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AN ISHIKAWA TYPE PERTURBED ITERATIVE ALGORITHM FOR A SYSTEM OF VARIATIONAL INCLUSIONS WITH (H, η)-MONOTONE OPERATORS

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Abstract: In this paper, we propose an Ishikawa type perturbed iterative algorithm for solving a system of variational inclusions with (H, η)-monotone operators in Hilbert spaces. We also prove the existence, uniqueness of the solution to the considered system and discuss the convergence of the proposed algorithm.

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1. Introduction

In 1998, Huang [1] considered a new class of generalized nonlinear implicit quasi-variational inclusions, presented an Ishikawa type perturbed iterative algorithm and analyzed the convergence of the proposed algorithm. In this paper, we consider a new system of variational inclusions involving (H, η)-monotone operators in Hilbert spaces, introduce the sequences of resolvent operators and propose an Ishikawa type perturbed algorithm for approximating the solutions of the system, the convergence of the proposed algorithm is also discussed. The result in this paper improves and extends many results in the corresponding

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literatures, see [2].

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot\rangle$. Let $2^{\mathcal{H}}$ denote the family of all nonempty subsets of $\mathcal{H}$.

Definition 2.1. Let $\eta:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ and $H:\mathcal{H}\to\mathcal{H}$ be two single-valued operators and $M:\mathcal{H}\to 2^{\mathcal{H}}$ be a multi-valued operator. Then M is said to be:

(i) η-monotone if $\langle x-y,\eta(u,v)\rangle\geq 0,\ \forall\, u,v\in\mathcal{H}, x\in Mu,\ \ y\in Mv$;.

(ii) strictly η-monotone if M is η-monotone and the equality holds if and only if $u=v$.

(iii) strongly η-monotone if there exists some constant $r>0$ such that $\langle x-y,\eta(u,v)\rangle\geq r\|u-v\|^2,\ \forall\, u,v\in\mathcal{H}, x\in Mu,\ \ y\in Mv$;.

(iv) (H,η)-monotone if M is η-monotone and $(H+\lambda M)(\mathcal{H})=\mathcal{H}$, for any $\lambda>0$, where I stands for the identity operator.

Definition 2.2. Let $F:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$, $H,f:\mathcal{H}\to\mathcal{H}$ be three single-valued operators.

(i) F is said to be strongly monotone in the first argument with respect to $(H\circ f)$, the composition of H and f, if there exists some constant $\theta>0$ such that

$$\langle (H\circ f)(a_1)-(H\circ f)(a_2),F(a_1,b)-F(a_2,b)\rangle\geq\theta\|a_1-a_2\|^2,\quad\forall\, a_1,a_2,b\in\mathcal{H},$$

where $(H\circ f)(a)=H(f(a))$, $a\in\mathcal{H}$;

(ii) F is said to be Lipschitz continuous in the second argument if there exists some constant $\alpha>0$ such that $\|F(a,b_1)-F(a,b_2)\|\leq\alpha\|b_1-b_2\|$, $\forall\, a,b_1,b_2\in\mathcal{H}$.

Lemma 2.1. *Let $\eta:\mathcal{H}\times\mathcal{H}\to\mathcal{H}$ be a single-valued operator, $H:\mathcal{H}\to\mathcal{H}$ be a strictly η-monotone operator and $M:\mathcal{H}\to 2^{\mathcal{H}}$ be a (H,η)-monotone operator. Then the operator $(H+\lambda M)^{-1}:\mathcal{H}\to\mathcal{H}$ is single-valued for all $\lambda>0$.*

Proof. For any given $x\in\mathcal{H}$, let $u,v\in(H+\lambda M)^{-1}(x)$. It follows that $-Hu+x\in\lambda M(u)$, $\ -Hv+x\in\lambda M(v)$. The (H,η)-monotonicity of M implies that $\langle(-Hu+x)-(-Hv+x),\eta(u,v)\rangle\geq 0$, i.e., $\langle Hv-Hu,\eta(u,v)\rangle\geq 0$. The η-monotonicity of H implies that $u=v$. Thus, $(H+\lambda M)^{-1}$ is single-valued. This completes the proof. □

Definition 2.3. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a single-valued operator, $H : \mathcal{H} \to \mathcal{H}$ be a strictly η-monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a (H, η)-monotone operator. Then the resolvent operator $R^{H,\eta}_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is defined by $R^{H,\eta}_{M,\lambda}(u) = (H + \lambda M)^{-1}(u)$, $u \in \mathcal{H}$, where $\lambda > 0$ is a constant.

