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ON S-CONVEXITY AND OUR PAPER WITH IJPAM

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Abstract: In this paper, we review the results of the paper H-H inequality for S-convex functions, published in the prestigious academic vehicle IJPAM. Substantial part of those results will have to be nullified. Most of the time, the mistakes have been inherited from other authors' work, so that we are also providing argumentation for the nullification of those authors' results in this paper in an indirect way.

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1. Introduction

This paper brings the following sections:

- Introduction;
- Definitions and symbols;
- Foundational theorems that we make use of;
- List of the results being analyzed in this paper and due remarks;
- Actual mathematical results contained in H-H inequality for S-convex functions;
- Conclusion;
- References.

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2. Definitions and Symbols

Symbols (see [1]):

- K_s^1 stands for the set of s-convex classes of type 1;
- K_s^2 stands for the set of s-convex classes of type 2;
- $0 < s \leq 1$ is the index designating one of the s-convex classes;
- $K_1^1 \equiv K_1^2$ and both classes are the same as the convex class of functions;
- s_1 stands for the s value designating a class in K_s^1;
- s_2 stands for the s value designating a class in K_s^2.

Definitions (see [1], [2]):

Definition 1. A function $f : X- > \Re$ is told to be $S-$convex in the first sense if

$$f(\lambda x + (1-\lambda^s)^{\frac{1}{s}} y) \leq \lambda^s f(x) + (1-\lambda^s) f(y) \quad (1)$$

$\forall x, y \in X, \forall \lambda \in [0,1]$, where $X \subseteq \Re_+$.

Definition 2. A function $f : X- > \Re$, for which $|f(x)| = f(x)$, is told to belong to K_s^2, for some allowed and fixed value of s, if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds $\forall \lambda \in [0,1]; \forall x, y \in X; s : 0 < s \leq 1; X \subseteq \Re_+$.

Definition 3. A function $f : X- > \Re$, for which $|f(x)| = -f(x)$, is told to belong to K_s^2, for some allowed and fixed value of s, if the inequality

$$f(\lambda x + (1-\lambda)y) \leq \lambda^{\frac{1}{s}} f(x) + (1-\lambda)^{\frac{1}{s}} f(y)$$

holds $\forall \lambda \in [0,1]; \forall x, y \in X; s : 0 < s \leq 1; X \subseteq \Re_+$.

Remark 1. If the complementary inequality to one of the two inequalities above is verified for some function f, then such a function is told to be s_2-concave.

3. Foundational Theorems that we Make Use of

Theorem 3.1. *If f is a bounded function defined in a closed and bounded interval $[a, b]$ and f is continuous except at countably many points, then f is Riemann integrable (see [5]).*

Theorem 3.2. *If f is a bounded function defined in a closed and bounded interval $[a, b]$ and f is Riemann integrable, then f is continuous in $[a, b]$ except, possibly, at countably many points (see [5]).*

Theorem 3.3. *If g is a function defined in $[a, b]$ such that $g(x) < f(x)$ in $[a, b]$, then $\int_a^b g(x)dx < \int_a^b f(x)dx$ (see [5]).*

4. List of the Results Being Analyzed in this Paper and due Remarks

1. (see [3], p. 568) For any $f : [a, b]- > \Re$, f being convex and continuous in $[a, b]$, it is always true that

   $$f(\lambda a + (1 - \lambda)b) \leq \frac{1}{b - a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2},$$

   for each pre-determined value of λ, $\lambda \in [0, 1]$;

   Pointed problems: We have deduced this result because of the proof presented in the paper of McAndrew and Dragomir (see [4]), proof that appears by the page 567 of [3]. However, such a proof is wrong and contains basic mistakes, some of which have been pointed by us throughout time (see our remarks on other results attained by Dragomir et al.). Notice, for instance, that, at the beginning of the proof located at the page 567 of [3], Dragomir et al. make use of a severely well known consequence of the definition of derivative of a function. y, there, is a variable, and the theorem can only make sense if y is a variable. Unfortunately, by the seventh line of our reproduction of that proof, Dragomir et al. replace y with a constant value, $0.5(a + b)$, without presenting any sound mathematical justification for such a move, what is obviously unacceptable. Because the proof in which we have based ourselves is wrong, our result is also wrong, in principle. Basic geometric reasoning tells us that the result is actually unacceptable (left side). Oh, well, if we want to 'pervert' the rules of Mathematics a bit and assume that it is possible to write generically about derivatives over a point in that piece of the proof, rather than writing about deriving the function generically first then about applying the derivative function to that point, we have to adapt all to the x in the formula as well, trivially, what has to mean that x does not belong to (a, b) anymore, but to $(a, 0.5(a + b))$ instead, at most, because there is an assumption that $x \neq y$ (otherwise the ratio would not be defined) and $y > x$ in the derivative formula. Notwithstanding, if we assume what we have just suggested, the resulting inequality is not what we would like it to be.

