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TWO-PHASE APPROACH FOR SOLVING TYPE-2 FUZZY LINEAR PROGRAMMIG PROBLEM

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Abstract: Type-2 fuzzy sets are fuzzy sets whose membership values are fuzzy sets on the interval [0,1]. This concept was proposed by Zadeh, as an extension of fuzzy sets. Type-2 fuzzy sets possess a great expressive power and are conceptually quite appealing. In this paper we solve the Type-2 fuzzy linear programming problems by using two-phase method. by introducing a new ranking function to compare Type-2 fuzzy numbers on the basis of graded mean integration representation.

AMS Subject Classification: 90C70
Key Words: Type-2 fuzzy set, Type-2 fuzzy linear programming problem, fuzzy number

1. Introduction

The concept of a Type-2 fuzzy set was introduced by Zadeh [1] as an extension of the concept of an ordinary fuzzy set. A Type-2 fuzzy set is characterized by a fuzzy membership function, i.e., the membership value for each element of this set is a fuzzy set in [0,1], unlike an ordinary fuzzy set where the membership value is a crisp number in [0,1].Type-2 fuzzy sets allow us to handle linguistic uncertainties (as typified by the adage “words can mean different things to different people [2]) as well as numerical uncertainties. A fuzzy relation of higher Type has been regarded as one way to increase the fuzziness of a relation and, according to Hisdal [3]“increased fuzziness in a description means increased ability to handle inexact information in a logically correct manner”. Accord-

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ing to Jhon [4] "Type-2 fuzzy sets allow for linguistic grades of membership, thus assisting in knowledge representation, and they also offer improvement on inferring with Type-2 fuzzy sets". Type-2 fuzzy sets have already been used in a number of applications, including decision making, solving fuzzy relation equations, and pre-processing of data. In Section 2 definition of generalized trapezoidal fuzzy number is reviewed. In Section 3 the definition of Type-2 fuzzy number, proposed ranking function and arithmetic operations on Type-2 fuzzy numbers are given. In Section 4, formulation, definition of Type-2 fuzzy linear programming problem (Type-2FLPP) and some theorems are discussed. To explain the proposed two phase simplex method a numerical example is solved. In the last section conclusion is discussed.

2. Generalized Trapezoidal Fuzzy Number

The membership function $\mu_A(x)$ of the generalized trapezoidal fuzzy number $(a_1, a_2, a_3, a_4; w_A)$ can be expressed as:

$$\mu_A(x) = \begin{cases} \mu_{AL}(x), & a_1 \leq x \leq a_2, \\ w_A, & a_2 \leq x \leq a_3, \\ \mu_{AR}(x), & a_3 \leq x \leq a_4, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mu_{AL}(x) : [a_1, a_2] \to [o, w]$, is continuous and strictly increasing, and $\mu_{AR}(x) : [a_3, a_4] \to [o, w]$, is continuous and strictly decreasing. Since $\mu_{AL}(x)$ and $\mu_{AR}(x)$ are continuous and monotonic, therefore the inverse functions of $\mu_{AR}(x)$ and $\mu_{AL}(x)$ exist. The inverse functions of $\mu_{AL}(x)$ and $\mu_{AR}(x)$ can be denoted by μ_{AL}^{-1} and μ_{AR}^{-1}, respectively.

Since μ_{AL}^{-1} and μ_{AR}^{-1} are continuous on $[o, w]$, therefore $\int_0^w \mu_{AL}^{-1}{}^{(y)} dy$ and $\int_0^w \mu_{AR}^{-1}{}^{(y)} dy$ exist.

In the case of a generalized trapezoidal fuzzy number $A = (a_1, a_2, a_3, a_4; w_A)$, $\mu_{AL}^{(x)} = \dfrac{w_A(x - a_1)}{(a_2 - a_1)}$, and $\mu_{AR}^{(x)} = \dfrac{w_A(a_4 - x)}{(a_4 - a_3)}$.

The inverse functions $\mu_{AL}^{-1}{}^{(y)}$ and $\mu_{AR}^{-1}{}^{(y)}$ can be analytically expressed as $\mu_{AL}^{-1}{}^{(y)} = a_1 + \dfrac{y(a_2 - a_1)}{w_A}$ and $\mu_{AR}^{-1}{}^{(y)} = a_4 + \dfrac{y(a_3 - a_4)}{w_A}$.

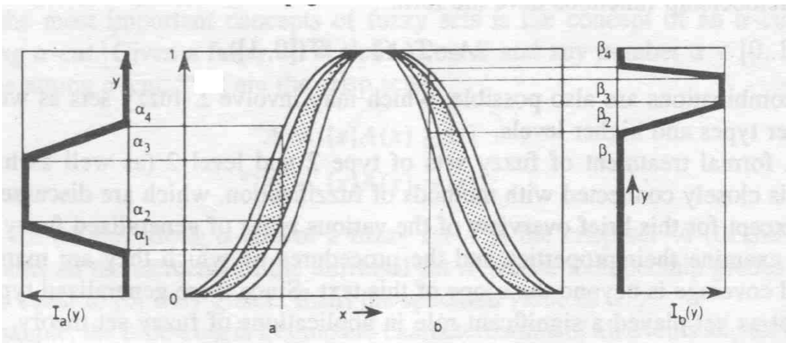

Figure 1: Illustration of the concept of a fuzzy set of Type-2

3. Type-2 Fuzzy Numbers and Their Arithmetic Operations

3.1. Type-2 Fuzzy Numbers

Definition 3.1.1. (see Zadeh) A Type-2 fuzzy set is a fuzzy set whose membership values are fuzzy sets on $[0, 1]$.

