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OPTIMAL CONVEX COMBINATION BOUNDS OF THE CENTROIDAL AND HARMONIC MEANS FOR THE SEIFFERT MEAN

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Abstract: We find the greatest value α and the least value β such that the double inequality

$$\alpha T(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta T(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$. Here $T(a,b)$, $H(a,b)$ and $P(a,b)$ denote the Centroidal, harmonic, and the Seiffert means of two positive numbers a and b, respectively.

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Key Words: optimal convex combination bound, Centroidal mean, harmonic mean, the Seiffert mean

1. Introduction

For $a, b > 0$ with $a \neq b$ the Seiffert means $P(a,b)$ was introduced by Seiffert [1,2] as follows:

$$P(a,b) = \frac{a-b}{4\arctan(\sqrt{a/b}) - \pi}. \tag{1.1}$$

Recently, the inequalities for means have been the subject of intensive research [3-20]. In particular, many remarkable inequalities for the Seiffert mean can be found in the literature [4,15-20].

Let $T(a,b) = 2(a^2+ab+b^2)/3(a+b), H(a,b) = 2ab/(a+b), A(a,b) = (a+b)/2, G(a,b) = \sqrt{ab}, I(a,b) = 1/e(b^b/a^a)^{1/(b-a)}$ and $L(a,b) = (b-a)/(logb-loga)$ be the centroidal, harmonic, arithmetic, geometric, identric and logarith-

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mic means of two positive real numbers a and b with $a \neq b$. Then

$$\min\{a,b\} < H(a,b) < G(a,b) < L(a,b) < I(a,b) < A(a,b) < T(a,b) < \max\{a,b\}. \quad (1.2)$$

In [1], Seiffert proved

$$L(a,b) < P(a,b) < I(a,b)$$

for all $a, b > 0$ with $a \neq b$.

The following bounds for the Seiffert mean $P(a,b)$ in terms of the power mean $M_r(a,b) = ((a^r + b^r)/2)^{1/r} (r \neq 0)$ were presented by Jagers in [18]:

$$M_{1/2} < P(a,b) < M_{2/3}(a,b) \quad (1.3)$$

for all $a, b > 0$ with $a \neq b$.

Hästö [20] found the sharp lower bound for the Seiffert mean as follow:

$$M_{log2/log\pi}(a,b) < P(a,b) \quad (1.4)$$

for all $a, b > 0$ with $a \neq b$.

In [3], Seiffert proved

$$P(a,b) > \frac{3A(a,b)G(a,b)}{A(a,b) + 2G(a,b)} \quad \text{and} \quad P(a,b) > \frac{2}{\pi}A(a,b) \quad (1.5)$$

for all $a, b > 0$ with $a \neq b$.

In [4], the authors found the greatest value α and the least value β such that the double inequality $\alpha A(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta A(a,b) + (1-\beta)H(a,b)$ holds for all $a, b > 0$ with $a \neq b$.

The purpose of the present paper is to find the greatest value α and the least value β such that the double inequality

$$\alpha T(a,b) + (1-\alpha)H(a,b) < P(a,b) < \beta T(a,b) + (1-\beta)H(a,b)$$

holds for all $a, b > 0$ with $a \neq b$.

2. Main Results

Theorem 2.1. *The double inequality*

$$\alpha T(a,b)+(1-\alpha)H(a,b)<P(a,b)<\beta T(a,b)+(1-\beta)H(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha\leqslant\frac{3}{2\pi}$ *and* $\beta\geqslant\frac{5}{8}$.

Proof. Firstly, we prove that

$$P(a,b)<\frac{5}{8}T(a,b)+\frac{3}{8}H(a,b), \tag{2.1}$$

$$P(a,b)>\frac{3}{2\pi}T(a,b)+\left(1-\frac{3}{2\pi}\right)H(a,b), \tag{2.2}$$

for all $a,b>0$ with $a\neq b$.

