Abstract: The orbit Dirichlet series for maps are studied. The main results find the orbit Dirichlet series of prime powers of maps with polynomial orbit growth.

AMS Subject Classification: 11M41

Key Words: periodic orbits, orbit Dirichlet series, powers of maps

1. Introduction

A Dirichlet series is any series of the form

\[ \sum_{n=1}^{\infty} \frac{a_n}{n^s} \]

where \( s \) and \( a_n \) are complex numbers, \( n = 1, 2, 3, \cdots \). The most famous Dirichlet series is

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \]

the Riemann zeta function (see details in [1]).

In 2009, Pakapongpun and Ward [2] studied the orbit Dirichlet series

\[ d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s} \]

where \( T \) is a continuous mapping \( T : X \to X \), \( X \) is a compact metric space and \( O_n(T) \) is the number of closed orbit of length \( n \) under \( T \).
Recently, Pakapongpun studied $d_{T \times T \times \cdots \times T}(s)$ where there are $m$ terms in the cartesian product (the detailed calculation is in [3]).

The orbit Dirichlet series $d_{T^p}(s)$ is studied in this paper, where $O_n(T) = n^a$ and $p$ is a prime number. This is a simple example of a set of problems studied in [2] and [3].

2. Preliminary Notes

Let $T$ be a map. A closed orbit $\tau$ of length $|\tau|$ is a set of the form

$$\{x, Tx, T^2x, \cdots, T^{|\tau|}x = x\}$$

with cardinality $|\tau|$. Write $O_n(T)$ for the number of closed orbits of length $n$ under $T$. We always assume that $O_n(T) < \infty$ for all $n \geq 1$.

The number of points of period $n$ is

$$F_n(T) = \sum_{d|n} dO_d(T),$$

and the relation with the number of closed orbit is given by

$$O_n(T) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right)F_d(T)$$

where $\mu : \mathbb{N} \to \mathbb{R}$ is the M"{o}bius function (see Definition 2.3 below).

The orbit Dirichlet series associated to $T$ is

$$d_T(s) = \sum_{n=1}^{\infty} \frac{O_n(T)}{n^s}.$$

Notice that if $O_n(T) = n^a$, then $d_T(s) = \zeta(s - a)$.

The asymptotic behaviour of expression like

$$\pi_T(N) = \#\{\tau : |\tau| \leq N\} = \sum_{n \leq N} O_n(T),$$

where $\tau$ is a closed orbit of length $|\tau| = n$ is studied in dynamical systems (see details in [1]).

The number of orbits of length $n$ under an iterate $T^p$ for some prime $p$ will be shown in the next theorem.
Theorem 2.1. Let $p$ be a prime. Then

$$O_n(T^p) = \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T) & \text{if } p \mid n. \end{cases}$$

Proof. We know that

$$O_n(T^p) = \frac{1}{n} \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_{pd}(T), \quad \text{since } F_{n}(T^p) = F_{pn}(T).$$

It $p \nmid n$ then

$$\sum_{d \mid pn} \mu \left( \frac{pn}{d} \right) F_d(T) = \sum_{d \mid pn, p \nmid d} \mu \left( \frac{pn}{d} \right) F_d(T) + \sum_{d \mid pn, p \mid d} \mu \left( \frac{pn}{d} \right) F_d(T)
= \sum_{d \mid pn, p \nmid d} \mu \left( \frac{pn}{d} \right) F_d(T) - \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_d(T)
= \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_{pd}(T) - \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_d(T).$$

Thus,

$$O_n(T^p) = \frac{1}{n} \sum_{d \mid pn} \mu \left( \frac{pn}{d} \right) F_d(T) + \frac{1}{n} \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_d(T)
= pO_{pn}(T) + O_n(T).$$

If $p \mid n$ then

$$O_n(T^p) = \frac{1}{n} \sum_{d \mid n} \mu \left( \frac{n}{d} \right) F_{pd}(T).$$

Let $pd = k$, $\frac{k}{p} = d$, so $k \mid pn$. Then

$$O_n(T^p) = \frac{1}{n} \sum_{d \mid pn} \mu \left( \frac{pn}{d} \right) F_d(T)
= pO_{pn}(T).$$

So,

$$O_n(T^p) = \begin{cases} pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\ pO_{pn}(T) & \text{if } p \mid n. \end{cases}$$
Definition 2.2. For real or complex $k$ and integer $n \geq 1$ we define
\[ \sigma_k(n) = \sum_{d|n} d^k \]
the sum of the $k^{th}$ power of the divisor of $n$. When $k = 1$, $\sigma_1(n)$ is the sum of divisors of $n$; this is often denoted by $\sigma(n)$.

Definition 2.3. The Möbius function $\mu : \mathbb{N} \rightarrow \mathbb{R}$ is defined by
\[
\mu(n) = \begin{cases} 
1 & \text{if } n = 1; \\
(-1)^k & \text{if } a_1 = a_2 = \cdots = a_k = 1; \\
0 & \text{if any } a_i > 1,
\end{cases}
\]
where for $n > 1$, we write $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, $a_i \geq 1$ for the factorization of $n$ into distinct primes.

