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LANDSBERG AND BERWALD SPACES OF DIMENSION TWO WITH GENERALIZED (α,β)-METRIC

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Abstract: M. Matsumoto introduced the concept of (α,β) -Metric in the year, 1972. We have in 1999, the concept of generalized (α,β) -metric by taking $\alpha,\beta^{1)},\beta^{2)}.....\beta^{m)}$ where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a purely Riemannian metric and $\beta^{1)},\beta^{2)}.....\beta^{m)}$ all are one form $(\beta^{r)}=b_i^{r)}y^i)$. In the present paper, we had studied the condition under which a two-dimensional generalized (α,β) - Metric be a Landsberg and Berwald spaces in which the main scalar I plays an important role.

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1. Introduction

A Finsler space is called a Landsberg space, if the covariant derivative $C_{hij|k}=0$ of C-torsion tensor $\dot\partial_i\dot\partial_j\dot\partial_k(\frac{L^2}{4})$ with Cartan connection satisfies $C_{hij|k}(x,y)y^k=0$, and a Berwald space is characterized by $C_{hij|0}=0$ [3, 4, 5].We do not know any concrete example of a Landsberg space which is not a Berwald space yet. Berwald spaces are specially interesting and important because the connection is linear, and many examples of Berwald spaces are known.

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The concept of generalized (α, β) - Metric was proposed by authors of [1] in 1999. The authors of [2, 6] found some important result regarding generalized (α, β) - Metric, such as Berwald connection, difference vector, geodesic and main scalar in two-dimensional case. In the present paper, we had studied the condition under which a two-dimensional generalized (α, β) - Metric be a Landsberg and Berwald spaces in which the main scalar I plays an important role.

2. Preliminaries

Let $F^n = (M^n, L(x,y))$ be an n-dimensional Finsler space with a fundamental metric L(x, y). The metric is called generalized (α, β) - Metric if L is a (1)p-homogeneous function $L(\alpha, \beta^{1)}, \beta^{2)}.....\beta^{m)})$ where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a purely Riemannian metric on the underlying manifold M^n and $\beta^{r)} = b_i^r y^i$, $r = 1, 2, ..., m$ all are linearly independent 1-form in M^n.

For, generalized (α, β) - Metric,

$$L = L(\alpha, \beta^{1)}, \beta^{2)}.....\beta^{m)}) \tag{1}$$

Differentiating equation (1) with respect to y^i, the normalized supporting element $l_i = \dot{\partial}_i L$ is given by,

$$l_i = \frac{L_\alpha}{\alpha} Y_i + \sum_{r=1}^{m} L_{\beta^{r)}} b_i^{r)} \tag{2}$$

where, $L_\alpha = \frac{\partial L}{\partial \alpha}$, $L_{\beta^{r)}} = \frac{\partial L}{\partial \beta^{r)}}$, $Y_i = a_{ij} y^j$ and $\sum$ denotes the summation over r which varies from 1 to m.

Differentiating equation (2) with respect to y^j, the angular metric $h_{ij} = L\dot{\partial}_i \dot{\partial}_j L$ and the fundamental tensor $g_{ij} = \dot{\partial}_i \dot{\partial}_j (\frac{L^2}{2}) = h_{ij} + l_i l_j$ and (h)hv-torsion tensor C_{ijk} is given by,

$$h_{ij} = pa_{ij} + \sum_{r,s=1}^{m} q_{rs)} b_i^{r)} b_j^{s)} + \sum_{r=1}^{m} q_{r)} \{b_i^{r)} Y_j + b_j^{r)} Y_i\} + qY_i Y_j \tag{3}$$

$$g_{ij} = pa_{ij} + \sum_{r,s=1}^{m} p_{rs)} b_i^{r)} b_j^{s)} + \sum_{r=1}^{m} p_{r)} \{b_i^{r)} Y_j + b_j^{r)} Y_i\} + p_0 Y_i Y_j \tag{4}$$