Without loss of generality, we assume $a>b$. Let $t=\sqrt{a/b}>1$ and $p\in\{\frac{5}{8},\frac{3}{2\pi}\}$. Then (1.1) leads to

$$\begin{aligned}&\frac{1}{b}[pT(a,b)+(1-p)H(a,b)-P(a,b)]\\ =\;&pT(t^2,1)+(1-p)H(t^2,1)-P(t^2,1)\\ =\;&\frac{2p(t^4+t^2+1)+6(1-p)t^2}{3(t^2+1)(4\arctan t-\pi)}f(t),\end{aligned} \tag{2.3}$$

where

$$f(t)=4\arctan t-\pi-\frac{3(t^4-1)}{2p(t^4+t^2+1)+6(1-p)t^2}. \tag{2.4}$$

Simple computations lead to

$$\lim_{t\to1^+}f(t)=0, \tag{2.5}$$

$$\lim_{t\to+\infty}f(t)=\pi-\frac{3}{2p}, \tag{2.6}$$

$$f'(t)=\frac{g_1(t)}{(1+t^2)[2p(t^4+t^2+1)+6(1-p)t^2]^2} \tag{2.7}$$

where

$$\begin{aligned}g_1(t)\;&=16p^2t^8-(36-24p)t^7+32p(3-2p)t^6-(36+24p)t^5\\ &+48(3+2p^2-4p)t^4-(36+24p)t^3\\ &+32p(3-2p)t^2-(36-24p)t+16p^2.\end{aligned} \tag{2.8}$$

Now we divide the proof into two cases:

Case 1. If $p = \frac{5}{8}$. (2.8) leads to

$$g_1(t) = \frac{(t-1)^4}{4}(25t^4 + 16t^3 + 54t^2 + 16t + 25) > 0 \tag{2.9}$$

for $t > 1$. (2.9) and (2.7) imply $f'(t) > 0$, thus $f(t)$ is strictly increasing for $t > 1$. Then inequality (2.1) follows from (2.3)-(2.5).

Case 2. If $p = \frac{3}{2\pi}$, Then from (2.8) we get

$$\lim_{t \to 1^+} g_1(t) = 0, \tag{2.10}$$

$$\lim_{t \to +\infty} g_1(t) = +\infty. \tag{2.11}$$

$$g_1'(t) = 128p^2t^7 - 84(3-2p)t^6 + 192p(3-2p)t^5 - 60(3+2p)t^4 + 192(3+2p^2-4p)t^3 - 36(3+2p)t^2 + 64p(3-2p)t - (36-24p), \tag{3.12}$$

$$\lim_{t \to 1^+} g_1'(t) = 0, \tag{2.13}$$

$$\lim_{t \to +\infty} g_1'(t) = +\infty. \tag{2.14}$$

$$\begin{aligned} g_1''(t) &= 896p^2t^6 - 504(3-2p)t^5 + 960p(3-2p)t^4 - 240(3+2p)t^3 \\ &+576(3+2p^2-4p)t^2 - 72(3+2p)t + 64p(3-2p), \end{aligned} \tag{2.15}$$

$$\lim_{t \to 1^+} g_1''(t) = 1152p - 720 = \frac{1152 \times 3}{2\pi} - 720 < 0, \tag{2.16}$$

$$\lim_{t \to +\infty} g_1''(t) = +\infty. \tag{2.17}$$

$$\begin{aligned} g_1'''(t)|_{p=\frac{3}{2\pi}} &= 216[\tfrac{56}{\pi^2}t^5 - 35(1-\tfrac{1}{\pi})t^4 + \tfrac{80}{\pi}(1-\tfrac{1}{\pi})t^3 - 10(1+\tfrac{1}{\pi})t^2 \\ &+16(1+\frac{3}{2\pi^2} - \frac{2}{\pi})t - (1+\frac{1}{\pi})], \end{aligned} \tag{2.18}$$

Let

$$\begin{aligned} g_2(t) &= \tfrac{56}{\pi^2}t^5 - 35(1-\tfrac{1}{\pi})t^4 + \tfrac{80}{\pi}(1-\tfrac{1}{\pi})t^3 - 10(1+\tfrac{1}{\pi})t^2 \\ &+16(1+\frac{3}{2\pi^2} - \frac{2}{\pi})t - (1+\frac{1}{\pi}). \end{aligned} \tag{2.19}$$