Definition 3.1.2. The Type-2 fuzzy sets are defined by functions of the form $\mu_{\tilde{A}} : X \to \chi([0, 1])$, where $\chi([0, 1])$ denotes the set of all ordinary fuzzy sets that can be defined within the universal set [0,1]. An example [5] of a membership function of this Type is given in Figure 1.

Definition 3.1.3. Let $\tilde{A}$ be a Type-2 fuzzy set defined in the universe of discourse R.

If the following conditions are satisfied:

(i) $\tilde{A}$ is normal,

(ii) $\tilde{A}$ is a convex set,

(iii) The support of $\tilde{A}$ is closed and bounded, then $\tilde{A}$ is called a Type-2 fuzzy number.

Definition 3.1.4. A Type-2 trapezoidal fuzzy number $\tilde{A}$ on R is given by $\tilde{A} = \{(x, (\mu_A^L(x), \mu_A^M(x), \mu_A^N(x), \mu_A^U(x)))\}$, $x \in R$ and $\mu_A^L(x) \leq \mu_A^M \leq \mu_A^N(x) \leq \mu_A^U$, for all $x \in R$. Denote $\tilde{A} = (A^L, A^M, A^N, A^U)$, where A^L, A^M, A^N, A^U are same Type of fuzzy numbers.

Definition 3.1.5. If $A^M = A^N$, then $\tilde{A} = (A^L, A^M, A^U)$ is called a Type-2 triangular fuzzy number.

Definition 3.1.6. If $A^M = A^N = A^U$, then $\tilde{A} = [A^L, A^U]$ is called interval-valued fuzzy number.

Definition 3.1.7. If $A^L = A^M = A^N = A^U$, then $\tilde{A}$ is an ordinary fuzzy number.

3.2. Arithmetic Operations on Type-2 Fuzzy Numbers

Let

$$\begin{aligned}\tilde{A} = (A^L, A^M, A^N, A^U) = ((a_1^L, a_2^L, a_3^L, a_4^L; \lambda_A), (a_1^M, a_2^M, a_3^M, a_4^M; \sigma_A),\\ (a_1^N, a_2^N, a_3^N; a_4^N; \tau_A), (a_1^U, a_2^U, a_3^U, a_4^U; \rho_A)),\end{aligned}$$

and

$$\begin{aligned}\tilde{B} = (B^L, B^M, B^N, B^U) = ((b_1^L, b_2^L, b_3^L, b_4^L; \lambda_B), (b_1^M, b_2^M, b_3^M, b_4^M; \sigma_B),\\ (b_1^N, b_2^N, b_3^N b_4^N; \tau_B), (b_1^U, b_2^U, b_3^U b_4^U; \rho_B)),\end{aligned}$$

be two Type-2 trapezoidal fuzzy numbers. Then we define

$$\begin{aligned}\tilde{A} + \tilde{B} = ((a_1^L + b_1^L, a_2^L + b_2^L, a_3^L + b_3^L, a_4^L + b_4^L; \min\{\lambda_A, \lambda_B\}),\\ (a_1^M + b_1^M, a_2^M + b_2^M, a_3^M + b_3^M, a_4^M + b_4^M; \min\{\sigma_A, \sigma_B\}),\\ (a_1^N + b_1^N, a_2^N + b_2^N, a_3^N + b_3^N, a_4^N + b_4^N; \min\{\tau_A, \tau_B\}),\\ (a_1^U + b_1^U, a_2^U + b_2^U, a_3^U + b_3^U, a_4^U + b_4^U; \min\{\rho_A, \rho_B\})),\end{aligned}$$

$$\begin{aligned}\tilde{A} - \tilde{B} = ((a_1^L - b_4^L, a_2^L - b_3^L, a_3^L - b_2^L, a_4^L - b_1^L; \min\{\lambda_A, \lambda_B\}),\\ (a_1^M - b_4^M, a_2^M - b_3^M, a_3^M - b_2^M, a_4^M - b_1^M; \min\{\sigma_A, \sigma_B\}),\\ (a_1^N - b_4^N, a_2^N - b_3^N, a_3^N - b_2^N, a_4^N - b_1^N; \min\{\tau_A, \tau_B\}),\\ (a_1^U - b_4^U, a_2^U - b_3^U, a_3^U - b_2^U, a_4^U - b_1^U; \min\{\rho_A, \rho_B\})),\end{aligned}$$

$$\begin{aligned}k\tilde{A} = ((ka_1^L, ka_2^L, ka_3^L, ka_4^L; \lambda_A), (ka_1^M, ka_2^M, ka_3^M, ka_4^M; \sigma_A),\\ (ka_1^N, ka_2^N, ka_3^N, ka_4^N; \tau_A), (ka_1^U, ka_2^U, ka_3^U, ka_4^U; \rho_A)),\end{aligned}$$

if $k \geq 0$ and $k \in R$,

$$k\tilde{A} = ((ka_4^L, ka_3^L ka_2^L, ka_1^L; \lambda_A), (ka_4^M, ka_3^M, ka_2^M, ka_1^M; \sigma_A), \\ (ka_4^N, ka_3^N ka_2^N, ka_1^N; \tau_A), (ka_4^U, ka_3^U, ka_2^U, ka_1^U; \rho_A)),$$

if $k < 0$ and $k \in R$,

$$\tilde{A} \times \tilde{B} = ((a_1^L b_1^L, a_2^L b_2^L, a_3^L b_3^L, a_4^L b_4^L; \min\{\lambda_A, \lambda_B\}), \\ (a_1^M b_1^M, a_2^M b_2^M, a_3^M b_3^M, a_4^M b_4^M; \min\{\sigma_A, \sigma_B\}), \\ (a_1^N b_1^N, a_2^N b_2^N, a_3^N b_3^N, a_4^N b_4^N; \min\{\tau_A, \tau_B\}), \\ (a_1^U b_1^U, a_2^U b_2^U, a_3^U b_3^U, a_4^U b_4^U; \min\{\rho_A, \rho_B\})),$$