$$2pC_{ijk} = \sum_{r,s,t=1}^{m} R_{rst)} b_i^{r)} b_j^{s)} b_k^{t)} + \pi_{(ijk)} \{(h_{ij} B_k) + \sum_{r,s=1}^{m} R_{rs)} (Y_i b_j^{r)} b_k^{s)}) \tag{5}$$

$$+\sum_{r=1}^{m} R_{r)}(Y_iY_jb_k^{r)})\} + RY_iY_jY_k$$

where, $p = \frac{LL_\alpha}{\alpha}$, $q_{rs)} = LL_{\beta^{r)}\beta^{s)}}$, $q_{r)} = \frac{LL_\alpha\beta^{r)}}{\alpha}$,
$q = \frac{1}{\alpha^2}(LL_{\alpha\alpha} - \frac{LL_\alpha}{\alpha})$, $p_{rs)} = q_{rs)} + L_{\beta^{r)}}L_{\beta^{s)}} = LL_{\beta^{r)}\beta^{s)}} + L_{\beta^{r)}}L_{\beta^{s)}}$,
$p_{r)} = q_{r)} + \frac{L_\alpha L_{\beta^{r)}}}{\alpha} = \frac{1}{\alpha}(LL_{\alpha\beta^{r)}} + L_\alpha L_{\beta^{r)}})$, $p_0 = q + \frac{L_\alpha^2}{\alpha^2}$
$= \frac{1}{\alpha^2}(L_\alpha^2 + LL_{\alpha\alpha} - \frac{LL_\alpha}{\alpha})$, $R_{rst)} = pp_{rs)\beta^{t)}} - 3p_{t)}q_{rs)}$,
$R_{rs)} = pp_{r)}\beta^{s)} - p_0q_{rs)} - 2p_{s)}q_{r)}$, $R_{r)} = pp_{0\beta^{r)}} - p_{r)}q - 2p_0q_{r)}$,
$R = \frac{pp_{0\alpha}}{\alpha} - 3p_0q$, $B_k = p_0Y_k + \sum_{r=1}^{m} p_{r)}b_k^{r)}$.

Since, $h_{ij}y^j = 0$, from homogeneity of L, we have,

$$p + \sum_{r=1}^{m} q_{r)}\beta^{r)} + q\alpha^2 = 0, \quad \sum_{r,s=1}^{m} q_{rs)}\beta^{s)} + q_r\alpha^2 = 0 \tag{6}$$

$$\sum_{r,s=1}^{m} p_{rs)}\beta^{s)} + p_r\alpha^2 = LL_{\beta^{r)}}, \quad \sum_{r=1}^{m} p_{r)}\beta^{r)} + p_0\alpha^2 = 0 \tag{7}$$

The inverse matrix g^{ij} of g_{ij} is given by,

$$g^{ij} = \frac{1}{p}a^{ij} - \sum_{r,s=1}^{m} K_{rs)}b^{r)i}b^{s)j} - \sum_{r=1}^{m} K_{r)}\{b^{r)i}Y^j + b^{r)j}Y^i\} - KY^iY^j \tag{8}$$

where, a^{ij} is the inverse matrix of $a_{ij}, K_{rs)}, K_{r)}, K$ are certain scalars and $b^{r)i} = a^{ij}b_j^{r)}$.

3. Landsberg and Berwald Spaces with Generalized Generalized (α, β)-Metric

The covariant differentiation in the Levi-Civita connection $\gamma_{jk}^i(x)$ of R^n is denoted by the semi-colon. Let us list the symbols here for the later use:-

$$\delta^{rs}(b^{r)})^2 = a^{ij}b_i^{r)}b_j^{r)}, \quad 2E_{ij}^{r)} = b_{i;j}^{r)} + b_{j;i}^{r)}, \quad 2F_{ij}^{r)} = b_{i;j}^{r)} - b_{j;i}^{r)}$$

$$E_j^{r)i} = a^{i\mu}E_{\mu j}^{r)}, \quad F_j^{r)i} = a^{i\mu}F_{\mu j}^{r)}, \quad E_i^{r)} = b_\mu^{r)}E_i^{r)\mu}, \quad F_i^{r)} = b_\mu^{r)}F_i^{r)\mu}$$