Then

$$\lim_{t \to 1^+} g_2(t) = \frac{72}{\pi} - 30 < 0, \tag{2.20}$$

$$\lim_{t\to+\infty} g_2(t) = +\infty. \tag{2.21}$$

$$\begin{aligned} g_2'(t) &= \tfrac{280}{\pi^2}t^4 - 140(1-\tfrac{1}{\pi})t^3 + \tfrac{240}{\pi}(1-\tfrac{1}{\pi})t^2 - 20(1+\tfrac{1}{\pi})t \\ &+16(1+\frac{3}{2\pi^2}-\frac{2}{\pi}), \end{aligned} \tag{2.22}$$

$$\lim_{t\to1^+} g_2'(t) = \frac{64}{\pi^2} + \frac{328}{\pi} - 144 < 0, \tag{2.23}$$

$$\lim_{t\to+\infty} g_2'(t) = +\infty. \tag{2.24}$$

$$g_2''(t) \quad = 20[\tfrac{56}{\pi^2}t^3 - 21(1-\tfrac{1}{\pi})t^2 + \tfrac{24}{\pi}(1-\tfrac{1}{\pi})t - (1+\tfrac{1}{\pi})] = 20g_3(t), \tag{2.25}$$

where

$$g_3(t) = \frac{56}{\pi^2}t^3 - 21(1-\frac{1}{\pi})t^2 + \frac{24}{\pi}(1-\frac{1}{\pi})t - (1+\frac{1}{\pi}). \tag{2.26}$$

So,

$$\lim_{t\to1^+} g_3(t) = \frac{32}{\pi^2} + \frac{44}{\pi} - 22 < 0, \tag{2.27}$$

$$\lim_{t\to+\infty} g_3(t) = +\infty. \tag{2.28}$$

$$g_3'(t) = 6[\frac{28}{\pi^2}t^2 - 7(1-\frac{1}{\pi})t + \frac{4}{\pi}(1-\frac{1}{\pi})] = 6g_4(t), \tag{2.29}$$

where

$$g_4(t) = \frac{28}{\pi^2}t^2 - 7(1-\frac{1}{\pi})t + \frac{4}{\pi}(1-\frac{1}{\pi}). \tag{2.30}$$

By simple computation, we have

$$\lim_{t\to1^+} g_4(t) = \frac{24}{\pi^2} + \frac{11}{\pi} - 7 < 0, \tag{2.31}$$

$$\lim_{t\to+\infty} g_4(t) = +\infty. \tag{2.32}$$

$$g_4'(t) = \frac{56}{\pi^2}t - 7(1-\frac{1}{\pi}), \tag{2.33}$$

$$\lim_{t\to1^+} g_4'(t) = \frac{56}{\pi^2} - 7(1-\frac{1}{\pi}) > 0, \tag{2.34}$$

$$g_4''(t) = \frac{56}{\pi^2} > 0, \tag{2.35}$$

From (2.35) and (2.34) we clearly see that $g_4'(t) > 0$ for $t > 1$, hence $g_4(t)$ is strictly increasing in $[1, +\infty)$. It follows from (2.31)and (2.32) together with the monotonicity of $g_4(t)$ that there exists $\lambda_1 > 1$ such that $g_4(t) < 0$ for $t \in [1, \lambda_1)$

and $g_4(t) > 0$ for $t \in (\lambda_1, +\infty)$, hence from (2.29) $g_3(t)$ is strictly decreasing in $[1, \lambda_1]$ and strictly increasing in $[\lambda_1, +\infty)$.

From (2.27) and (2.28) together with the monotonicity of $g_3(t)$ we know that there exists $\lambda_2 > 1$ such that $g_3(t) < 0$ for $t \in [1, \lambda_2)$ and $g_3(t) > 0$ for $t \in (\lambda_2, +\infty)$, hence from (2.25) $g_2'(t)$ is strictly decreasing in $[1, \lambda_2]$ and strictly increasing in $[\lambda_2, +\infty)$.

