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STUDY ON SINGULAR SYSTEMS OF INDEX THREE VIA RUNGE-KUTTA METHOD OF ORDER-10

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Abstract: Singular systems of index three are investigated using Single Term Walsh Series (STWS) technique and Runge-Kutta method of order-10 (RK-10). Singular systems are remodeled as initial value problems and have been solved using RK-10. The obtained results are compared with the exact solutions and are found to be very accurate. The proposed method is very simple to implement in a digital computer and the solutions can be found for any length of time. Numerical examples are presented.

AMS Subject Classification: 93C15, 42C05, 65L06
Key Words: singular systems, single-term Walsh series, Runge-Kutta method

1. Introduction

Singular systems are of considerable research interest in recent days. Campbell [2] employed orthogonal functions to the analysis of solutions of singular systems. Rao et al., [4] introduced the Single-Term Walsh Series (STWS) approach to remove the invonveniences in Walsh Function method. Palanisamy and Balachandran [5] proposed STWS approach to time-varying singular systems. Balachandran and Murugesan [6] demonstrated the efficiency of STWS technique through different types of systems. Butcher and Wanner [9] surveyed the development of Runge-Kutta methods. Murugesan et al., [10] analyzed different second order systems via Runge-Kutta method. Hairer [8] presented 17-stage, Tenth order Runge-Kutta method (RK-10). Campbell [3] discussed a general solution of the index two linear singular system. Balachandran and

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Murugesan [1] established that the STWS method fails to work for singular systems of index three. This paper presents the results of the singular system of index three using STWS and Runge-Kutta method of order-10.

2. Singular Systems

Singular systems have wide applications in control problems, economics, demography, large-scale systems, power systems, interconnected systems, robotics, aircraft dynamics, neutral delay systems, bio-medical engineering systems and network theory, etc.,[11]. Several researchers have adopted different techniques to study the higher index singular systems [12], [13] and [14].

Consider the linear time-varying singular system

$$K(t)\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

with $x(0) = x_0$, where the $n \times n$ matrix $K(t)$ is singular for all t, $A(t)$ is an $n \times n$ matrix, $B(t)$ is an $n \times r$ matrix, $x(t)$ is an n-state vector and $u(t)$ is an r-input vector. If the matrix $K(t)$ is nilpotent of index v, that is $K^v = 0$ and $K^{v-1} \neq 0$, then the index of $K(t)$ is v (for definition of index, one can refer [7]).

3. Single-Term Walsh Series (STWS)

For the time-varying singular system, the following recursive relationships are obtained by using the STWS method [1],[5].

With this STWS approach, the given function is expanded as single term Walsh sries in the normalized interval $\tau \in [0, 1)$, which corresponds to $t \in [0, 1/m)$ by defining $t = \tau/m$, m being any integer

$$\begin{aligned} C_i &= \big[M_i - \frac{S_i}{2m}\big]^{-1} G_i, \\ B_i &= \frac{C_i}{2} + x(i-1), \\ x(i) &= C_i + x(i-1), \\ G_i &= \big[S_i x(i-1) + Y_i H_i\big]/m, \end{aligned} \tag{2}$$

where

$$M_i = m \int_{(i-1)/m}^{i/m} K(t)dt, \quad S_i = m \int_{(i-1)/m}^{i/m} A(t)dt,$$

$$Y_i = m\int_{(i-1)/m}^{i/m} B(t)dt, \quad H_i = m\int_{(i-1)/m}^{i/m} u(t)dt.$$

and $i = 1, 2, 3, \ldots$ the interval number, $x(i)$ gives the discrete values of the state and B_i gives the block-pulse values of the state to any length of time. The value of $'m'$ can be selected as large enough to increase the accuracy.

4. Runge-Kutta Method of Order-10 (RK-10)

Runge-Kutta methods have been used by many researchers to determine numerical solutions for problems, which are modeled as initial value problems involving differential equations that arise in the fields of Science and Engineering. RK methods have become very popular, both as computational techniques and in research applications [9]. The method was developed by Runge in the mid-1890s and extended by Kutta, a few years later. They developed algorithms, which solve differential equations efficiently and yet are the equivalent of approximating the exact solutions by matching n terms of the Taylor series expansion.