The Berwald connection BΓ G_{jk}^i, G_j^i of F^n plays leading roles in the present paper. The difference tensor B_{jk}^i [6] of G_{jk}^i from γ_{jk}^i is given by

$$G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$$

and

$$G^i_j = \gamma^i_{j0} + B^i_j, \quad 2G^i = \gamma^i_{00} + 2B^i$$

where, $B^i_j = \dot{\partial}_j B^i$, $B^i_{jk} = \dot{\partial}_k B^i_j$ and '0' denotes transvection by y^i . It is noted that the Cartan connection also has the non-linear connection G^i_{jk} common to BΓ. $B^i(x, y)$ is called the difference vector.

Since, BΓ is L-metrical, $L(\alpha, \beta^{1)}, \beta^{2)}.....\beta^{m)})$ satisfies,

$$L_{|i} = \partial_i L - (\dot{\partial}_r L)G^{r)}_i = 0 = L_\alpha \alpha_{|i} + \textstyle\sum_{r=1}^m L_{\beta^{r)}} \beta^{r)}_{|i}$$

where, $_|$ is h-covariant derivative, and so,

$$\alpha_{|i} = \frac{(\sum_{r=1}^m L_{\beta^{r)}} \beta^{r)}_{|i})}{L_\alpha} \tag{9}$$

It is observed that $\beta^{r)}_{|i} = b^{r)}_{h|i} y^h = (b^{r)}_{h;i} - b^{r)}_\mu B^\mu_{hi}) y^h$, which emplies,

$$\beta^{r)}_{|i} y^i = E^{r)}_{00} - 2b^{r)}_\mu B^\mu \tag{10}$$

For the scalar $(b^{r)})^2$, we have

$$(b^{r)})^2_{|i} y^i = (\partial_i (b^{r)})^2) y^i = (b^{r)})^2_{;i} y^i$$

$$= 2b^{r)\mu} (E^{r)}_{\mu i} + F^{r)}_{\mu i}) y^i$$

Then, we have

$$(b^{r)})^2_{|i} y^i = 2b^{r)\mu} (E^{r)}_0 + F^{r)}_0) y^i \tag{11}$$

Proposition 1. *If $\alpha^2 \equiv 0 (mod \beta^{r)})$, for every r = 1, 2,..., m, that is $a_{ij}(x) y^i y^j$ contains $b^{r)}_i(x) y^i$ as a factor, then the dimension n is equal to two and $(b^{r)})^2$ vanishes. In this case we have $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta^{r)} \delta$, and $d_i b^{r)i} = 2$.*

Proof. from the assumption we have another differential one form $\delta = d_i(x) y^i$ satisfying $\alpha^2 = \beta^{r)} \delta$, for every r = 1,2,...,m, that is

$$a_{ij} = \frac{(b^{r)}_i d_j + b^r_j d_i)}{2} \tag{12}$$

Thus, we have rank $(a_{ij}) < 3$. So that n must be equal to 2. Since $\det(a_{ij}) = -\frac{(b_1^{r)}d_2+b_2^r d_1)^2}{4}$, the Riemannian metric α is not positive-definite and δ is not proportional to $\beta^{r)}$. The transvection of (12) by $b^{r)j}$ gives, $(2 - d_j b^{r)j})b_i^{r)} = (b^{r)})^2 d_i$, which implies $(b^{r)})^2 d_i = 0$, $d_i b^{r)i} = 2$.

Proposition 2. *We consider the two-dimensional case:*

1. *If* $(b^{r)})^2 \neq= 0$, *then there exist a sign* $e = \pm 1$ *and* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \frac{\beta^{r)}\beta^{s)}}{(b^{r)})^2} + e\delta^2$ *and* $d_i b^{r)i} = 0$.