Lemma 2.2. *Let $\eta : H \times H \to H$ be Lipschitz continuous with constant $\tau > 0$, $H : \mathcal{H} \to \mathcal{H}$ be strongly η-monotone with constant $\gamma > 0$ and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be (H, η)-monotone. Then the resolvent operator $R^{H,\eta}_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $\frac{\tau}{\gamma} > 0$, i.e., $\|R^{H,\eta}_{M,\lambda}(u) - R^{H,\eta}_{M,\lambda}(v)\| \le \frac{\tau}{\gamma}\|u - v\|, \forall\, u, v \in \mathcal{H}$, where $\lambda > 0$ is a constant.*

Proof. The proof can be obtained from the definition of $R^{H,\eta}_{M,\lambda}$, the η-monotone of M and the Lipschitz continuity of η and the η-monotonicity of H. □

3. A System of Variational Inclusions with (H, η)-Monotone Operators

In this section, we consider a system of variational inclusions with (H, η)-monotone operators in Hilbert spaces. Let $F, G, \eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, $H_1, H_2, f, g : \mathcal{H} \to \mathcal{H}$ be seven operators, let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a (H_1, η)-monotone operator and $N : \mathcal{H} \to 2^{\mathcal{H}}$ be a (H_2, η)-monotone operator. Consider the problem of finding $a, b \in \mathcal{H}$ such that

$$\begin{aligned} 0 &\in F(a, b) + M(f(a)), \\ 0 &\in G(a, b) + N(g(b)). \end{aligned} \tag{1}$$

Lemma 3.1. *Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be a single-valued operator, $H_1, H_2 : \mathcal{H} \to \mathcal{H}$ be two strictly η-monotone operators, and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be (H_1, η)-monotone and $N : \mathcal{H} \to 2^{\mathcal{H}}$ be (H_2, η)-monotone. Then $(a, b) \in \mathcal{H} \times \mathcal{H}$ is a solution of problem (1) if and only if (a, b) satisfies*

$$\begin{aligned} f(a) &= R^{H_1,\eta}_{M,\rho}[H_1(f(a)) - \rho F(a, b)], \\ g(b) &= R^{H_2,\eta}_{N,\lambda}[H_2(g(b)) - \lambda G(a, b)], \end{aligned} \tag{2}$$

where $\rho > 0, \lambda > 0$ are two constants.

Proof. The results can be obtained directly from the definition of resolvent operator. □

Theorem 3.1. *Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous with constant $\sigma > 0$. Let $H_1 : \mathcal{H} \to \mathcal{H}$ be strongly η-monotone, Lipschitz continuous with constants $\gamma_1 > 0, \tau_1 > 0$, respectively, and $H_2 : \mathcal{H} \to \mathcal{H}$ be strongly η-monotone, Lipschitz continuous with constants $\gamma_2 > 0, \tau_2 > 0$, respectively. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be (H_1, η)-monotone and $N : \mathcal{H} \to 2^{\mathcal{H}}$ be (H_2, η)-monotone. Let $f, g : \mathcal{H} \to \mathcal{H}$ be Lipschitz continuous with constants $\xi_1 > 0$ and $\xi_2 > 0$, respectively, and strongly monotone with constants $\zeta_1 > 0, \zeta_2 > 0$, respectively. Let $F : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an operator, F is strongly monotone with respect to $(H_1 \circ f)$ and Lipschitz continuous in the first argument with constants $r_1 > 0, s_1 > 0$, respectively, F is Lipschitz continuous in the second argument with constant $\theta_1 > 0$. Let $G : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be an operator, G is strongly monotone with respect to $(H_2 \circ g)$ and Lipschitz continuous in the second argument with constants $r_2 > 0, s_2 > 0$, respectively, G is Lipschitz continuous in the first argument with constant $\theta_2 > 0$. If there exist two constants $\rho > 0, \lambda > 0$ such that*