Theorem 2.4. Suppose that the series $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ does not converge for all $s$ or diverge for all $s$. Then there exists a real number $x_a$, called the abscissa of absolute convergence, such that the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely if $\Re > x_a$ but does not converge absolutely if $\Re < x_a$.

Theorem 2.5. (Perron's formula). Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ be absolutely convergent for $\Re(s) > x_a$. Then for $c > x_a$,
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(z) \frac{n^z}{z} \, dz = \sum_{n \leq N}^{*} f(n)
\]
for any $N > 0$. Here $\sum^{*}$ means that the last term in the sum must be multiplied by $1/2$ when $N$ is an integer.

Definition 2.2, 2.3, and Theorem 2.4, 2.5 are taken from [1].

3. Main Results

Theorem 3.1. Let $T$ be map, let $p$ be a prime number and let $k \in \mathbb{N}_0$. If $d_T(s) = \zeta(s - k)$, then
1. $d_T(p)(s) = (p^{k+1} + 1 - p^{k-s}) \zeta(s - k),$
2. the abscissa of convergence of $d_T$ is $k + 1$, and
3. $\pi_T(N) \sim (p^{k+1} + 1 - p^{-1}) \frac{N^{k+1}}{k+1}.$
Proof. 1. Clearly, $d_T(s) = \zeta(s - k)$ if and only if $O_n(T) = n^k$, and we know that

$$O_n(T^p) = \frac{1}{n} \sum_{d|n} \mu \left( \frac{n}{d} \right) F_{pd}(T)$$

$$= \frac{1}{n} \sum_{d|n} \mu \left( \frac{n}{d} \right) \sigma(pd)$$

$$= \left\{ \begin{array}{ll}
pO_{pn}(T) + O_n(T) & \text{if } p \nmid n; \\
pO_{pn}(T) & \text{if } p | n; \end{array} \right. \text{ by Theorem 2.1}$$

$$= \left\{ \begin{array}{ll}
p^{k+1}n^k + n^k & \text{if } p \nmid n; \\
p^{k+1}n^k & \text{if } p | n. \end{array} \right.$$

Thus,

$$d_{T^p}(s) = \sum_{n=1}^{\infty} \frac{p^{k+1}n^k}{n^s} + \sum_{p|n} \frac{n^k}{n^s}$$

$$= \sum_{n=1}^{\infty} \frac{p^{k+1}n^k}{n^s} + \sum_{n=1}^{\infty} \frac{n^k}{n^s} - \sum_{n=1}^{\infty} \frac{(pn)^k}{(pn)^s}$$

$$= \left( p^{k+1} + 1 \right) \zeta(s - k) - p^{k-s} \zeta(s - k)$$

$$= \left( p^{k+1} + 1 - p^{-1} \right) \frac{N^k}{k+1}.$$ 

Hence, the absissa of convergence $d_{T^p}$ is $k + 1$.

2. As we know the abscissa of convergence of the Riemann zeta function is 1, so the abscissa of convergence of $d_T(s) = \zeta(s - k)$ is $k + 1$ (and $(p^{k+1} + 1 - p^{k-s})$ converges for any $s \in \mathbb{C}$). Hence, the abscissa of convergence $d_{T^p}$ is $k + 1$.

3. Notice that $F(s) = \sum_{n=1}^{\infty} \frac{O_n(T^p)}{n^s} = d_{T^p}(s)$ is absolutely convergent for $\Re(s) > k + 1$. If $\Re(s) > k + 1 - \epsilon$ for some small $\epsilon > 0$, then

$$\sum_{n \leq N} O_n(T^p) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} F(z) \frac{N^z}{z} dz \quad \text{(by Perron’s formula)}$$

$$= Res \left( d_{T^p}(s) \frac{N^z}{z} \right)_{z=k+1}$$

$$= \left( p^{k+1} + 1 - p^{-1} \right) \frac{N^k}{k+1}.$$ 

Hence

$$\pi_{T^p}(N) \sim \left( p^{k+1} + 1 - p^{-1} \right) \frac{N^k}{k+1}. \quad \square$$
Example 3.2. Let $d_T(s) = \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Then

$$d_{T^3}(s) = \left(\sigma(3) - \frac{(\sigma(3) - 3)}{3^s}\right) \zeta(s) = \left(4 - \frac{1}{3^s}\right) \zeta(s)$$

and the abscissa of convergence $d_{T^3}$ is 1. Moreover,

$$\pi_{T^p}(N) = \sum_{n \leq N} O_n(T^p) \sim \text{Res} \left( d_{T^p}(s) \frac{N^s}{s} \right)_{s=1} = (4 - \frac{1}{3})N = \frac{11}{3}N.$$

Theorem 3.1 is a simple example of more general results that properties of the orbit Dirichlet series for products and iterates of maps. These results allow analytic methods to be used to find orbit growth asymptotics.

Acknowledgments

I would like to thank you to Tom Ward for his comments and suggestions on this work and this work is supported by Faculty of Science, Burapha University, Thailand.

References