Runge-Kutta methods have always been considered to be the excellent tools for the numerical integration of ordinary differential equations. The fact that RK methods are self-starting, easy to program and extremely accurate and versatile in ODE problems has led to their continuous analysis and use in mathematical research.

Consider the initial value problem

$$y' = f(x, y) \quad \text{with} \quad y(x_0) = y_0.$$

The general expression for the s-stage Runge Kutta method is

$$\begin{aligned} y_{n+1} &= y_n + h\sum_{i=1}^{s} b_i k_i, \qquad (3)\\ k_i &= f\big(x_n + c_i h, y_n + h\sum_{j=1}^{s} a_{i,j}\ k_j\big), \end{aligned}$$

where

$$c_i = \sum_{i=1}^{s} a_{i,j} \quad i = 1, 2, \ldots s.$$

The Butcher tableau is of the form

Hairer[8] proposed a Runge Kutta method of order 10. The coefficients which are not mentioned are zero. The Butcher coefficients are

$$
\begin{array}{c|cccccc}
c_1 & a_{11} & & & & & \\
c_2 & a_{21} & a_{22} & & & & \\
c_3 & a_{31} & a_{32} & a_{33} & & & \\
\vdots & \vdots & \vdots & \vdots & & & \\
c_{s-1} & a_{s1} & a_{s2} & a_{s3} & \dots & a_{s,s-1} & \\
c_s & a_{s1} & a_{s2} & a_{s3} & \dots & a_{s,s-1} & a_{s,s} \\
\hline
 & b_1 & b_2 & b_3 & \dots & b_{s-1} & b_s
\end{array}
$$

$b_1 = 0.0333333333333333333333$	$b_2 = -0.0333333333333333333333$
$b_3 = -0.12$	$b_6 = -0.13$
$b_7 = -0.18$	$b_9 = 0.27742918851774317650 8$
$b_{10} = 0.18923747814892349015 8$	$b_{11} = 0.27742918851774317650 8$
$b_{12} = 0.18923747814892349015 8$	$b_{13} = 0.13$
$b_{14} = 0.18$	$b_{15} = 0.12$
$b_{16} = 0.0333333333333333333333$	$b_{17} = 0.0333333333333333333333$
$c_2 = 0.5$	$c_3 = 0.526509100141612162751$
$c_4 = 0.789763650212418244126$	$c_5 = 0.393923570125671611283$
$c_6 = 0.7666539862535488$	$c_7 = 0.289763650212418244126$
$c_8 = 0.108477689219566212940$	$c_9 = 0.357384241759677451843$
$c_{10} = 0.882527661964732346426$	$c_{11} = 0.642615758240322548157$
$c_{12} = 0.117472338035267653574$	$c_{13} = 0.7666539862535488$
$c_{14} = 0.289763650212418244126$	$c_{15} = 0.526509100141612162751$
$c_{16} = 0.5$	$c_{17} = 1$
$a_{2,1} = 0.5$	$a_{3,1} = 0.249297267609681978013$
$a_{3,2} = 0.277211832531930184738$	$a_{4,1} = 0.197440912553104561032$
$a_{4,3} = 0.592322737659313683095$	$a_{5,1} = 0.197320548628702140900$

5. Numerical Examples

Example 1. Consider the singular system discussed in [1], by taking $K(t) = \begin{pmatrix} 0 & -t & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{pmatrix}$, $A(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B(t) = 0$ with $x(0) = \begin{pmatrix} 0 \\ 1/e \\ 1/e \end{pmatrix}$

Exact solution is

$$x_1(t) = -te^{-(t+1)}, \quad x_2(t) = e^{-(t+1)}, \quad x_3(t) = e^{-(t+1)}. \tag{4}$$

Using the equations (2),(3) and (4), the discrete time solution by STWS, RK-10 and exact solutions are calculated along with absolute errors and are presented in Table 1.