$$\begin{aligned}
&\gamma_2\gamma_1\sqrt{1 - 2\zeta_1^2 + \xi_1^2} + \sigma\gamma_2\sqrt{\tau_1^2\xi_1^2 + \rho^2 s_1^2 - 2\rho r_1} + \sigma\lambda\theta_2\gamma_1 < \gamma_1\gamma_2,\\
&\gamma_1\gamma_2\sqrt{1 - 2\zeta_2^2 + \xi_2^2} + \sigma\gamma_1\sqrt{\tau_2^2\xi_2^2 + \lambda^2 s_2^2 - 2\lambda r_2} + \sigma\rho\theta_1\gamma_2 < \gamma_1\gamma_2,
\end{aligned} \tag{3}$$

then, problem (1) has a unique solution.

Proof. For any given $\rho > 0$ and $\lambda > 0$, define $T_\rho : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $S_\lambda : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ by $T_\rho(a, b) = a - f(a) + R_{M,\rho}^{H_1,\eta}[H_1(f(a)) - \rho F(a, b)]$ and $S_\lambda(a, b) = b - g(b) + R_{N,\lambda}^{H_2,\eta}[H_2(g(b)) - \lambda G(a, b)], a, b \in H$. It follows from Lemma 2.2 that, for any $(a_1, b_1), (a_2, b_2) \in \mathcal{H} \times \mathcal{H}$,

$$\begin{aligned}
\|T_\rho(a_1, b_1) - T_\rho(a_2, b_2)\| &\leq \|a_1 - a_2 - (f(a_1) - f(a_2))\|\\
&\quad + \frac{\sigma}{\gamma_1}\|H_1(f(a_1)) - H_1(f(a_2)) - \rho(F(a_1, b_1) - F(a_2, b_2))\|.
\end{aligned} \tag{4}$$

Since f is strongly monotone and Lipschitz continuous, we have $\|a_1 - a_2 - (f(a_1) - f(a_2))\|^2 \leq (1 - 2\zeta_1 + \xi_1^2)\|a_1 - a_2\|^2$. According to the assumptions made in Theorem 3.1, we obtain

$$\begin{aligned}
&\|H_1(f(a_1)) - H_1(f(a_2)) - \rho(F(a_1, b_1) - F(a_2, b_2))\|\\
&\qquad \leq \|H_1(f(a_1)) - H_1(f(a_2)) - \rho(F(a_1, b_1) - F(a_2, b_1))\|\\
&\qquad + \rho\|F(a_2, b_1) - F(a_2, b_2)\|,
\end{aligned} \tag{5}$$

$$\begin{aligned}
&\|H_1(f(a_1)) - H_1(f(a_2)) - \rho(F(a_1, b_1) - F(a_2, b_1))\|^2\\
&\qquad \leq (\tau_1^2\xi_1^2 + \rho^2 s_1^2 - 2\rho r_1)\|a_1 - a_2\|^2
\end{aligned} \tag{6}$$

and $\|F(a_2,b_1)-F(a_2,b_2)\| \le \theta_1\|b_1-b_2\|$. By combining the above inequalities, we have $\|T_\rho(a_1,b_1)-T_\rho(a_2,b_2)\| \le (\sqrt{1-2\zeta_1+\xi_1^2}+\frac{\sigma}{\gamma_1}\sqrt{\tau_1^2\xi_1^2+\rho^2 s_1^2-2\rho r_1})\|a_1-a_2\| + \frac{\sigma\rho\theta_1}{\gamma_1}\|b_1-b_2\|$. Similarly, for the operator S_λ, we have $\|S_\lambda(a_1,b_1)-S_\lambda(a_2,b_2)\| \le (\sqrt{1-2\zeta_2+\xi_2^2}+\frac{\sigma}{\gamma_2}\sqrt{\tau_2^2\xi_2^2+\lambda^2 s_2^2-2\lambda r_2})\|b_1-b_2\|+\frac{\sigma\lambda\theta_2}{\gamma_2}\|a_1-a_2\|$. Adding these two inequalities, we have $\|T_\rho(a_1,b_1)-T_\rho(a_2,b_2)\|+\|S_\lambda(a_1,b_1)-S_\lambda(a_2,b_2)\| \le K(\|a_1-a_2\|+\|b_1-b_2\|)$, where