2. *If* $(b^{r)})^2 = 0$, *then there exist* $\delta = d_i(x)y^i$ *such that* $\alpha^2 = \beta^{r)}\delta$, $d_i b^{r)i}$, *for every r = 1, 2,.., m.*

Proof. 1. Since $h_{ij} = a_{ij} - \frac{b_i^{r)}b_j^{s)}}{(b^{r)})^2}$, satisfies $h_{ij}b^{r)j} = 0$, for every r, s=1, 2,.., m. We have rank$(h_{ij}) = 1$. Thus we may assume $h_{11} = 0$, for instance, because $h_{11} = h_{22} = 0$, imply $h_{12} = 0$, a contradiction.
Then, we can get e and $d_1 (\neq 0)$ from $h_{11} = e(d_1)^2$ and d_2 from $h_{12} = d_1 d_2$. In consequence, we have $h_{22} = e(d_2)^2$, hence

$$h_{ij} = e d_i d_j \tag{13}$$

Contracting (13) by $b^{r)i}$, we get,

$$d_i b^{r)i} = 0$$

2. If $b_1^{r)}b_2^{r)} \neq 0$, then $a_{11} = b_1^{r)}\, d_1$ and $a_{22} = b_2^{r)}d_2$. Now, putting a = $\det(a_{ij})$, then $(b^{r)})^2 = a^{ij}b_i^{r)}b_j^{s)} = 0$, leads to,

$$\frac{b_1^{r)}d_2+b_2^{r)}d_1}{2} = a_{12}$$

$$= \frac{1}{b_1^{r)}b_2^{r)}}[a_{22}(b_1^{r)})^2 + a_{11}(b_2^{r)})^2$$

$$= \frac{a}{2b_1^{r)}b_2^{r)}}[a^{11}(b_1^{r)})^2 + a^{22}(b_2^{r)})^2$$

Thus, we get,

$$a_{ij} = \frac{(b_i^{r)}d_j + b_j^{r)}d_i)}{2} \tag{14}$$

Contracting (14) by$b^{r)i}$, we get,

$$d_i b^{r)i} = 2$$

Next, if $b_1^{r)} \neq 0$ and $b_2^{r)} = 0$, then

$$(b^{r)})^2 = b_1^{r)} b^{r)1}, \quad b^{r)1} = 0 = a^{11} b_1^{r)}, \quad a^{11} = 0 = \frac{a_{22}}{a}$$

Thus $a_{22} = 0$. Then $a_{11} = b_1^{r)} d_1$, and $2a_{12} = b_1^{r)} d_2$ give d_1, d_2.
If there are two functions f(x), g(x) satisfying $fa^2 + g(^{r)})^2 = 0$, then,

Proposition 3. *We consider the two-dimensional case,*

1. If there are two functions f(x), g(x) satisfying $g\alpha^2 + f(\beta^{r)})^2 = 0$, then, f = g = 0 or f = g, $(b^{r)})^2 = 0$.

2. If there exists two 1-forms λ, μ satisfying $\lambda\alpha^2 + \mu(\beta^{r)})^2 = 0$, then:

i. $(b^{r)})^2 \neq 0, \lambda = \mu = 0$,

ii. $(b^{r)})^2 = 0, \lambda = f\beta^{r)}, \mu = -f\delta$, where f = f(x) and δ is one in 2. of Proposition 2.

Proof. 1. is trivial. **2.** is verified from proposition 2.