Example 2. Consider the singular system discussed in [1], by taking

$a_{5,3} = 0.29508333409267185371\,1$ $a_{5,4} = -0.09848031259570238332\,77$

$a_{5,3} = 0.2950833340926718537\,11$

$a_{5,4} = -0.098480312595702383327\,7$

$a_{6,1} = 0.13131341734446152007\,6$

$a_{6,4} = 0.11015443953863850704\,0$

$a_{6,5} = 0.52518612937044877288\,4$

$a_{7,1} = 0.13420034184632240619\,3$

$a_{7,4} = 0.69608870328807690807\,9$

$a_{7,5} = 0.25049772157033937535\,2$

$a_{7,6} = -0.79102311649232044549\,8$

$a_{8,1} = 0.07221827418966214544\,8$

$a_{8,5} = -0.05833632293645503691\,26$

$a_{8,6} = 0.00304755766857449437\,925$

$a_{8,7} = 0.09154818029778461002\,86$

$a_{9,1} = 0.03125500813516561706\,20$

$a_{9,6} = 0.00010912382154241994\,6873$

$a_{9,7} = 0.15672575863099501516\,4$

$a_{9,8} = 0.16929435117197439967\,0$

$a_{10,1} = 0.01190660441467503214\,45$

$a_{10,6} = 0.28343708202460654811\,2$

$a_{10,7} = -0.41631216757056131505\,6$

$a_{10,8} = 0.26464633394974300483\,7$

$a_{10,9} = 0.73884980914626907638\,8$

$a_{11,1} = 0.02340657369133544937\,17$

$a_{11,6} = 0.09449313018949618022\,40$

$a_{11,7} = -0.27287205590195641900\,6$

$a_{11,8} = 0.22402204611559220741\,0$

$a_{11,9} = 0.60438144107513509541\,6$

$a_{11,10} = -0.03081537692927996525\,86$

$a_{12,1} = 0.04544377531017636994\,08$

$a_{12,6} = -0.00118799667186441567\,723$

$a_{12,7} = 0.01203565499092811348\,03$

$a_{12,8} = 0.07512690298764792405\,91$

$a_{12,9} = -0.01822092409888456904\,12$

$a_{12,10} = -0.00025715285408406504\,2855$

$a_{12,11} = 0.00453207837134829585\,506$

$a_{13,1} = 0.17840108640043642929\,2$

$a_{13,4} = 0.11015443953863850704\,0$

$a_{13,5} = 0.52518612937044877288\,4$

$a_{13,6} = -0.48914859182043621280\,3$

$a_{13,7} = 0.93244361263513573303\,8$

$a_{13,8} = -0.77447505343983952540\,9$

$a_{13,9} = -1.05490217813935824270$

$a_{13,10} = 0.13104671203415715450\,9$

$a_{13,11} = 0.58704977759948739226\,7$

$a_{13,12} = 0.62089805207487879188\,1$

$a_{14,1} = 0.13022080660049779349\,6$

$a_{14,4} = 0.69608870328807690807\,9$

$a_{14,5} = 0.25049772157033937535\,2$

$a_{14,6} = -0.75894898712960734266\,2$

$a_{14,7} = -0.17151720846348838357\,7$

$a_{14,8} = -0.37021767367890670468\,8$

$a_{14,9} = 0.12498100857474734780\,2$

$a_{14,10} = 0.00335310924837267073\,965$

$a_{14,11} = -0.00663254613676153581\,907$

$a_{14,12} = 0.42911657312161790471\,4$

$a_{14,13} = -0.03717785678246978931\,08$

$a_{15,1} = 0.24929726760968197801\,3$

$a_{15,2} = 0.27721183253193018473\,8$

$a_{15,6} = -0.14594059593608521818\,5$

$a_{15,7} = -0.79901589351102947535\,8$

$a_{15,13} = 0.14594059593608521818\,5$

$a_{15,14} = 0.79901589351102947535\,8$

$a_{16,1} = 0.5$

$a_{16,3} = -0.80709707609534109325\,1$

$a_{16,15} = 0.80709707609534109325\,1$

$a_{17,1} = 0.05732079543205754123\,21$

$a_{17,2} = -0.5$

$a_{17,3} = -0.89747016339485512084\,6$

$a_{17,6} = -1.03991004922695343354$

$a_{17,7} = -0.40735701428838580902\,2$

$a_{17,8} = -0.18283023664074184966\,3$

$a_{17,9} = -0.33365927064922502113\,7$

$a_{17,10} = 0.39564854237605792400\,1$

$a_{17,11} = 0.69505704945998228178\,0$

$a_{17,12} = 0.27148737645738323911\,1$

$a_{17,13} = 0.58542373486658975681\,1$

$a_{17,14} = 0.95881907221323537042\,9$

$a_{17,15} = 0.89747016339485518420\,6$

$a_{17,16} = 0.5$