$$K = \max\{\sqrt{1-2\zeta_1+\xi_1^2}+\frac{\sigma}{\gamma_1}\sqrt{\tau_1^2\xi_1^2+\rho^2 s_1^2-2\rho r_1}+\frac{\sigma\lambda\theta_2}{\gamma_2},$$
$$\sqrt{1-2\zeta_2+\xi_2^2}+\frac{\sigma}{\gamma_2}\sqrt{\tau_2^2\xi_2^2+\lambda^2 s_2^2-2\lambda r_2}+\frac{\sigma\rho\theta_1}{\gamma_1}\}.$$

Define $||\cdot||_*$ on $\mathcal{H}\times\mathcal{H}$ by $||(a,b)||_* = ||a||+||b||, \forall\,(a,b)\in\mathcal{H}\times\mathcal{H}$. It is clear that $(\mathcal{H}\times\mathcal{H}, ||\cdot||_*)$ is a Banach space. For any given $\rho>0, \lambda>0$, define $Q_{\rho,\lambda}:\mathcal{H}\times\mathcal{H}\to\mathcal{H}\times\mathcal{H}$ by $Q_{\rho,\lambda}(a,b)=(T_\rho(a,b),S_\lambda(a,b)), (a,b)\in\mathcal{H}\times\mathcal{H}$. By (3) we know that $0<K<1$, it follows that $\|Q_{\rho,\lambda}(a_1,b_1)-Q_{\rho,\lambda}(a_2,b_2)\|_* \le K\|(a_1,b_1)-(a_2,b_2)\|_*$. This proves that $Q_{\rho,\lambda}$ is a contraction mapping, hence, there exists a unique $(a,b)\in\mathcal{H}\times\mathcal{H}$ such that $Q_{\rho,\lambda}(a,b)=(a,b)$, i.e., $a=T_\rho(a,b)=a-f(a)+R_{M,\rho}^{H_1,\eta}[H_1(f(a))-\rho F(a,b)]$, $b=S_\lambda(a,b)=b-g(b)+R_{N,\lambda}^{H_2,\eta}[H_2(g(b))-\lambda G(a,b)]$. By Lemma 3.1, (a,b) is the unique solution of problem (1). □

4. An Ishikawa Type Perturbed Iterative Algorithm and Convergence Analysis

Algorithm 4.1 (ITPIA). Under the conditions of Theorem 3.1, for given $(a_0,b_0)\in\mathcal{H}\times\mathcal{H}$, the iterative sequences $\{a_n\}$, $\{u_n\}$, $\{b_n\}$ and $\{v_n\}$ are generated by the following iterative schemes

$$\begin{aligned} a_{n+1} =& (1-\alpha_n)a_n+\alpha_n[u_n-f(u_n)+R_{M,\rho}^{H_1^{(n)},\eta}[H_1(f(u_n)) \\ &-\rho F(u_n,b_n)]+\alpha_n p_n, \\ u_n =& (1-\beta_n)a_n+\beta_n[a_n-f(a_n)+R_{M,\rho}^{H_1^{(n)},\eta}[H_1(f(a_n)) \\ &-\rho F(a_n,b_n)]+\beta_n q_n, \end{aligned} \tag{7}$$