As a plus, notice that the HH inequality, so that this is another criticism to the trial of proof presented in [4], may be proved without much difficulty, for instance using the geometric idea of integral... : There is no need to think of anything else, then, for the basic rule of Science is 'making it simple if it can be'. We propose that this result be **entirely nullified**.

2. (see [3], p. 570) Let f be an almost everywhere continuous s_2-convex function obeying the following conditions: $Im_f \subseteq \Re_+, D_f \subseteq I \subseteq \Re, \{a, b\} \subset I, a < b$. We then have that:

$$2^{s-1} f(0.5(a + b)) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{s + 1}.$$

Pointed problems: In the proof provided by us for the right side of the above inequality, justified in this step because $f \in K_s^2$, we have that

$$f((1-\lambda)x+\lambda(x+\delta)) \leq (1-\lambda)^s f(x)+\lambda^s f(x+\delta), \forall \lambda \in [0, 1], x \in D_f, \delta > 0.$$

As a special instance of this inequality, we have:

$$f((1 - \lambda)a + \lambda b) \leq (1 - \lambda)^s f(a) + \lambda^s f(b), \forall \lambda \in [0, 1].$$

There is some huge hesitation here, by the time of this step, step that is present in many deductions of Dragomir et al.: We now would like to make λ, previously treated by us as a constant, play the role of a variable. However, if it is just swapping its qualifier from constant to variable, why would we not have called it variable in the definition? Basically, in the traditional definition of Convexity, we are supposed to have x and y (only) playing the role of variables. Thus, we cannot change the definition to constants a and b instead of variables x and y or, as we have made it be, x and[1] $(x+\delta)$. If λ were a variable, we would be obliged to have our domain

[1]With a lot of extra analytical work, these things might be possible, but the results would be severely distinct from the claimed results.

at least in some part of $\Re^2$ as well. Notice that f is not a function of λ, λ is used in order for us to get a domain member for f, which will be a function of another variable, that has disappeared by means of insertion of a and b. Because λ is not the variable of f, to integrate f, we need to do it over an unknown variable in the situation we find f in the inequality we now discuss. Therefore, unfortunately, we cannot proceed from here the way we suggested before (see [3]). As any other reasoning would lead to an upper bound that differs from the one we currently hold, we'd better **nullify this result entirely**. There was also a typo in our published theorem: Our deduction involves 0 and 1 as limits of the integral, but we used a and b in that instance of application of the definition.

3. (see [3], p. 571) For any choice of couple of elements of the domain of any convex function, which be continuous almost everywhere, we have that:

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} f(x)dx \leq \frac{f(x_1)+f(x_2)}{s+1}.$$

Pointed problems: The same problems that we have mentioned in this paper, which appear in our just nullified theorem, appear here. Therefore, we should **nullify this result entirely**.

4. (see [3], p. 572) Via geometric deduction, for $f \in K_s^2$, $f : [a,b]- > \Re$, f continuous almost everywhere, we have:

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(0.5(a+b))}{2^{s-1}}.$$

Pointed problems: The proposal of ours in [3] is not so wrong in what is found there described with words, but it is atrociously wrong in what is there described mathematically. Basically, we wished to refer to the integral reasoning, therefore we should obviously simply have restricted ourselves to picking the supreme of the function in the interval. In mathematical terms:

$$f \in K_s^2, f : [a,b]- > \Re \implies \exists \sup f(x).$$

$$\sup f(x) = k \implies f(x) \leq k \implies f(x) < k + \epsilon, \epsilon \in \Re_+^* \implies$$

$$\implies \int_a^b f(x)dx < (k+\epsilon)(b-a).$$

Obs.: The last step in the proof just described has been attained with the aid of the theorem 3.3, from the Section 3, from this very paper.

The correct geometric deduction must then be: $f \in K_s^2$, $f : [a,b] - > \Re$, $\sup f(x) = k$, $\epsilon \in \Re_+^*$, f continuous almost everywhere $\implies \frac{1}{b-a}\int_a^b f(x)dx$ $< k + \epsilon$.

Our thoughts, as for [3], have originated in the confusion present in the works of Dragomir et al., which ends up affecting us badly after we decide to respect the approval of all those pieces of theory blindly only because they were found 'published' in academic journals, therefore approved by several mathematicians in the world, by entire editorial boards... . It all resumes, basically, to accepting the moves 'a constant may become a variable' and 'a variable may become a constant' in Mathematics... .

Trivially, such thoughts are absurd. We can only apologize for this 'temporary' equivocated 'acceptance' of the unacceptable in our writings. We must also add that there is a typo in our paper (see [3]): Instead of $f(0.5(a+b))$, it should have been $f(a) + f(b)$ in our numerator. Thus, this result will be **entirely nullified** by now.

