Abstract: We construct Eberlein almost periodic functions $f_j : J \rightarrow H$ so that $||f_1(\cdot)||$ is not ergodic and thus not Eberlein almost periodic and $||f_2(\cdot)||$ is Eberlein almost periodic, but $f_1$ and $f_2$ are not pseudo almost periodic, the Parseval equation for them fails, where $J = \mathbb{R}_+$ or $\mathbb{R}$ and $H$ is a Hilbert space. This answers several questions posed by Zhang and Liu [18].

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1. Introduction and Notation

Recently Zhang and Liu [18] asked, whether for Hilbert space valued Eberlein almost periodic $f : \mathbb{R} \rightarrow H$ (see §2) a Parseval equation holds (Fourier coefficients for such $f$ are always defined by [14, Theorem 2.4, for $\mathbb{R}_+$]); this would imply that such $f$ are pseudo almost periodic (see (2.8)). If additionally the range $f(\mathbb{R})$ is relatively norm compact, this is true by results of Goldberg and Irvin [9, Proposition 2.9].

Here we show by examples, that without $f(\mathbb{R})$ relatively compact the $f$ is in general no longer pseudo almost periodic, one has no Parseval equation.
Throughout this paper, $\mathbb{R}^+ = [0, \infty)$, $J \in \{\mathbb{R}^+, \mathbb{R}\}$, $X$ real or complex Banach space; for $f : J \to X$, $f_s(t) := f(s + t)$, $||f|| := ||f(t)||$, $C_b(J, X) = \{f : J \to X : f$ continuous, $||f|| = \sup_{t \in J} ||f(t)|| < \infty\}$, $C_{ub}(J, X) = \{f \in C_b(J, X) : f$ uniformly continuous\}, $AP(\mathbb{R}, X) =$ almost periodic functions [1, p. 3], [16, p. 18-19], $AP(\mathbb{R}^+, X) = AP(\mathbb{R}, X)|_{\mathbb{R}^+}$.

2. Eberlein and Pseudo Almost Periodic Functions

A function $f : J \to X$ is called Eberlein almost periodic if $f \in C_b(J, X)$ and orbit $O(f) := \{f_s : s \in J\}$ is relatively weakly compact in $C_b(J, X)$ (see [8, Definition 10.1, p. 232], [6, Definition 1.4], [12, p. 467], [5, Definition 2.1, p. 138])

$$EAP(J, X) := \{f : f$ Eberlein weakly almost periodic\}, \hspace{1cm} (2.1)$$

$$EAP_0(J, X) := \{f \in EAP(J, X) : 0 \in \text{weak closure of } O(f)\}, \hspace{1cm} (2.2)$$

$$EAP_{rc}(J, X) := \{f \in EAP(J, X) : f(J) \text{ relatively norm compact in } X\}. \hspace{1cm} (2.3)$$

By [2], Theorem 2.3.4 and Theorem 2.4.7, [14] one has

$$EAP(J, X) \subset \mathcal{E}(J, X) \cap C_{ub}(J, X), \hspace{1cm} (2.4)$$

where

$$\mathcal{E}(J, X) := \{f \in L^1_{loc}(J, X) : \text{to } f \text{ exists } x \in X,\}$$

with

$$||\frac{1}{T} \int_s^{s+T} f(t) dt - x|| \to 0 \text{ as } T \to \infty, \text{ uniformly in } s \in J\}, \hspace{1cm} (2.5)$$

then $m_B(f) := x$ is called the Bohr-mean.

For $J$ and $X$ as in Section 1 one has a decomposition theorem [13], p. 18, (in $f = g + h$ the $g \in AP(J, X)$, $h \in EAP_0(J, X)$ are unique)

$$EAP(J, X) = AP(J, X) \oplus EAP_0(J, X). \hspace{1cm} (2.6)$$

The class of pseudo almost periodic functions introduced by Zhang [16], [17], Definition 5.1, p. 57, [3], (1.1) is given by

$$PAP_0(\mathbb{R}, X) := \{f \in C_b(\mathbb{R}, X),$$
\[ m_B(|f|) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(t)| \, dt \text{ exists } = 0 \}, \quad (2.7) \]

similarly for \( \mathbb{R}_+ \), with \( m_B(|f|) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} |f(t)| \, dt \),

\[ \text{PAP}(J, X) := \text{AP}(J, X) \oplus \text{PAP}_0(J, X). \quad (2.8) \]

Now by [9], Proposition 2.9, one has

\[ f \in \text{EAP}_{rc}(J, X) \text{ implies } |f| \in \text{EAP}(J, \mathbb{C}), \text{ and so } |f|^2 \in \text{EAP}(J, \mathbb{C}), \quad (2.9) \]

by [8], Theorem 12.1, p. 234.