$$\begin{aligned} b_{n+1} =& (1-\alpha_n)b_n+\alpha_n[v_n-g(v_n)+R_{N,\lambda}^{H_2^{(n)},\eta}[H_2(g(v_n)) \\ &-\lambda G(a_n,v_n)]+\alpha_n d_n, \\ v_n =& (1-\beta_n)b_n+\beta_n[b_n-g(b_n)+R_{N,\lambda}^{H_2^{(n)},\eta}[H_2(g(b_n)) \\ &-\lambda G(a_n,b_n)]+\beta_n f_n, \end{aligned} \tag{8}$$

where $\alpha_n, \beta_n \in [0,1]$, $n = 0,1,2\cdots$, $\sum_{n=0}^{\infty} \alpha_n = \infty$. $H_1^{(n)} : \mathcal{H} \to \mathcal{H}$ are strongly η-monotone operators with constants $\gamma_1^{(n)} > 0$, $n = 0,1,2,\cdots$, $H_2^{(n)} : \mathcal{H} \to \mathcal{H}$ are strongly η-monotone operators with constants $\gamma_2^{(n)} > 0$, $n = 0,1,2,\cdots$ and $\gamma_1^{(0)} = \gamma_1, \gamma_2^{(0)} = \gamma_2, \gamma_1^{(n)} \uparrow \infty, \gamma_2^{(n)} \uparrow \infty$. $\{p_n\}, \{q_n\}, \{d_n\}, \{f_n\}$ are sequences in $\mathcal{H}$ which are introduced in order to take into account any possible inexact computation.

Lemma 4.1. *Let $\{c_n\}, \{t_n\}, \{w_n\}$ be sequences of nonnegative numbers satisfying the following conditions: there exists n_0 such that $c_{n+1} \leq (1-\lambda_n)c_n + \lambda_n t_n + w_n, \forall\, n \geq n_0$, where $\lambda_n \in [0,1], \sum_{n=0}^{\infty} \lambda_n = \infty, \lim_{n\to\infty} t_n = 0, \sum_{n=0}^{\infty} w_n < \infty$, then $\lim_{n\to\infty} c_n = 0$.*

Theorem 4.1. *Assume that the conditions of Theorem 3.1 hold, and $\zeta_1 > \frac{\xi_1^2}{2}$, $\zeta_2 > \frac{\xi_2^2}{2}$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \lim_{n\to\infty} d_n = \lim_{n\to\infty} f_n = 0$,*

$$R_{M,\rho}^{H_1^{(n)},\eta}[H_1(f(a)) - \rho F(a,b)] \to R_{M,\rho}^{H_1,\eta}[H_1(f(a)) - \rho F(a,b)], \text{ as } n \to \infty,$$

$$R_{N,\lambda}^{H_2^{(n)},\eta}[H_2(g(b)) - \lambda G(a,b)] \to R_{N,\lambda}^{H_2,\eta}[H_2(g(b)) - \lambda G(a,b)], \text{ as } n \to \infty.$$

Then the sequences $\{a_n\}, \{b_n\}$ generated by Algorithm 4.1 converge strongly to the unique solution (a,b) of problem (1).

Proof. Since by Theorem 3.1, problem (1) admits a unique solution, suppose that $(a,b) \in \mathcal{H}\times\mathcal{H}$ is the unique solution of problem (1). According to Lemma 3.1, we have $a = (1-\alpha_n)a + \alpha_n[a - f(a) + R_{M,\rho}^{H_1,\eta}[H_1(f(a)) - \rho F(a,b)], a = (1-\beta_n)a + \beta_n[a - f(a) + R_{M,\rho}^{H_1,\eta}[H_1(f(a)) - \rho F(a,b)], b = (1-\alpha_n)b + \alpha_n[b - g(b) + R_{N,\lambda}^{H_2,\eta}[H_2(g(b)) - \lambda G(a,b)], b = (1-\beta_n)b + \beta_n[b - g(b) + R_{N,\lambda}^{H_2,\eta}[H_2(g(b)) - \lambda G(a,b)]$. Therefore,