$$\tilde{A}/\tilde{B} = \Big(\Big(\frac{a_1^L}{b_4^L}, \frac{a_2^L}{b_3^L}, \frac{a_3^L}{b_2^L}, \frac{a_4^L}{b_1^L}; \min\{\lambda_A, \lambda_B\}\Big), \Big(\frac{a_1^M}{b_4^M}, \frac{a_2^M}{b_3^M}, \frac{a_3^M}{b_2^M}, \frac{a_4^M}{b_1^M}; \min\{\sigma_A, \sigma_B\}\Big), \\ \Big(\frac{a_1^N}{b_4^N}, \frac{a_2^N}{b_3^N}, \frac{a_3^N}{b_2^N}, \frac{a_4^N}{b_1^N}; \min\{\tau_A, \tau_B\}\Big), \Big(\frac{a_1^U}{b_4^U}, \frac{a_2^U}{b_3^U}, \frac{a_3^U}{b_2^U}, \frac{a_4^U}{b_1^U}; \min\{\rho_A, \rho_B\}\Big)\Big).$$

3.3. The Proposed Ranking Function

Definition 3.3.1. We can define the ranking function of $\tilde{A} = (A^L, A^M, A^N, A^U)$ based on the graded mean integration representation as

$$\mathbb{R}(\tilde{A}) = \Bigg[\Big\{\int_0^{\lambda_A} y\mu_{AL}^{-1L} dy + \int_0^{\lambda_A} y\mu_{AR}^{-1L} dy\Big\} / \int_0^{\lambda_A} y dy \\ + 2\Big\{\int_0^{\sigma_A} y\mu_{AL}^{-1M} dy + \int_0^{\sigma_A} y\mu_{AR}^{-1M} dy\Big\} / \int_0^{\sigma_A} y dy \\ + 2\Big\{\int_0^{\tau_A} y\mu_{AL}^{-1N} dy + \int_0^{\tau_A} y\mu_{AR}^{-1N} dy\Big\} / \int_0^{\tau_A} y dy \\ + \Big\{\int_0^{\rho_A} y\mu_{AL}^{-1U} dy + \int_0^{\rho_A} y\mu_{AR}^{-1U} dy\Big\} / \int_0^{\rho_A} y dy\Bigg] / 36,$$

where μ_{AL}^{-1L}, μ_{AL}^{-1M}, μ_{AL}^{-1N} and μ_{AL}^{-1U} are inverse functions of μ_{AL}^L, μ_{AL}^M, μ_{AL}^N and μ_{AL}^U, respectively, and μ_{AR}^{-1L}, μ_{AR}^{-1M}, μ_{AR}^{-1N} and μ_{AR}^{-1U} are inverse functions of μ_{AR}^L, μ_{AR}^M, μ_{AR}^N and μ_{AR}^U, respectively.

Remark 3.3.1. Let

$$\tilde{A} = ((a_1^L, a_2^L, a_3^L, a_4^L; \lambda_A), (a_1^M, a_2^M, a_3^M, a_4^M; \sigma_A), (a_1^N, a_2^N, a_3^N; a_4^N; \tau_A), \\ (a_1^U, a_2^U, a_3^U, a_4^U; \rho_A))$$

be a Type-2 trapezoidal fuzzy number then

$$\mathbb{R}(\tilde{A}) = (a_1^L + 2a_2^L + 2a_3^L + a_4^L + 2a_1^M + 4a_2^M + 4a_3^M + 2a_4^M + 2a_1^N + 4a_2^N + 4a_3^N \\ + 2a_4^N + a_1^U + 2a_2^U + 2a_3^U . + a_4^U)/36.$$

Let $F(R)$ be the set of all Type-2 normal trapezoidal fuzzy numbers. One convenient approach for solving Type-2FLPP is based on the concept of comparison of fuzzy numbers by use of ranking function. An effective approach for ordering the elements of $F(R)$ is to define a linear ranking function $\mathbb{R} : F(R) \to R$ which maps each fuzzy number in to R.

Definition 3.3.2. We define orders on $F(R)$ by

$$\mathbb{R}(\tilde{A}) \geq \mathbb{R}(\tilde{X}), \quad \text{if and only if} \quad (\tilde{A}) \underset{\mathbb{R}}{\geq} (\tilde{B}),$$

$$\mathbb{R}(\tilde{A}) \leq \mathbb{R}(\tilde{X}), \quad \text{if and only if} \quad (\tilde{A}) \underset{\mathbb{R}}{\leq} (\tilde{B}),$$

$$\mathbb{R}(\tilde{A}) = \mathbb{R}(\tilde{X}), \quad \text{if and only if} \quad (\tilde{A}) \underset{\mathbb{R}}{=} (\tilde{B}).$$

Definition 3.3.3. We consider the linear ranking function

$$R(\tilde{A}) = (a_1^L + 2a_2^L + 2a_3^L + a_4^L + 2a_1^M + 4a_2^M + 4a_3^M + 2a_4^M + 2a_1^N + 4a_2^N \\ + 4a_3^N + 2a_4^N + a_1^U + 2a_2^U + 2a_3^U + a_4^U)/36,$$

where

$$\bar{A} = ((a_1^L, a_2^L, a_3^L, a_4^L; \lambda_A), (a_1^M, a_2^M, a_3^M, a_4^M; \sigma_A), (a_1^N, a_2^N, a_3^N; a_4^N; \tau_A), \\ (a_1^U, a_2^U, a_3^U, a_4^U; \rho_A)).$$