So, if \( X = \text{complex Hilbert space } H \), the polarisation formula ([11, p. 24, (2)]) yields \( (f(\cdot), g(\cdot))_H \in \text{EAP}(J, \mathbb{C}) \) if \( f, g \in \text{EAP}_{rc}(J, H) \), (2.4) shows that \( (f, g) := m_B(f(\cdot), g(\cdot))_H \) is well defined. With this (semi-definite) scalar product one gets [9, Theorems 5.2 and 5.7] a Parseval equation for \( f \in \text{EAP}_{rc}(J, H) \).

So, (2.6), (2.4) and [9], Corollary 4.19, give

\[ \text{EAP}_{rc}(\mathbb{R}, H) = \text{AP}(\mathbb{R}, H) + \{ f \in \text{EAP}_{rc}(\mathbb{R}, H) : (f, f) = 0 \}, \]

with \( (f, f) = 0 \) if and only if \( m_B(|f|) = 0 \), \( f \in \text{EAP}_{rc}(\mathbb{R}, H) \). \quad (2.10)

So, with (2.8) one gets for any complex Hilbert space \( H \)

\[ \text{EAP}_{rc}(\mathbb{R}, H) \subset \text{PAP}(\mathbb{R}, H). \quad (2.11) \]

Without the “range relatively compact” however all this is no longer true.

### 3. Examples

For the following we need a converse of Mazur’s theorem (see [15], p. 120, Theorem 2), namely

**Proposition 3.1.** A sequence \( (x_n) \subset X \) weakly converges to \( x \in X \) if and only if, for each subsequence \( (x'_n) \) of \( (x_n) \), there exists a sequence \( (y_n) \) of finite convex combinations of the elements of \( (x'_n) \) with \( ||y_n - x|| \to 0 \) as \( n \to \infty \).

For a proof see [4], Proposition 1.8, p. 17.

**Example 3.2.** For \( J = \mathbb{R}_+ \) or \( \mathbb{R} \) and \( H \) infinite dimensional Hilbert space there exists \( f \in \text{EAP}_0(J, H) \) so that \( m_B(|f|) \) and \( m_B(|f|^2) \) do not exist, \( m_B(|f|) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} ||f(t)|| \, dt \) respectively \( \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} ||f(t)|| \, dt \) if \( J = \mathbb{R}_+ \) respectively \( \mathbb{R} \). So \( |f| \) and \( |f|^2 = (f(\cdot), f(\cdot))_H \) are not in \( \text{EAP}(J, \mathbb{C}) \).
Theorem. Choose an orthonormal sequence $(e_n)_{n \in \mathbb{N}}$ from $H$. Define $h : \mathbb{R} \to H$ by $h := 0$ on $(-\infty, \frac{1}{2}, n) \text{ and on } [n, n + \frac{1}{2}]$, with $h(n - \frac{1}{2}) = 0$, $n \in \mathbb{N}$. Then $h$ is well defined and $\in C_{ub}(\mathbb{R}, H)$.

Define further $\phi : \mathbb{R} \to [0, 1]$ for given $I_n = [\alpha_n, \beta_n]$, $\beta_n = \alpha_{n+1} \in \mathbb{N}$, $\alpha_n < \beta_n$, $n \in \mathbb{N}$, $I_1 = [0, 1]$, as follows:

$\phi := 0$ on $(-\infty, 0]$ and all $I_n$ with odd $n$, $\phi = 1$ on $[\alpha_2n + \frac{1}{10}, \beta_2n - \frac{1}{10}]$ and $\phi$ linear on $[\alpha_2k, \alpha_2k + \frac{1}{10}]$ and $[\beta_2k - \frac{1}{10}, \beta_2k]$, $k \in \mathbb{N}$. Then also $\phi$ is well defined and $\in C_{ub}(\mathbb{R}, \mathbb{R})$.