In two-dimensional case, a Finsler space is a Landsberg space iff its main scalar I(x, y) satisfies $I_{|i} y^i = 0$ [3] and is a Berwald space iff $I_{|i} = 0$. The main scalar I [2] of two-dimensional generalized (α, β)-metric is given by,

$$I = \frac{eL}{2TM} \sum_{r=1}^{m} T_{\beta^{r)}} D^{r)} \tag{15}$$

where, $T = e\epsilon(\frac{LM}{\alpha})^2, M^2 = e\epsilon L[\frac{L_\alpha}{\alpha} + e\sum_{r,s=1}^m L_{\beta^{r)}\beta^{s)}} \beta^{r)}\beta^{s)}], (D^{r)})^2 = e[(b^{r)})^2 - (\frac{\beta^{r)}}{\alpha})^2]$ and e, ϵ are sign with respect to Riemannian frame and Finslerian frame respectively. The h-covariant differentiation of (15), is given by,

$$I_{|i} = \frac{eL[\sum_{r=1}^m (T_{\beta^r} D^{r)})_{|i} 2TM - 2(TM)_{|i} \sum_{r=1}^m (T_{\beta^r} D^{r)})]}{(2TM)^2} \tag{16}$$

where,

$$M_{|i} = \frac{e\epsilon L}{2M}[\frac{L_{\alpha|i}}{\alpha} + \frac{1}{\alpha^2}\sum_{r=1}^m L_{\beta^{r)}}(b_{0;i}^{r)} - b_\mu^{r)} B_i^\mu) + \sum_{r,s=1}^m (L_{\beta^{r)}\beta^{s)}})_{|i} \beta^{r)}\beta^{s)} + L_{\beta^{r)}\beta^{s)}}(b_{0;i}^{r)} - b_\mu^{r)} B_i^\mu)\beta^{s)} + L_{\beta^{r)}\beta^{s)}}(b_{0;i}^{s)} - b_\mu^{s)} B_i^\mu)\beta^{r)})]$$

$$T_{|i} = \frac{2e\epsilon L^2 M}{\alpha^2}[M_{|i} + \frac{M\sum_{r=1}^m L_{\beta^{r)}}(b_{0;i}^{r)} - b_\mu^{r)} B_i^\mu)}{\alpha L_\alpha}]$$

$$D_i^{r)} = \frac{e}{2D^{r)}}[2b^{r)j}(E_{ij}^{r)} + F_{ij}^{r)}) - \frac{2\beta^{r)}(b_{0;i}^{r)} - b_\mu^{r)} B_i^\mu)}{\alpha^2} - \frac{2}{\alpha L_\alpha}(\frac{\beta^{r)}}{\alpha})^2 \sum_{r=1}^m L_{\beta^{r)}}(b_{0;i}^{r)} - b_\mu^{r)} B_i^\mu)]$$

Thus a generalized (α, β) Finsler space is a Berwald space, iff

$$\sum_{r=1}^{m}(T_{\beta^r}D^{r)})_{|i}2TM - 2(TM)_{|i}\sum_{r=1}^{m}(T_{\beta^r}D^{r)}) = 0 \tag{17}$$

Contracting equation (17) by y^i, we get,

$$\sum_{r=1}^{m}(T_{\beta^r}D^{r)})_{|0}2TM - 2(TM)_{|0}\sum_{r=1}^{m}(T_{\beta^r}D^{r)}) = 0 \tag{18}$$

where,

$$M_{|0} = \tfrac{e\epsilon L}{2M}[\tfrac{L_{\alpha|0}}{\alpha} + \tfrac{1}{\alpha^2}\textstyle\sum_{r=1}^{m} L_{\beta^{r)}}(E^{r)}_{00} - b^{r)}_{\mu}B^{\mu}) + \sum_{r,s=1}^{m}(L_{\beta^{r)}\beta^{s)}})_{|0}\beta^{r)}\beta^{s)} + L_{\beta^{r)}\beta^{s)}}(E^{r)}_{00} - b^{r)}_{\mu}B^{\mu})\beta^{s)} + L_{\beta^{r)}\beta^{s)}}(E^{s)}_{00} - b^{s)}_{\mu}B^{\mu})\beta^{r)})]$$

$$T_{|0} = \tfrac{2e\epsilon L^2 M}{\alpha^2}[M_{|0} + \tfrac{M\sum_{r=1}^{m} L_{\beta^{r)}}(E^{r)}_{00} - b^{r)}_{\mu}B^{\mu}_{i})}{\alpha L_{\alpha}}]$$

$$D^{r)}_0 = \tfrac{e}{2D^{r)}}[(E^{r)}_0 + F^{r)}_0) - \tfrac{2\beta^{r)}(E^{r)}_{00} - b^{r)}_{\mu}B^{\mu})}{\alpha^2} - \tfrac{2}{\alpha L_{\alpha}}(\tfrac{\beta^{r)}}{\alpha})^2\textstyle\sum_{r=1}^{m} L_{\beta^{r)}}(E^{r)}_{00} - b^{r)}_{\mu}B^{\mu})]$$