5. (see [3], p. 573) Let f be an almost everywhere continuous s_2-convex function obeying the following conditions: $Im_f \subseteq \Re_+, D_f \subseteq I \subset \Re, \{a,b\} \subset I, a < b$. We then have that:

$$2^{s-1}f(0.5(a+b)) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq 2^{1-s}f(0.5(a+b)).$$

Pointed problems: The right side of this inequality has already been nullified by us in this paper (see the previous items). The left side is not correct either. In visiting the third page of [6], we find a series of basic mistakes. From the first and second lines, already, departing from inequality (2.2): x and y are variables appearing together in a simple sum, which has only x and y as addends. With Dragomir et al.'s suggestion, we actually get $x = ta + (1-t)b$ and $y = tb + (1-t)a$, what leads to $x + y = a + b$, therefore to a single possible evaluation of the inequality (2.2), rather than a set of them (variables), being it all unacceptable mathematically from this point onwards. Notice that one new variable is inserted when Dragomir et al. supposedly simply 'apply' the definition of s_2-convexity to the now 'constant' argument. Such has been observed before by us in their work, and is fully unacceptable mathematically, leading to the nullification of our previously 'claimed-to-be' left bound for this inequality. Therefore, this result must be **entirely nullified** as well.

6. (see [3], p. 574) $f : [a, a+\lambda]-> \Re$, $\lambda \leq 1$, f being convex, it is always true that

$$f\left(\frac{2a+\lambda}{2}\right) \leq \frac{1}{\lambda}\int_a^{a+\lambda} f(t)dt \leq (a+0.5\lambda)\lambda f(x) + (1-0.5\lambda-a)\lambda f(y).$$

   Pointed problems: This theorem has been obtained using the equivocated reasoning of making λ, from the definition of $S-$convexity, play the role of 'variable'. Another atrocious step in our proof has been rewriting, following [6], our constants a and b as functions of λ, own a, and own b, obviously absurdity of no dimension in Real Analysis the way it has been done. The other atrocious step here present is what we have called PROOF Z in recent work (review of our paper with Aequationes Mathematicae, which is replacing one of the variables in the derivative definition with a constant value). Therefore, this result must be **entirely nullified**.

7. (see [3], p. 576) Let f be an almost everywhere continuous s_2-convex function obeying the following conditions: $Im_f \subseteq \Re_+, D_f \subseteq I \subseteq \Re$, $\{a,b\} \subset I$, $a < b$. We then have that:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}.$$

   Pointed problems: The right side has already been opposed to by us in this very paper. Even if the left bound may be kept, we have not yet presented a sound mathematical proof of that, so that this result must be **entirely nullified**.

8. (see [3], p. 578) Let f be an almost everywhere continuous s_1-convex function in an interval $I \subset \Re_+$ and let $\{a,b\} \subset I$ with $a < b$. We then have that:

$$f(a) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f(b).$$

   Pointed problems: This theorem could be seen as a direct derivation of a basic theorem from Real Analysis, the theorem 3.3 of this paper, Section 3, provided we assume that I is a compact interval. Because any s_1-convex function is non-decreasing (in principle, because we have not yet nullified this theorem), and we are assuming that our s_1-convex function would be defined in both bounds of a compact, there is an infimum to the function in the lower bound of its interval of definition and a supremum in its upper bound. That will also be true for each compact contained in I. We now consider both extremes, attained in the just proposed way, as equivalent to constant functions in the domain interval. Next, we apply the theorem and are obliged to accept the boundaries.

5. Actual Mathematical Results Contained in H-H Inequality for S-Convex Functions

Let f be an almost everywhere continuous s_1-convex function in a compact interval $I \subseteq \Re+$ and assume that $\{a, b\} \subset I$, as well as $a < b$. We then have that:

$$f(a) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq f(b).$$

Obs.: The result above still depends on our revision of the result attained by other researchers to be of mathematical use (that every s_1-convex function is non-decreasing).

6. Conclusion

In this paper, we have managed to provide the scientific community with a revision of our IJPAM paper. Our revision does imply the nullification of a large amount of previous results, also of other researchers, given the horrible mathematical mistakes that have remained ‘unnoticed’ by the editors of the journals publishing them, as well as by the vast majority of the scientific community this far.

All our claimed results from the IJPAM paper will have to be nullified.

One of the results for that paper may be saved if re-stated, provided we revise one particular theorem, stated by other researchers (that every s_1-convex function is non-decreasing), and agree with its proof or provide alternative sound proof for it.

It is also possible that the inequality involving the lower bound for the H-H inequality may be kept as it is for s_2-convexity, but we still need to produce a sound proof for it.

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