To get a non-ergodic $\phi$, choose the $I_n$ recursively with $I_1 = [0, 1]$ as follows (Zorn's Lemma):

If $I_1, \ldots, I_{2k}$ are defined, take $\alpha_{2k+1} := \beta_{2k}$ and $\beta_{2k+1}$ such that $\frac{\alpha_{2k}}{\beta_{2k+1}} < \frac{1}{5}$;

If $I_1, \ldots, I_{2k-1}$ are defined, take $\alpha_{2k} := \beta_{2k-1}$ and $\beta_{2k}$ such that

$$\frac{\beta_{2k} - \alpha_{2k} - \frac{1}{5} - 2}{\beta_{2k}} > \frac{3}{4}.$$ 

Finally, define $f := \phi h | J$, $\in C_{ub}(J, H)$. Then

$$\lim_{T \to \infty} \inf \frac{1}{2T} \int_{-T}^{T} |f(t)| \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} |f(t)| \, dt \leq$$

$$\lim_{T \to \infty} \inf \frac{1}{2T} \int_{0}^{T} \phi(t) \, dt \leq \lim_{k \to \infty} \frac{1}{\beta_{2k+1}} \int_{0}^{\beta_{2k+1}} \phi(t) \, dt \leq \frac{1}{5},$$

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f|^2(t) \, dt = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{T} \|\phi(t)h(t)\|^2 \, dt \geq$$

$$\lim_{k \to \infty} \frac{1}{\beta_{2k}} \int_{0}^{\beta_{2k}} |\phi|^2(t) |h|^2(t) \, dt \geq \lim_{k \to \infty} \frac{1}{\beta_{2k}} [\beta_{2k} - \alpha_{2k} - \frac{1}{5} - 2] \geq \frac{3}{4}.$$ 

Since $|f|^2 \leq |f|$, the above shows that $m_B(|f|)$ and $m_B(|f|^2)$ do not exist, $J = \mathbb{R}$ or $\mathbb{R}^+$. 

$f \in EAP(J, H)$: With the Eberlein-Smulian theorem [7], p. 430, Theorem 1, one has to show: To each sequence $(b_n)$ from $J$ there exists a subsequence $(a_m)$ and $g \in V := C_b(J, H)$ with $f_{b_m} \to g$ weakly in $V$. Now if $(b_n)$ is bounded, there exists a subsequence $(c_k)$ and $c \in J$ with $c_k \to c$, so $f_{c_k} \to f_c$ uniformly on $J$ and so even in the norm of $V$, since $f \in C_{ub}(J, H)$.

Now assume $a_m \to \infty$; by taking a further subsequence, one can assume $a_{m+1} - a_m > 1$ and $a_1 \geq 1$, $m \in \mathbb{N}$. To apply Proposition 3.1, let $(c_k)$ be any
subsequence of \( (a_m) \) and \( \varepsilon > 0 \), then there exist \( q, k_1, \ldots, k_q \in \mathbb{N} \) with \( \frac{1}{q} < \varepsilon^2 \), \( c_{k_j+1} - c_{k_j} > 1 \) and \( c_{k_1} \geq 1, 1 \leq j \leq q \). Then the \( c_{k_j} \) are in different \([n - \frac{1}{2}, n + \frac{1}{2}]\) intervals for different \( j \), so for any \( t \in \mathbb{R} \), \( f_{c_{k_j}}(t) = f(c_{k_j} + t) = r_j t e_p(j,t) \) with \( 0 \leq r_j t \leq 1 \) and \( p(i,t) < p(j,t) \) if \( i < j \) and \( c_{k_j} + t > \frac{1}{2} \). With \( i_0 \) minimal if such \( i \) exist and \( \theta_{k_j} = \frac{1}{q}, 1 \leq j \leq q \), else \( = 0 \), one gets

\[
\left\| \sum_{j=1}^{k_q} \theta_{k_j} f(c_{k_j} + t) \right\|_H^2 = \left\| \sum_{j=1}^{q} \frac{1}{q} f(c_{k_j} + t) \right\|_H^2 =
\]

\[
\left\| \sum_{j=io}^{q} r_{j,t} e_p(j,t) \right\|_H^2 = \frac{1}{q^2} \sum_{j=io}^{q} (r_{j,t})^2 \leq \frac{q}{q^2} = \frac{1}{q};
\]

if no such \( i_0 \) exists, the above sum is even \( 0 \leq \frac{1}{q} \).