Remark 3.3.2. Note that $\mathbb{R}((\tilde{0}, \tilde{0}, \tilde{0}, \tilde{0})) = \mathbf{0}$ and $\mathbb{R}((\tilde{1}, \tilde{1}, \tilde{1}, \tilde{1})) = \mathbf{1}$,

$$\mathbb{R}(k\tilde{A}) = k\mathbb{R}(\tilde{A}) \quad \text{and} \quad \mathbb{R}(\tilde{A} + k\tilde{B}) = \mathbb{R}(\tilde{A}) + k\mathbb{R}(\tilde{B})$$

4. Type-2 Fuzzy Linear Programming Problem (Type-2FLPP)

Consider the Type-2 trapezoidal fuzzy linear programming problem

$$\max \sum_{j=1}^{n} c_j \tilde{x}_j,$$

$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \underset{R}{\leq} \tilde{b}_I, \quad \tilde{x}_j \underset{\mathbb{R}}{\geq} \tilde{0}, \tag{4.0}$$

where $\tilde{0} = ((\tilde{0}, \tilde{0}, \tilde{0}, \tilde{0}))$.

Definition 4.1. We say that fuzzy vector $\tilde{x} \quad \in [F(R)]^n$ is a fuzzy feasible solution to (4.0) if $\tilde{x}$ satisfies the constraints of the problem.

Definition 4.2. A fuzzy feasible solution $\tilde{x}_*$ is a fuzzy optimal solution for (4.0) if for all fuzzy feasible solution $\tilde{x}$ for (4.0) we have $c\tilde{x} \underset{\mathbb{R}}{\leq} c\tilde{x}_*$

4.1. Fuzzy Basic Feasible Solution

Consider the Type-2 fuzzy linear programming problem

$$\text{Maximize } \tilde{Z} \underset{\mathbb{R}}{=} \sum_{j=1}^{n} c_j \tilde{x}_j,$$

$$\text{Subject to } \sum_{j=1}^{n} a_{ij}\tilde{x}_j \underset{\mathbb{R}}{=} \tilde{b}_j, \quad \tilde{x}_I \underset{\mathbb{R}}{\geq} \tilde{0}. \tag{4.1}$$

Assume rank $[a_{ij}]$ = m. Partition $[a_{ij}]$ as [BN] where B, $m \times m$, is non singular, therefore rank $[B] = m$. Let y_j be the solution of $By = a_j$, where a_i, is the j-th column of the coefficient matrix $[a_{ij}]$. The basic solution $\tilde{x}_B = (\tilde{x}_{B1}, \tilde{x}_{B2} \ldots .\tilde{x}_{Bm})^T$ is a solution of $\sum_{j=1}^{n} a_{ij}\tilde{x}_j \underset{\mathbb{R}}{=} \tilde{b}_j$

We call $\tilde{x} \underset{\mathbb{R}}{=} (\tilde{x}_B^T \tilde{x}_N^T)$, a fuzzy basic solution corresponding to the basis B. If $\tilde{x}_B \underset{\mathbb{R}}{\geq} \tilde{0}$,then the fuzzy basic solution is feasible and the corresponding fuzzy objective value is $\tilde{Z} \underset{\mathbb{R}}{=} c_B\tilde{x}_B$ where $c_B = (c_{B1,}c_{B2}, \ldots, c_{Bm})$.

Now define $Z_j = c_B y_j = c_B B^{-1} a_j \leq c_j$ for all $1 \leq j \leq n$. For any basic index $j = B_i$, $1 \leq i \leq m$ we have $B^{-1}a_j = e_i$ where $e_i = [0, 0, \ldots, 0, 1, \ldots, 0]^T$ is the i-th unit vector.

Since $Be_i = [a_{B1}, \dots a_{Bi} \dots a_{Bm}]e_i = a_{Bi} = a_j$.

We have $Z_j - c_j = c_B B^{-1} a_j - c_j = c_B e_i - c_j = c_{Bi} - c_i = c_i - c_i$, $Z_j - c_j = 0$ for all $j = B_i$.

Theorem 4.1.1. *Assume the linear programming problem with Type-2 normal trapezoidal fuzzy number variables*

$$\min \tilde{Z} \underset{\mathbb{R}}{=} c\tilde{x} \quad s.t \quad A\tilde{x} = \tilde{b}; \tilde{x} \geq \tilde{0} \tag{4.2}$$

is non degenerate and B is a feasible basis. A fuzzy basic feasible solution $\tilde{x}_B = B^{-1}\tilde{b} \underset{\mathbb{R}}{\geq} \tilde{0}$, $\tilde{X}_N \underset{\mathbb{R}}{=} \tilde{0}$ is optimal to the Type-2LPP if and only if $Z_j \underset{\mathbb{R}}{=} c_B B^{-1} a_j \leq c_j$ for all j, $1 \leq j \leq n$.

Theorem 4.1.2. *The Type-2FLP problems*

$$\max \tilde{Z} \underset{\mathbb{R}}{=} c\tilde{x} \text{ s.t } A\tilde{x} \underset{\mathbb{R}}{\leq} \tilde{b}; \tilde{x} \underset{\mathbb{R}}{\geq} \tilde{0} \tag{4.5}$$

$$\text{and } \max \mathbb{R}(\tilde{Z}) = c\mathbb{R}(\tilde{x}) \text{ s.t } A(\tilde{x}) \leq \mathbb{R}(\tilde{b}); \mathbb{R}(\tilde{x}) \geq \tilde{0} \tag{4.6}$$

are equivalent, where $\mathbb{R}$ is a linear ranking function, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T \in [F(R)]^n$;

$$\tilde{b} = (\tilde{b}_1, \tilde{b}_2, \dots, \tilde{b}_m)^T \in [F(R)]^m;\ A = [a_{ij}]_{m \times n};\ c = (c_1, c_2, \dots, c_n)^T \in R^n;$$