$$K(t) = \begin{pmatrix} 0 & -t & 0 \\ 1 & 0 & t \\ 0 & 1 & 0 \end{pmatrix},\ A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ B(t) = 0 \text{ with } x(0) = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

			$x_1(t)$		
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	-0.071626	0.000000	0.071626	-0.071626	0.000000
0.50	-0.111565	-8.829107	-8.717542	-0.111565	0.000000
0.75	-0.130330	523.860324	523.990654	-0.130330	0.000000
1.00	-0.135335	-26357.826201	-26357.690866	-0.135335	0.000000
			$x_2(t)$		
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.286505	0.367879	0.081374	0.286505	0.000000
0.50	0.223130	-2.575156	-2.798286	0.223130	0.000000
0.75	0.173774	100.431087	100.257313	0.173774	0.000000
1.00	0.135335	-3590.135466	-3590.270801	0.135335	0.000000
			$x_3(t)$		
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.286505	-0.367879	-0.654384	0.286505	0.000000
0.50	0.223130	23.912163	23.689033	0.223130	0.000000
0.75	0.173774	-847.962112	-848.135886	0.173774	0.000000
1.00	0.135335	30372.494543	30372.359208	0.135335	0.000000

Table 1

Exact solution is

$$x_1(t) = te^{-t}, \quad x_2(t) = e^{-t}, \quad x_3(t) = -e^{-t} \tag{5}$$

Using the equations (2),(3) and (5), the discrete time solution by STWS, RK-10 and exact solutions are calculated along with absolute errors and are presented in Table 2.

Example 3. Consider the time-invariant singular system discussed in [2], by taking $K(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u(t) = 1 + 4t - 3t^2 - 2e^t$, with $x(0) = \begin{pmatrix} 7 \\ -1 \\ 1 \end{pmatrix}$.

Exact solution is

$$x_1(t) = 6e^t + 3t^2 + 2t + 1, \; x_2(t) = 4e^t + 3t^2 + 2t - 5, \; x_3(t) = 2e^t + 3t^2 - 4t - 1. \tag{6}$$

Using the equations (2),(3) and (6), the discrete time solution by STWS, RK-10 and exact solutions are calculated along with absolute errors and are presented in Table 3.

	$x_1(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.194700	0	-0.194700	0.194700	0.000000
0.50	0.303265	24	23.696735	0.303265	0.000000
0.75	0.354275	-1424	-1424.354275	0.354275	0.000000
1.00	0.367879	71648	71647.632121	0.367880	0.000001
	$x_2(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.778801	1	0.221199	0.778801	0.000000
0.50	0.606531	-7	-7.606531	0.606531	0.000000
0.75	0.472367	273	272.527633	0.472367	0.000000
1.00	0.367879	-9759	-9759.3678790	0.367879	0.000000
	$x_3(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	-0.778801	1	1.778801	-0.778801	0.000000
0.50	-0.606531	-65	-64.393469	-0.606531	0.000000
0.75	-0.472367	2305	2305.472367	-0.472367	0.000000
1.00	-0.367879	-82561	-82560.632121	-0.367879	0.000000

Table 2

	$x_1(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	9.391653	2.866687	-6.524966	9.391653	0.000000
0.50	12.642328	35.894457	23.252129	12.642328	0.000000
0.75	16.889500	-34.279519	-51.169019	16.889500	0.000000
1.00	22.309691	111.388320	89.078629	22.309691	0.000000
	$x_2(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.823602	0.024660	-0.798942	0.823602	0.000000
0.50	3.344885	4.656036	1.311151	3.344885	0.000000
0.75	6.655500	4.499471	-2.156029	6.655500	0.000000
1.00	10.873127	13.482383	2.609256	10.873127	0.000000
	$x_3(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	0.755551	0.669407	-0.086144	0.755551	0.000000
0.50	1.047443	1.040727	-0.006716	1.047443	0.000000
0.75	1.921500	1.826733	-0.094767	1.921500	0.000000
1.00	3.436564	3.418776	-0.017788	3.436564	0.000000

