $B \subset C$. Since $P \in \langle S\rangle$, we have $P_i \subset \langle V \cup B\rangle$ for all i. Since $V \subset \langle\{P_1, \dots, P_x\}\rangle$, we get $\langle B\rangle \cap \langle\{P_1, \dots, P_x\}\rangle \neq \emptyset$. Since $d \geq 2g + xr$ and C is linearly normal, Riemann-Roch gives $\cup_{i=1}^x B_i = C \cap \langle \cup_{i=1}^x B_i\rangle$. Write $B = (B\cap) \cup_{i=1}^x B_i)) \sqcup B'$ and $\sharp(B') \leq r$ (we allow the case $B' = \emptyset$). We have $B \cup (\cup_{i=1}^x B_i) = B' \cup (\cup_{i=1}^x B_i)$. Since $d \geq 2g - 1 + (x + 1)r$, the set $B' \cup (\cup_{i=1}^x B_i)$ is linearly independent.

Hence $\langle B'\rangle \cap \langle \cup_{i=1}^{x} B_i\rangle = \emptyset$ and $\langle B\rangle \cap \langle \cup_{i=1}^{x} B_i\rangle = \langle B \cap (\cup_{i=1}^{x} B_i)\rangle$. Hence $\langle (B \cap \cup_{i=1}^{x} B_i)\rangle \cap \langle \{P_1, \dots, P_x\}\rangle \neq \emptyset$. Set $B_i' := B \cap B_i$. Since $B \neq B_i$ and $\sharp(B_i) \leq \sharp(B_i)$, we have $B \cap B_i \subsetneqq B_i$. Take $B_i' \subset B_i$ such that $\sharp(B_i') = r - 1$ and $B \cap B_i' \subseteq B_i'$. Hence $\langle \cup_{i=1}^{x} B_i'\rangle \cap \{P_1, \dots, P_x\} \neq \emptyset$. Since $B_i \cap B_j = \emptyset$ for all $i \neq j$ and $\cup_{i=1}^{x} B_i$ is linearly independent, $\langle \cup_{i=1}^{x} B_i'\rangle$ has codimension x in $\langle \cup_{i=1}^{x} B_i\rangle$. Since $P_i \in \langle B_i\rangle$ and $P_i \notin \langle B_i'\rangle$, we have $\langle B_i\rangle = \langle B_i' \cup \{P_i\}\rangle$. Thus $\langle \cup_{i=1}^{x} B_i\rangle$ is spanned by the union of $\langle \cup_{i=1}^{x} B_i'\rangle$ and $\langle \{P_1, \dots, P_x\}\rangle$. Since $\langle \cup_{i=1}^{x} B_i'\rangle \cap \langle \{P_1, \dots, P_x\}\rangle \neq \emptyset$ and $\langle \cup_{i=1}^{x} B_i'\rangle$ has codimension x in $\langle \cup_{i=1}^{x} B_i\rangle$, we obtained a contradiction.

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