$$\tilde{b}_i = ((b_{i1}^L, b_{i2}^L, b_{i3}^L, b_{i4}^L), (b_{i1}^M, b_{i2}^M, b_{i3}^M, b_{i4}^M), (b_{i1}^N, b_{i2}^N, b_{i3}^N, b_{i4}^N), (b_{i1}^U, b_{i2}^U, b_{i3}^U, b_{i4}^U)) \in F(R);$$

$$\tilde{x}_i = ((x_{i1}^L, x_{i2}^L, x_{i3}^L, x_{i4}^L), (x_{i1}^M, x_{i2}^M, x_{i3}^M, x_{i4}^M), (x_{i1}^N, x_{i2}^N, x_{i3}^N, x_{i4}^N), (x_{i1}^U, x_{i2}^U, x_{i3}^U, x_{i4}^U));$$

$$\mathbb{R}(\tilde{x}) = (\mathbb{R}(\tilde{x}_1), \mathbb{R}(\tilde{x}_2), \dots, \mathbb{R}(\tilde{x}_n)) \in R^n.$$

Proof. $\tilde{Z} \underset{\mathbb{R}}{=} c\tilde{x}$ iff $\mathbb{R}(\tilde{Z}) = \mathbb{R}(c\tilde{x})$ iff $\mathbb{R}(\tilde{Z}) = \mathbb{R}(\sum_{j=1}^{n} c_j\tilde{x}_j) = \sum_{j=1}^{n} \mathbb{R}(c_j\tilde{x}_j) = \sum_{j=1}^{n} c_j\mathbb{R}(\tilde{x}_j)$

$$\text{Therefore} \quad \tilde{Z} \underset{\mathbb{R}}{=} c\tilde{x} \text{ iff } \mathbb{R}(\tilde{Z}) = c\mathbb{R}(\tilde{x}); \tag{4.7}$$

$$\mathbb{R}(A\tilde{x}) = \mathbb{R}(\sum_{j=1}^{n} a_{ij}\tilde{x}_j) = (\sum_{j=1}^{n} \mathbb{R}(a_{ij}\tilde{x}_j) = \sum_{j=1}^{n} a_{ij}\mathbb{R}(\tilde{x}_j);$$

$$\mathbb{R}(A\tilde{x}) = A\mathbb{R}(\tilde{x}). \tag{4.8}$$

Therefore from (4.7) and (4.8) we say that (4.5) and (4.6) are equivalent. □

Let $\tilde{x} \in (F(R))^n$ be any feasible solution for (4.4) therefore $\sum\limits_{j=1}^{n} a_{ij}\mathbb{R}\tilde{x}_j \leq \mathbb{R}(\tilde{b}_i)$, $\mathbb{R}\tilde{x}_j \geq 0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Using the linear ranking function

$$\mathbb{R}(\tilde{x}_j) = (x^L_{i1} + 2x^L_{i2} + 2x^L_{i3} + x^L_{i4} + 2x^M_{i1} + 4x^M_{i2} + 4x^M_{i3} + 2x^M_{i4} + 2x^N_{i1} + 4x^N_{i2} + 4x^N_{i3} + 2x^N_{i4} + x^U_{i1} + 2x^U_{i2} + 2x^U_{i3} + 2x^U_{i4})$$

we have

$$\sum_{j=1}^{n} a_{ij}(x^L_{i1} + 2x^L_{i2} + 2x^L_{i3} + x^L_{i4} + 2x^M_{i1} + 4x^M_{i2} + 4x^M_{i3} + 2x^M_{i4} + 2x^N_{i1} + 4x^N_{i2} + 4x^N_{i3} + 2x^N_{i4} + x^U_{i1} + 2x^U_{i2} + 2x^U_{i3} + 2x^U_{i4}) \leq (b^L_{i1} + 2b^L_{i2} + 2b^L_{i3} + b^L_{i4} + 2b^M_{i1} + 4b^M_{i2} + 4b^M_{i3} + 2b^M_{i4} + 2b^N_{i1} + 4b^N_{i2} + 4b^N_{i3} + 2b^N_{i4} + b^U_{i1} + 2b^U_{i2} + 2b^U_{i3} + 2b^U_{i4}),$$

and

$$(x^L_{i1} + 2x^L_{i2} + 2x^L_{i3} + x^L_{i4} + 2x^M_{i1} + 4x^M_{i2} + 4x^M_{i3} + 2x^M_{i4} + 2x^N_{i1} + 4x^N_{i2} + 4x^N_{i3} + 2x^N_{i4} + x^U_{i1} + 2x^U_{i2} + 2x^U_{i3} + 2x^U_{i4}) \geq 0, \quad i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n.$$

Now for any optimal solution $\tilde{x}_*$ for (4.5) we have $c\tilde{x}_* \underset{\mathbb{R}}{\geq} c\tilde{x}$ for any feasible solution $\tilde{x}$ of (4.5);
or $\mathbb{R}(c\tilde{x}_*) \geq \mathbb{R}(c\tilde{x})$;
or $\mathbb{R}(c\tilde{x}_*) \geq c\mathbb{R}(\tilde{x})$;
or $\sum\limits_{j=1}^{n} c_j\mathbb{R}(\tilde{x}_{j*}) \geq \sum\limits_{j=1}^{n} c_j\mathbb{R}(\tilde{x}_j)$;
or $\sum\limits_{j=1}^{n} c_j(x^L_{i1*} + 2x^L_{i2*} + 2x^L_{i3*} + x^L_{i4*} + 2x^M_{i1*} + 4x^M_{i2*} + 4x^M_{i3*} + 2x^M_{i4*} + 2x^N_{i1*} + 4x^N_{i2*} + 4x^N_{i3*} + 2x^N_{i4*} + x^U_{i1*} + 2x^U_{i2*} + 2x^U_{i3*} + 2x^U_{i4*}) \geq \sum\limits_{j=1}^{n} c_j(x^L_{i1} + 2x^L_{i2} + 2x^L_{i3} + x^L_{i4} + 2x^M_{i1} + 4x^M_{i2} + 4x^M_{i3} + 2x^M_{i4} + 2x^N_{i1} + 4x^N_{i2} + 4x^N_{i3} + 2x^N_{i4} + x^U_{i1} + 2x^U_{i2} + 2x^U_{i3} + 2x^U_{i4})$.