Table 3

Example 4. Consider the time-invariant singular system discussed in [3], by taking $K(t) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, $A(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u(t) = t^2 + t + 1$, with $x(0) = \begin{pmatrix} 11 \\ 3 \\ 1 \end{pmatrix}$.

Exact solution is

$$x_1(t) = t^2 + 5t + 11, \quad x_2(t) = t^2 + 5t + 3, \quad x_3(t) = t^2 + t + 1 \tag{7}$$

Using the equations (2),(3) and (7), the discrete time solution by STWS, RK-10 and exact solutions are calculated along with absolute errors and are presented in Table 4.

	$x_1(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	12.312500	6.625000	-5.687500	12.312500	0.000000
0.50	13.750000	35.750000	22.000000	13.750000	0.000000
0.75	15.312500	-33.708333	-49.020833	15.312500	0.000000
1.00	17.000000	103.666667	86.666667	17.000000	0.000000
	$x_2(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	4.312500	3.958333	-0.354167	4.312500	0.000000
0.50	5.750000	6.416667	0.666667	5.750000	0.000000
0.75	7.312500	6.291667	-1.020833	7.312500	0.000000
1.00	9.000000	10.333333	1.333333	9.000000	0.000000
	$x_3(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	1.312500	1.291667	-0.020833	1.312500	0.000000
0.50	1.750000	1.750000	0.000000	1.750000	0.000000
0.75	2.312500	2.291667	-0.020833	2.312500	0.000000
1.00	3.000000	3.000000	0.000000	3.000000	0.000000

Table 4

Example 5. Consider the time-invariant singular system discussed in [3], by taking $K(t) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$, $A(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $u(t) = e^t + t + 2$

with $x(0) = \begin{pmatrix} -3 \\ 1 \\ -3 \end{pmatrix}$.

Exact solution is

$$x_1(t) = -3e^t - t, \quad x_2(t) = e^t - t, \quad x_3(t) = -e^t - t - 2 \tag{8}$$

Using the equations (2),(3) and (8), the discrete time solution by STWS, RK-10 and exact solutions are calculated along with absolute errors and are presented in Table 5.

	$x_1(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	-4.102076	722.148997	726.251073	-4.102076	0.000000
0.50	-5.446164	-2993.006896	-2987.560732	-5.446164	0.000000
0.75	-7.101000	6789.379851	6796.480851	-7.101000	0.000000
1.00	-9.154845	-12149.457657	-12140.302812	-9.154845	0.000000
	$x_2(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	1.034025	-44.166950	-45.200975	1.034025	0.000000
0.50	1.148721	97.469946	96.321225	1.148721	0.000000
0.75	1.367000	-140.178128	-141.545128	1.367000	0.000000
1.00	1.718282	194.354219	192.635937	1.718282	0.000000
	$x_3(t)$				
t	Exact solution	STWS	STWS Error	RK-10	RK-10 Error
0.25	-3.534025	-0.522203	3.011822	-3.534025	0.000000
0.50	-4.148721	-4.145363	0.003358	-4.148721	0.000000
0.75	-4.867000	-1.850866	3.016134	-4.867000	0.000000
1.00	-5.718282	-5.709388	0.008894	-5.718282	0.000000

Table 5

6. Conclusion

From the tables 1-5, it is observed that the solution obtained by the STWS method yields little error whereas RK-10 method is absolutely error free. Also, it is observed that the singular system of index three problem is converted into initial value problem and then RK-10 method is working well, but the STWS method fails to work in the same problem and is discussed in [1]. In fact, it is noted in [1], the STWS method diverges completely for the singular system of index three. It is observed that Runge-Kutta method of order-10 **(RK-10)** is more suitable for solving index three singular systems.

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