This holds for any \( t \in \mathbb{R} \), so \( \left\| \sum_{j=1}^{k_q} \theta_{k_j} f_{c_{k_j}} \right\|_V < \varepsilon \). Therefore by Proposition 3.1 indeed \( f_{c_k} \rightarrow 0 \) weakly in \( V, J = \mathbb{R} \) or \( \mathbb{R}_+ \).

The case \( a_m \rightarrow -\infty \) ( \( J = \mathbb{R} \)) follows similarly.

\( f \in EAP_0(J, H) \): For \( (b_n) = (n) \) the above shows \( 0 \in \text{weak closure of orbit } O(f) \).

\( |f| \) and \( (f(.),f(.)) \) not Eberlein almost periodic follows with (2.4). \( \square \)

Since for the \( f \) of Example 3.2 the Bohr mean \( m_B(|f|^2) \) does not exist, one has no Parseval equation.

\( EAP(J, X) \subset PAP(J, X) \) is also false, already for \( X = \text{Hilbert space} \):

Assume \( f \in EAP(J, X) \subset PAP(J, X) \). Then \( f = g + h, g \in AP(J, X), h \in PAP_0(J, X) \); now for \( f \in EAP_0(J, X) \) one can show that all Fourier coefficients vanish (for \( J = \mathbb{R}_+ \) see [14, Theorem 2.4]), for \( h \) the same holds , implying \( g = 0 \), then \( f = h \in PAP_0(J, X) \) and so the existence of \( m_B(|f|) \), a contradiction for \( f \) of Example 3.2.

The proof of Example 3.2 works also for \( X = l^p(N, \mathbb{C}), 1 < p \leq \infty \) and \( c_0 \), so \( EAP(J, X) \subset PAP(J, X) \) is also false for these \( X \).

Since for any \( f \in EAP(J, X) \) the range \( f(J) \) is relatively weakly compact, and if \( X = l^1 = l^1(M, \mathbb{C}) \), any \( M \), this implies \( f(J) \) relatively norm compact [10, p. 281 (2)], one has

\[
EAP(J, l^1) = EAP_{rc}(J, l^1) \subset PAP(J, l^1).
\]

**Example 3.3.** For \( J = \mathbb{R} \) or \( \mathbb{R}_+ \) and \( H \) separable infinite dimensional Hilbert space there exist \( f \in EAP_0(J, H) \) with \( |f|, |f|^2 \in AP(J, \mathbb{R}) \subset EAP(J, \mathbb{R}) \), but \( f(J) \) is not relatively compact, \( m_B(|f|) \) and \( m_B(|f|^2) \) exist and are \( > 0 \).
So a converse of (2.9) is not true, even with $|f|, |f|^2 \in EAP(J, \mathbb{R})$ the Parseval equation can fail, such $f$ need not be pseudo almost periodic.

**Proof.** Choose an orthonormal sequence $(e_n)_{n \in \mathbb{Z}}$ from $H$. Define $f : \mathbb{R} \to H$ by $f(n) := e_n$, $n \in \mathbb{Z}$, $f$ linear on $[n - \frac{1}{2}, n]$ and on $[n, n + \frac{1}{2}]$, with $f(n - \frac{1}{2}) = 0$, $n \in \mathbb{Z}$. Then $f$ is well defined and $\in C_{ub}(\mathbb{R}, H)$. One can prove that $f \in EAP_0(\mathbb{R}, H)$ as in the proof of Example 3.2. Obviously, $|f| \in C_{ub}(\mathbb{R}, \mathbb{R})$ has period 1 and so $|f| \in AP(\mathbb{R}, \mathbb{R}) \subset EAP(\mathbb{R}, \mathbb{R})$. $f|_{\mathbb{R}_+}$ has the same properties.

Added in proof: By communication from Kreulich and Ruess, they can construct to each bounded uniformly continuous $g : \mathbb{R} \to [0, \infty)$ an $f \in EAP(\mathbb{R}, H)$ with $|f| = g$ on $\mathbb{R}$.

**References**