Therefore $\tilde{x}_* = (\tilde{x}_{1*}, \tilde{x}_{2*}, \ldots, \tilde{x}_{n*})^T$ where

$$\tilde{x}_{j*} = ((x^L_{j1}, x^L_{j2}, x^L_{j3}, x^L_{j4}), (x^M_{j1}, x^M_{j2}, x^M_{j3}, x^M_{j4}), (x^N_{j1}, x^N_{j2}, x^N_{j3}, x^N_{j4}), (x^U_{j1}, x^U_{j2}, x^U_{j3}, x^U_{j4})) \in F(R)$$

is optimal for (4.5) if and only if

$$\begin{aligned}\tilde{x}_* = [\{((x^L_{11*}, x^L_{12*}, x^L_{13*}, x^L_{14*}), (x^M_{11*}, x^M_{12*}, x^M_{13*}, x^M_{14*}), (x^N_{11*}, x^N_{12*}, x^N_{13*}, x^N_{14*}),\\ (x^U_{11*}, x^U_{12*}, x^U_{13*}, x^U_{14*})), \dots\dots\dots\dots\dots\dots, ((x^L_{n1*}, x^L_{n2*}, x^L_{n3*}, x^L_{n4*}),\\ (x^M_{n1*}, x^M_{n2*}, x^M_{n3*} x^M_{n4*}), (x^N_{n1*}, x^N_{n2*}, x^N_{n3*}, x^N_{n4*}), (x^U_{n1*}, x^U_{n2*}, x^U_{n3*}, x^U_{n4*}))\}]^T \in R^{4n}\end{aligned}$$

is an optimal solution for (4.5).

4.2. Formulation of a Type-2 Fuzzy Linear Programming Problem

Consider the following crisp linear programming problem[[6]]. A factory produces n products X_j, $j = 1, 2, \dots, n$. Each product requires m processes A_i, $i = 1, 2 \dots, m$. Product X_j, through process A_i, requires a_{ij} hours, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$. Each process A_i provides b_j hours, $i = 1, 2, \dots, m$. Let the quantity produced for X_j, be x_j, $j = 1, 2, \dots, n$.

Then we get the following constraint functions

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_j,\ i = 1, 2, \dots, m.$$

In monopoly market, the monopolist can determine the profit c_j, $j = 1, 2, \dots, n$ and can get total profit $\sum_{j=1}^{n} c_j x_j$. Therefore, we have the following crisp linear programming objective function

$$\text{Maximize} Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij} x_j \leq b_j,\ i = 1, 2, \dots, m,\ x_j \geq 0.$$

This is a crisp linear programming problem.

Suppose that this optimal solution is the production quantity x_{j*}, $j = 1, 2, \dots, n$. The total profit $Z_* = \sum_{j=1}^{n} c_j x_{j*}$ is maximized. If in a plan period, a_{ij}, b_i, c_j, $j = 1, 2 \dots n$, $i = 1, 2 \dots m$ do not change, the result stays the same. That is to say in this period, the optimal solutions of the product X_j is the quantity x_{j*}, $j = 1, 2, \dots n$.

Consider the objective function of the crisp interval linear programming problem

$$Z = \sum_{j=1}^{n} [\underline{c}_j, \bar{c}_j] x_j$$

In a perfect competitive market, the profit $[\underline{c}_j, \bar{c}_j]$ in a plan period may fluctuate a little.

We can fuzzify $[\underline{c}_j, \bar{c}_j]$. Set $[\underline{c}_j, \bar{c}_j]$ to be a Type-2 trapezoidal fuzzy number,

$$\begin{aligned}\tilde{c}_j = ((&\underline{c}_j - \delta_{j4}, \underline{c}_j, \underline{c}_j \bar{c}_j, \bar{c}_j + \delta_{j5}; \lambda), (\underline{c}_j - \delta_{j3}, \underline{c}_j, \bar{c}_j, \bar{c}_j + \delta_{j6}; \sigma)),\\ &((\underline{c}_j - \delta_{j2}, \underline{c}_j, \bar{c}_j, \bar{c}_j + \delta_{j7}; \tau), (\underline{c}_j - \delta_{j1}, \underline{c}_j, \bar{c}_j, \bar{c}_j + \delta_{j8}; \rho)),\\ &\qquad\qquad\qquad\qquad\qquad\qquad j = 1, 2, \ldots, n.\end{aligned}$$

where $0 < \delta_{j4} < \delta_{j3} < \delta_{j2} < \delta_{j1}$, $0 < \delta_{j5} < \delta_{j6} < \delta_{j7} < \delta_{j8} < \delta_{j3}$. In the similar way, we can set $\tilde{b}_i$ and, $\tilde{a}_{ij}$ as a Type-2 fuzzy number.