$$\begin{aligned}\|a_{n+1} - a\| \leq &(1-\alpha_n)\|a_n - a\| + \alpha_n\sqrt{1 - 2\zeta_1 + \xi_1^2}\|u_n - a\| + \alpha_n\|p_n\| \\ &+ \alpha_n \frac{\sigma}{\gamma_1^{(n)}}\|H_1(f(u_n)) - \rho F(u_n,b_n) - (H_1(f(a)) - \rho F(a,b)\| \\ &+ \alpha_n c_n^1,\end{aligned} \tag{9}$$

where $c_n^1 = \|R_{M,\rho}^{H_1^{(n)}),\eta}[H_1(f(a)) - \rho F(a,b)] - R_{M,\rho}^{H_1,\eta}[H_1(f(a)) - \rho F(a,b)]\|$, and the last inequality comes from Lemma 2.2, the strong monotonicity and Lipschitz continuity of f. Furthermore, we have $\|H_1(f(u_n)) - \rho F(u_n,b_n) - (H_1(f(a)) - \rho F(a,b))\| \leq \sqrt{\tau_1^2\xi_1^2 + \rho^2 s_1^2 - 2\rho r_1}\|u_n - a\| + \rho\theta_1\|b_n - b\|$. Hence

$$\|a_{n+1}-a\| \leq (1-\alpha_n)\|a_n-a\| + \alpha_n h_n\|u_n-a\| + \alpha_n\|p_n\| + \alpha_n c_n^1 + \alpha_n \frac{\sigma\rho\theta_1}{\gamma_1^{(n)}}\|b_n-b\|, \quad (10)$$

where $h_n = \sqrt{1-2\zeta_1+\xi_1^2} + \frac{\sigma}{\gamma_1^{(n)}}\sqrt{\tau_1^2\xi_1^2+\rho^2 s_1^2-2\rho r_1}$. Similarly, by the iteration scheme we have the following estimation:

$$\|u_n-a\| \leq (1-\beta_n)\|a_n-a\| + \beta_n(\sqrt{1-2\zeta_1+\xi_1^2} + \frac{\sigma}{\gamma_1^{(n)}}\sqrt{\tau_1^2\xi_1^2+\rho^2 s_1^2-2\rho r_1}) \|a_n-a\| + \beta_n\|q_n\| + \beta_n c_n^1 + \beta_n\frac{\sigma}{\gamma_1^{(n)}}\rho\theta_1\|b_n-b\|. \quad (11)$$

It follows that

$$\|a_{n+1}-a\| \leq [1-\alpha_n+h_n\alpha_n(1-\beta_n+\beta_n h_n)]\|a_n-a\| + \alpha_n h_n(\beta_n c_n^1+\beta_n\|q_n\|) + \alpha_n\|p_n\| + \alpha_n c_n^1 + \alpha_n\rho\theta_1\frac{\sigma}{\gamma_1^{(n)}}(1+\beta_n h_n)\|b_n-b\|. \quad (12)$$

Condition (3) implies that $0 < h_n < 1$, $n = 0,1,2,\cdots$, hence, we obtain

$$\|a_{n+1}-a\| \leq [1-\alpha_n(1-h_n)]\|a_n-a\| + \alpha_n(1-h_n)\sigma_n + \frac{\alpha_n\rho\theta_1\sigma(1+\beta_n h_n)}{\gamma_1^{(n)}}\|b_n-b\| \quad (13)$$

where $\sigma_n = \frac{(1+h_n\beta_n)c_n^1+h_n\beta_n\|q_n\|+\|p_n\|}{1-h_n}$. Again, by repeating the similar arguments, we obtain

$$\begin{aligned}\|b_{n+1}-b\| &\leq [1-\alpha_n+\alpha_n k_n(1-\beta_n+\beta_n k_n)]\|b_n-b\| + \alpha_n k_n(\beta_n\|f_n\|+\beta_n c_n^2) \\ &\quad + \alpha_n\|d_n\| + \alpha_n c_n^2 + \alpha_n\lambda\theta_2\frac{\sigma}{\gamma_2^{(n)}}(1+k_n\beta_n)\|a_n-a\|, \quad (14)\end{aligned}$$