Thus

Theorem 1. *In, two-dimensional Finsler space with generalized (α, β)-metric, is a Berwald space iff equation (17) holds good and is a Landsberg space iff equation (18) holds good.*

4. Landsberg and Berwald Spaces with m-Generalized Kropina Metric

The m-generalized Kropina metric is defined as

$$L = \frac{\alpha^2}{\beta^{1)}} + \frac{\alpha^2}{\beta^{2)}} + \frac{\alpha^2}{\beta^{3)}} + ... + \frac{\alpha^2}{\beta^{m)}} = \sum_{r=1}^{m}\frac{\alpha^2}{\beta^{r)}} \tag{19}$$

The main scalar I in two-dimensional case is given by,

$$I = \frac{e(\sum_{r=1}^{m}\frac{\alpha^2}{\beta^{r)}})}{2TM}\sum_{r=1}^{m}T_{\beta^{r)}}D^{r)} \tag{20}$$

where,

$$M^2 = 2e\epsilon L(\textstyle\sum_{r=1}^{m} \frac{1}{\beta^{r)}} + 2eL), T = e\epsilon(\frac{LM}{\alpha})^2$$

The h-covariant derivative is given as,

$$I_{|i} = \frac{eL[\sum_{r=1}^{m}(T_{\beta^r} D^{r)})_{|i} 2TM - 2(TM)_{|i} \sum_{r=1}^{m}(T_{\beta^r} D^{r)})]}{(2TM)^2} \tag{21}$$

where,

$$M_{|i} = \tfrac{-e\epsilon L}{M} \tfrac{1}{(\beta^{r)})^2} \textstyle\sum_{r=1}^{m}(b_{0;i}^{r)} - b_{\mu}^{r)} B_i^{\mu})$$

$$T_{|i} = \tfrac{2e\epsilon L^2 M}{\alpha^2}[M_{|i} + \tfrac{M \sum_{r=1}^{m} L_{\beta^{r)}} (b_{0;i}^{r)} - b_{\mu}^{r)} B_i^{\mu})}{\alpha L_\alpha}]$$

Thus, a m-generalized Kropina metric is a Berwald space iff,

$$TM \sum_{r=1}^{m}(T_{\beta^r} D^{r)})_{|i} - (TM)_{|i} \sum_{r=1}^{m}(T_{\beta^r} D^{r)}) = 0 \tag{22}$$

Contracting equation (22) by y^i, we get,

$$\sum_{r=1}^{m}(T_{\beta^r} D^{r)})_{|0} 2TM - 2(TM)_{|0} \sum_{r=1}^{m}(T_{\beta^r} D^{r)}) = 0 \tag{23}$$

where,

$$M_{|0} = \tfrac{-e\epsilon L}{M} \tfrac{1}{(\beta^{r)})^2} \textstyle\sum_{r=1}^{m}(E_{00}^{r)} - b_{\mu}^{r)} B^{\mu})$$

$$T_{|0} = \tfrac{2e\epsilon L^2 M}{\alpha^2}[M_{|0} + \tfrac{M \sum_{r=1}^{m} L_{\beta^{r)}} (E_{00}^{r)} - b_{\mu}^{r)} B^{\mu})}{\alpha L_\alpha}]$$

Thus

Theorem 2. *In two-dimensional Finsler space with m-generalized Kropina metric is a Berwald space iff equation (22) holds good and is a Landsberg space iff equation (23) holds good.*

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