4.3. Two-Phase Method for Type-2FLP Problems

If the original constraint is an equation or is of the Type $(\geq)$ we may no longer have a ready starting basic feasible solution. In order to obtain an initial basic feasible solution, we put the given LPP into its standard form and then a non negative variable is added to the left side of each of equation that lacks the much needed starting basic variables. The so added variable is called an artificial variable. Since such artificial variables have no physical meaning from the standpoint of the original problem, the method will be valid only if we are able to force these variables to be out or at zero level when the optimum solution is attained. Consider the Type-2FLP problem

$$\max \tilde{z} = \tilde{c}x \text{ s.t } Ax \leq b; x \geq 0.$$

If all $\tilde{z}_j - \tilde{c}_j \geq \tilde{0}$ the basic feasible solution is optimal. When this is not the case, we select the column having the smallest $\tilde{z}_j - \tilde{c}_j$. Call this column k. If $a_{ik} \geq 0$ we shall insert x_k in to the basis.

The vector to be removed is computed from $\dfrac{x_{Br}}{x_{rk}} = \min\limits_{j} \Big\{ \dfrac{x_{Bi}}{x_{ik}} \Big| x_{ik} > 0 \Big\}$ column r of the basis is then removed.

Numerical Example 4.3.1.

$$\begin{aligned}\text{Maximize} \tilde{Z} = &((4,5,8,9),(3,5,8,10),(2,5,8,11),(1,5,8,12))\tilde{x}_1 +\\ &((5,6,10,11),(4,6,10,12),(3,6,10,13),(2,6,10,14))\tilde{x}_2,\end{aligned}$$

$\tilde{c}_B$			$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{-1}$
	Basis	x_B	x_1	x_2	s_1	s_2	a_1
$\tilde{-1}$	a_1	((1.8,2,4,4.2),(1.6,2,4,4.4),(1.4,2,4,4.6), (1.2,2,4,4.8))	3	2	-1	0	1
$\tilde{0}$	s_2	((2.9,3,5,5.1),(2.8,3,5,5.2),(2.7,3,5,5.3), (2.6,3,5,5.4))	2	1	0	1	0
	$Z_j - C_j$	((1.8,2,4,4.2),(1.6,2,4,4.4),(1.4,2,4,4.6),(1.2,2,4,4.8))	$\tilde{-3}$	$\tilde{-2}$	$\tilde{1}$	$\tilde{0}$	$\tilde{-1}$
	Rank		-3	-2	1	0	-1

Table 1

Subject to

$$3\tilde{x}_1 + 2\tilde{x}_2 \geq ((1.8, 2, 4, 4.2), (1.6, 2, 4, 4.4), (1.4, 2, 4, 4.6), (1.2, 2, 4, 4.8)),$$
$$2\tilde{x}_1 + \tilde{x}_2 \leq ((2.9, 3, 5, 5.1), (2.8, 3, 5, 5.2), (2.7, 3, 5, 5.3), (2.6, 3, 5, 5.4)),$$
$$\tilde{x}_1, \tilde{x}_2 \geq \bar{0}.$$

Phase I. It consist the following steps:

Step 1. Set up the problem in standard form. The original objective function is temporarily set aside during the Phase 1 solution. The given constraints, after the introduction of slack, surplus and artificial variables take of the form:

$$3\tilde{x}_1 + 2\tilde{x}_2 - \tilde{s}_1 + \tilde{a}_1 = ((1.8, 2, 4, 4.2), (1.6, 2, 4, 4.4), (1.4, 2, 4, 4.6), (1.2, 2, 4, 4.8)),$$
$$2\tilde{x}_1 + \tilde{x}_2 + \tilde{s}_2 = ((2.9, 3, 5, 5.1), (2.8, 3, 5, 5.2), (2.7, 3, 5, 5.3), (2.6, 3, 5, 5.4)),$$
$$\tilde{x}_1, \tilde{x}_2, \tilde{s}_1, \tilde{s}_2, \tilde{a}_1 \geq \tilde{0}.$$

The new objective function is Minimize $W = a_1$. Thus the problem for Phase 1 in the standard form becomes

$$\text{Max } Z^* = \tilde{0}\tilde{x}_1 + \tilde{0}\tilde{x}_2 + \tilde{0}\tilde{s}_1 + \tilde{0}\tilde{s}_2 - \tilde{a}_1.$$

$$3\tilde{x}_1 + 2\tilde{x}_2 - \tilde{s}_1 + \tilde{a}_1 = ((1.8, 2, 4, 4.2), (1.6, 2, 4, 4.4), (1.4, 2, 4, 4.6), (1.2, 2, 4, 4.8)),$$
$$2\tilde{x}_1 + \tilde{x}_2 + \tilde{s}_2 + \tilde{a}_1 = ((2.9, 3, 5, 5.1), (2.8, 3, 5, 5.2), (2.7, 3, 5, 5.3), (2.6, 3, 5, 5.4)),$$
$$\tilde{x}_1, \tilde{x}_2, \tilde{s}_1, \tilde{s}_2, \tilde{a}_1 \geq \tilde{0}.$$

Step 2. Find an initial basic feasible solution. We may write the first simplex tableau is Table 1.

Since $Z_j - C_j$ is negative under some column, the above table is not optimal. Since the ratio $R((1.8, 2, 4, 4.2), (1.6, 2, 4, 4.4), (1.4, 2, 4, 4.6), (1.2, 2, 4, 4.8))/3$ is minimum therefore a_1 is a leaving variable, and x_2 is entering variable. The next tableau is Table 2.

$\tilde{c}_B$			$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{-1}$
	Basis	x_B	x_1	x_2	x_3	x_4	x_5
$\tilde{0}$	x_1	((1.8,2,4,4.2),(1.6,2,4,4.4), (1.4,2,4,4.6), (1.2,2,4,4.8))/3	1	2/3	-1/3	0	1/3
$\tilde{0}$	s_2	((.3/3,1/3,11/3,11.7/3), (-.4/3,1/3,11/3,12.4/3), (-1.1/3,1/3,11/3,13.1/3), (-1.8/3,1/3,11/3,13.8/3))	0	-1/3	2/3	1	-2/3
	$Z_j - C_j$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{0}$	$\tilde{1}$
	Rank	0	0	0	0	0	1

Table 2

Since $Z_j - C_j$ is non-negative under all columns, the above table is optimal. Also, since $w_{min} = 0$ and no artificial variable appears in the basis, this tableau gives a basic feasible solution to the original problem.