where $k_n = \sqrt{1-2\zeta_2+\xi_2^2}+\frac{\sigma}{\gamma_2^{(n)}}\sqrt{\tau_2^2\xi_2^2+\lambda^2 s_2^2-2\lambda r_2}$, $c_n^2 = \|R_{N,\lambda}^{H_2^{(n)},\eta}[H_2(g(b))-\lambda G(a,b)] - R_{N,\lambda}^{H_2,\eta}[H_2(g(b))-\lambda G(a,b)]\|$. Condition (3) implies that $0 < k_n < 1$, $n = 0,1,2,\cdots$, hence, we obtain $\|b_{n+1}-b\| \leq [1-\alpha_n(1-k_n)]\|b_n-b\| + \alpha_n(1-k_n)\delta_n + \alpha_n\lambda\theta_2\frac{\sigma(1+\beta_n k_n)}{\gamma_2^{(n)}}\|a_n-a\|$, where $\delta_n = \frac{(1+k_n\beta_n)c_n^2+k_n\beta_n\|f_n\|+\|d_n\|}{1-k_n}$. Then we have $\|a_{n+1}-a\| + \|b_{n+1}-b\| \leq [1-\alpha_n(1-h_n-\lambda\theta_2\frac{\sigma(1+\beta_n k_n)}{\gamma_2^{(n)}})]\|a_n-a\| +$

$[1-\alpha_n(1-k_n-\rho\theta_1\frac{\sigma(1+\beta_n h_n)}{\gamma_1^{(n)}})]\|b_n-b\|+\alpha_n[(1-h_n)\sigma_n+(1-k_n)\delta_n]$. Set $m=\min\{1-\sqrt{1-2\zeta_1+\xi_1^2}, 1-\sqrt{1-2\zeta_2+\xi_2^2}\}$. Since $0<h_n, k_n<1, \beta_n\in[0,1]$, for all $n\geq 0$, $\{h_n\}, \{k_n\}, \{\beta_n\}$ are bounded, and because we assume that $\gamma_1^{(n)}\uparrow\infty, \gamma_2^{(n)}\uparrow\infty$ as $n\to\infty$, for some positive number ε satisfying $\varepsilon<m<1+\varepsilon$, there exists a positive integer n_1 such that for $n\geq n_1$, $h_n<\sqrt{1-2\zeta_1+\xi_1^2}+\frac{\varepsilon}{2}, \lambda\theta_2\frac{\sigma}{\gamma_2^{(n)}}(1+\beta_n k_n)<\frac{\varepsilon}{2}$ and $k_n<\sqrt{1-2\zeta_2+\xi_2^2}+\frac{\varepsilon}{2}, \rho\theta_1\frac{\sigma}{\gamma_1^{(n)}}(1+\beta_n h_n)<\frac{\varepsilon}{2}$. Therefore, $\|a_{n+1}-a\|+\|b_{n+1}-b\|\leq[1-\alpha_n(m-\varepsilon)](\|a_n-a\|+\|b_n-b\|)+\alpha_n(m-\varepsilon)\frac{(1-h_n)\sigma_n+(1-k_n)\delta_n}{m-\varepsilon}$. Because $\lim_{n\to\infty}p_n=\lim_{n\to\infty}q_n=\lim_{n\to\infty}d_n=\lim_{n\to\infty}f_n=0$, we derive that $\sigma_n\to 0, \delta_n\to 0$ as $n\to\infty$. Let $c_n=\|a_n-a\|+\|b_n-b\|, t_n=\frac{(1-h_n)\sigma_n+(1-k_n)\delta_n}{m-\varepsilon}, \lambda_n=(m-\varepsilon)\alpha_n$, then $c_{n+1}\leq(1-\lambda_n)c_n+\lambda_n t_n, \forall\, n\geq n_1$, and we have $\lambda_n\in[0,1]$, $n=0,1,2\cdots$, $\sum_{n=0}^{\infty}\lambda_n=\infty$, $t_n\to 0$ as $n\to\infty$. By Lemma 4.1, we have $c_n\to 0$ as $n\to\infty$. Hence, $a_n\to a$, $b_n\to b$ as $n\to\infty$. This completes the proof. □

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