From (2.23) and (2.24) together with the monotonicity of $g_2'(t)$ we clearly see that there exists $\lambda_3 > 1$ such that $g_2(t)$ is strictly decreasing in $[1, \lambda_3]$ and strictly increasing in $[\lambda_3, +\infty)$. It follows from (2.18) (2.20) and (2.21) together with the monotonicity of $g_2(t)$ that there exists $\lambda_4 > 1$ such that $g_1''(t)$ is strictly decreasing in $[1, \lambda_4]$ and strictly increasing in $[\lambda_4, +\infty)$.

From (2.16) and (2.17) together with the monotonicity of $g_1''(t)$ we can see that there exists $\lambda_5 > 1$ such that $g_1'(t)$ is strictly decreasing in $[1, \lambda_5]$ and strictly increasing in $[\lambda_5, +\infty)$. From (2.13) and (2.14) together with the monotonicity of $g_1'(t)$ we clearly see there exists $\lambda_6 > 1$ such that $g_1(t)$ is strictly decreasing in $[1, \lambda_6]$ and strictly increasing in $[\lambda_6, +\infty)$. Then (2.7) (2.10) and (2.11) imply that there exists $\lambda_7 > 1$ such that $f(t)$ is strictly decreasing in $[1, \lambda_7]$ and strictly increasing in $[\lambda_7, +\infty)$.

Note that (2.6) becomes

$$\lim_{t \to +\infty} f(t) = 0, \tag{2.36}$$

for $p = \frac{3}{2\pi}$.

It follows from (2.5) and (2.36) together with the monotonicity of $f(t)$ that

$$f(t) < 0, \tag{2.37}$$

for $t > 1$.

Therefore, inequality (2.2) follows from (2.3) and (2.4) together with (2.37).

Secondly, we prove that $\frac{5}{8}T(a,b) + \frac{3}{8}H(a,b)$ is the best possible upper convex combination bound of centroidal and harmonic means for the Seiffert mean P(a,b).

For any $t > 1$ and $\beta \in R$, we have

$$\begin{aligned} \beta T(t^2,1) + (1-\beta)H(t^2,1) - P(t^2,1) &= \tfrac{2}{3}\beta\tfrac{t^4+t^2+1}{t^2+1} + (1-\beta)\tfrac{2t^2}{t^2+1} - \tfrac{t^2-1}{4\arctan t - \pi} \\ &= \frac{h(t)}{3(t^2+1)(4\arctan t - \pi)}, \end{aligned} \tag{2.38}$$

where

$$h(t) = [2\beta(t^4+t^2+1) + 6(1-\beta)t^2](4\arctan t - \pi) - 3(t^4 - 1). \tag{2.39}$$

It follows from (2.39) that

$$h(1) = h'(1) = h''(1) = 0, \tag{2.40}$$

$$h'''(1) = 12(8\beta - 5). \tag{2.41}$$

If $\beta < \frac{5}{8}$, then (2.41) leads to

$$h'''(1) < 0. \tag{2.42}$$

From (2.42) and the continuity of $h'''(t)$ we see that there exists $\delta = \delta(\beta) > 0$ such that

$$h'''(t) < 0 \tag{2.43}$$

for $t \in [1, 1+\delta)$. Then (2.40) and (2.43) imply that

$$h(t) < 0 \tag{2.44}$$

for $t \in [1, 1+\delta)$.

Therefore, $\beta T(t^2, 1) + (1-\beta)H(t^2, 1) < P(t^2, 1)$ for $t \in (1, 1+\delta)$ follows from (2.38) and (2.44).

Finally, we prove that $\frac{3}{2\pi}T(a,b) + (1 - \frac{3}{2\pi})H(a,b)$ is the best possible lower convex combination bound of centroidal and harmonic means for the Seiffert mean P(a,b).

In fact, for $\alpha > \frac{3}{2\pi}$, we have

$$\lim_{t\to+\infty} \frac{\alpha T(1,x) + (1-\alpha)H(1,x)}{P(1,x)} = \frac{2\pi}{3}\alpha > 1. \tag{2.45}$$

Inequality (2.45) implies that for any $\alpha > \frac{3}{2\pi}$ there exists $X = X(\alpha) > 1$ such that $\alpha T(1,x) + (1-\alpha)H(1,x) > P(1,x)$ for $x \in (X, +\infty)$. □

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