Phase II. Phase II of the simplex method finds optimal solution to the original problem. Objective function for the initial table of Phase II is the objective function of the original. The initial table for Phase II computations is Table 3.

$\tilde{x}_2$ is an entering variable and $\tilde{x}_1$ is a leaving variable. The next tableau is Table 4.

$\tilde{s}_1$ is an entering variable and $\tilde{s}_2$ is leaving variable. The next tableau is Table 5.

Since all $Z_j - C_j \geq 0$, an optimum basic feasible solution has been reached. Hence, an optimum basic feasible solution to the given problem is $\tilde{x}_1 = \tilde{0}$ and $\tilde{x}_2 = ((2.9, 3, 5, 5.1), (2.8, 3, 5, 5.2), (2.7, 3, 5, 5.3), (2.6, 3, 5, 5.4))$, $\max \tilde{Z} = ((14.5, 18, 50, 56.1), (11.2, 18, 50, 62, 4), (8.1, 18, 50, 68.9), (5.2, 18, 50, 75.6))$, and $\mathbb{R}(\tilde{Z}) = 35.23$.

5. Conclusion

We introduced a new effective linear ranking function for solving Type-2 fuzzy linear programming problem by using two phase method. Since the new ranking function, gives more importance to the core of the fuzzy number, therefore it is better than the other ranking function. The two phase method, using the proposed new ranking function as linear ranking functions on Type-2 fuzzy numbers, appears to be natural extension of the results for linear programming problem with crisp data. The capabilities offered here will be useful for post optimal analysis.

$\tilde{c}_B$			((4,5,8,9), (3,5,8,10), (2,5,8,11), (1,5,8,12))	((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))	$\tilde{0}$	$\tilde{0}$
	Basis	$\tilde{x}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{s}_1$	$\tilde{s}_2$
((4,5,8,9), (3,5,8,10), (2,5,8,11), (1,5,8,12))	x_1	((1.8,2,4,4.2), (1.6,2,4,4.4), (1.4,2,4,4.6), (1.2,2,4,4.8))/3	1	2/3	-1/3	0
$\tilde{0}$	s_2	((.3/3,1/3,11/3,11.7/3), (-.4/3,1/3,11/3,12.4/3), (-1.1/3,1/3,11/3,13.1/3), (-1.8/3,1/3,11/3,13.8/3))	0	-1/3	2/3	1
	Z_j-C_j	((7.2,10,32,37.8), (4.8,10,32,44), (2.8,10,32,50.6), (1.2,10,32,57.6))/3	((-5,-3,3,5), (-7,-3,3,7), (-9,-3,3,9), (-11,-3,3,11))	((-25/3,-20/3,-2/3,1), (-10,,-20/3,-2/3,8/3), (-11,-20/3,-2/313/3), (-40/3,-20/3,-2/3,6))	((-9,-8,-5,-4), (-10,-8,-5,-3), (-11,-8,-5,-2), (-12,-8,-5,-1))/3	$\tilde{0}$
	Rank		0	-3.63	-2.07	0

Table 3

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$\tilde{c}_B$			((4,5,8,9), (3,5,8,10), (2,5,8,11), (1,5,8,12))	((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))	$\tilde{0}$	$\tilde{0}$
	Basis	$\tilde{x}_B$	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{S}$	$\tilde{s}_2$
((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))	$\tilde{x}_2$	((.9,1,2,2.1), (.8,1,2,2.2), (.7,1,2,2.3), (.6,1,2,2.4))	3/2	1	-1/2	0
$\tilde{0}$	$\tilde{s}_2$	((.8,1,4,4.2), (.6,1,4,4.4), (.4.1,4,4.6), (.2,1,4,4.8))	1/2	0	1/2	1
	Z_j-C_j	((9,12,40,46.2), (6.4,12,40,52.8), (4.2,12,40,59.8), (2.4,12,40,67.2))/2	((-1.5,1,10,12.5), (-4,1,10,13.5), (-6.5,1,10,17.5), (-9,1,10,20))	((-6,-4,4,6), (-8,-4,4,8), (-10,-4,4,10), (-12,-4,4,12))	((-5.5,-5,-3,-2.5), (-6,-5,-3,-2), (-6.5,-5,-3,-1.5), (-7,-5,-3,-1))	$\tilde{0}$
	Rank		5.42	0	-4	0

Table 4

			((4,5,8,9), (3,5,8,10), (2,5,8,11), (1,5,8,12))	((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))	$\tilde{0}$	$\tilde{0}$
$\tilde{c}_B$	Basis	$\tilde{x}_B$	$\tilde{x}_1$	$\tilde{x}_2$		
((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))	$\tilde{x}_2$	((2.9,3,5,5.1), (2.8,3,5,5.2), (2.7,3,5,5.3), (2.6,3,5,5.4)),	2	1	0	1
	$\tilde{s}_1$		1	0	1	2
	Z_j-C_j	((14.5,18,50,56.1), (11.2,18,50,62,4), (8.1,18,50,68.9), (5.2,18,50,75.6))	((1,4,15,18), (2,4,15,21), (-5,4,15,240, (-8,4,15,27))	((-6,-4,4,6), (-8,-4,4,8), (-10,-4,4,10), (-12,-4,4,12))	$\tilde{0}$	((5,6,10,11), (4,6,10,12), (3,6,10,13), (2,6,10,14))
	Rank	35.23	9.5	0	0	8

Table 5

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