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EIGENVALUES OF STURM-LIOUVILLE PROBLEMS AND THE ZEROS OF ENTIRE FUNCTIONS OF SINE TYPE

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Abstract: The main contribution of this paper is a result about the eigenvalues of two classes of Sturm-Liouville problems on a finite interval: if $\{\lambda_n\}_{n\geq 1}$ is the sequence of Dirichlet, respectively of Dirichlet-Robin eigenvalues of the canonical Sturm-Liouville operator with coefficient function $q \in L^2_{\mathbb{R}}(0,a)$ (and boundary parameter $\alpha \in \mathbb{R}$ - for the Dirichlet-Robin case), then $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$, and respectively $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$ are the zeros of some entire functions of sine type a.

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1. Introduction

The main concern of the paper is to show that the Dirichlet and respectively the Dirichlet-Robin eigenvalues of the canonical Sturm-Liouville operator with coefficient function $q \in L^2_{\mathbb{R}}(0,a)$ (and boundary parameter $\alpha \in \mathbb{R}$ - for the Dirichlet-Robin case) are closely related to the zeros of some entire functions of sine type (see Section 3). Section 2 provides the background material, and Lemmas 1 and 2 of Section 3 provide supporting details for the proofs of Theorems 1 and 2, which are the main results. As a corollary of our main result, two sets of sine functions are generated and they are shown to be Riesz bases of the space $L^2(0,a)$ (Section 4). In Section 5 the reader is made aware about a misleading route of reasoning that the sets of sine functions resulting from these sets of eigenvalues form Riesz bases.

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2. Preliminaries

In this section we summarize some known facts as they are useful information to refer to when developing the subsequent sections.

Firstly, we provide the reader with the necessary information about the eigenvalues, eigenfunctions, and the characteristic functions of the two classes of Sturm-Liouville differential operators we need. A canonical Sturm-Liouville differential operator with coefficient function (also called potential function) $q \in L^2_{\mathbb{R}}(0,a)$ has the form:

$$L^{(q)}[u] := -u'' + q(x)u, \;\; x \in (0,a).$$

We are interested in such operators studied in connection with two types of boundary conditions: Dirichlet boundary conditions at both ends of the interval

$$u(0) = 0 = u(a),$$

and respectively Dirichlet-Robin boundary conditions

$$u(0) = 0 = u'(a) + \alpha u(a),$$

where $\alpha \in \mathbb{R}$ is called the boundary parameter. Hence the domain of the operator will be by case $D(L^{(q)}) = \{u \in H^2(0,a), u(0) = 0 = u(a)\}$, or $D(L^{(q)}) = \{u \in H^2(0,a), u(0) = 0 = u'(a) + \alpha u(a)\}$. The definition we use here for the space $H^k(0,a)$, $k \geq 1$ is as in [6, pages 128, 147]:

$$H^k(0,a) = \{u \in C^{k-1}[0,a] | u^{(k-1)}(x) = C + \int_0^x v(s)ds, \text{ for some } C \in \mathbb{C}, v \in L^2(0,a)\}.$$

We say that $\lambda \in \mathbb{C}$ is a Dirichlet eigenvalue of $L^{(q)}$, if there exists a function $v \neq 0$, $v \in \{u \in H^2(0,a), u(0) = 0 = u(a)\}$, such that $L^{(q)}[v] = \lambda v$. And we say that $\lambda \in \mathbb{C}$ is a Dirichlet-Robin eigenvalue of $L^{(q)}$, if there exists a function $v \neq 0$, $v \in \{u \in H^2(0,a), u(0) = 0 = u'(a) + \alpha u(a)\}$, such that $L^{(q)}[v] = \lambda v$.

For $q \in L^2_{\mathbb{R}}(0,a)$ and $\lambda \in \mathbb{C}$ consider the canonical Sturm-Liouville differential equation:

$$-u''(x) + q(x)u(x) = \lambda u(x), \;\; x \in (0,a). \tag{2.1}$$

Denote by $C(\cdot; q, \lambda)$ the solution to (2.1) which satisfies the initial conditions:

$$u(0) = 1, \text{ and } u'(0) = 0, \tag{2.2}$$

and by $S(\cdot; q, \lambda)$ the solution to (2.1) which satisfies the initial conditions:

$$u(0) = 0, \text{ and } u'(0) = 1. \tag{2.3}$$

These solutions exist in the week sense (i.e. they are $H^2(0, a)$) and are unique as demonstrated in the direct Sturm-Liouville theory (see for example Theorem 4.4 in [6, page 130]).

We observe that $\lambda \in \mathbb{C}$ is a Dirichlet eigenvalue of $L^{(q)}$ *if and only if* $S(a; q, \lambda) = 0$. This is seen as follows: if λ is such an eigenvalue, then there exists a corresponding eigenfunction $v \neq 0$. That is $v \in H^2(0, a) - \{0\}$ satisfies (2.1), and $v(0) = 0 = v(a)$. It follows that $v'(0) \neq 0$, because otherwise, the first Dirichlet boundary condition $v(0) = 0$ and the fact that v satisfies (2.1) would imply that $v = 0$, as the only solution to such initial value problem. Hence $\frac{v}{v'(0)}$ is well defined and also satisfies (2.1), and in addition it satisfies (2.3). By the uniqueness of solution to such initial value problem, we get that $\frac{v}{v'(0)} = S(\cdot; q, \lambda)$, from which, using the second Dirichlet boundary condition $v(a) = 0$, we obtain $S(a; q, \lambda) = 0$. Conversely, if λ is such that $S(a; q, \lambda) = 0$, then by the above definition of $S(\cdot; q, \lambda)$ we obtain that $S(\cdot; q, \lambda)$ is a Dirichlet eigenfunction of $L^{(q)}$ that goes with λ. So, λ is a Dirichlet eigenvalue of $L^{(q)}$.

We also note that $\lambda \in \mathbb{C}$ is a Dirichlet-Robin eigenvalue of $L^{(q)}$ *if and only if* $S'(a; q, \lambda) + \alpha S(a; q, \lambda) = 0$. The arguments are similar to the above ones: if λ is such an eigenvalue, then there exists an associated eigenfunction v (i.e. $v \in H^2(0, a) - \{0\}$ satisfies (2.1) and $v(0) = 0 = v'(a) + \alpha v(a)$). Hence again $\frac{v}{v'(0)}$ is well defined and coincides with $S(\cdot; q, \lambda)$. It follows from this and the fact that v satisfies the Robin boundary condition $v'(a) + \alpha v(a) = 0$ that $S'(a; q, \lambda) + \alpha S(a; q, \lambda) = 0$. Conversely, if λ is such that $S'(a; q, \lambda) + \alpha S(a; q, \lambda) = 0$, then by the definition of $S(\cdot; q, \lambda)$ one has that $S(\cdot; q, \lambda)$ is a Dirichlet-Robin eigenfunction of $L^{(q)}$ associated with λ. Thus λ is a desired eigenvalue of $L^{(q)}$.

The functions

$$\begin{cases} \lambda \in \mathbb{C} \to S(a; q, \lambda) \\ \lambda \in \mathbb{C} \to S'(a; q, \lambda) + \alpha S(a; q, \lambda) \end{cases} \tag{2.4}$$

are known as the characteristic functions of the Sturm-Liouville operator $L^{(q)}$ with domain $D(L^{(q)}) = \{u \in H^2(0, a), u(0) = 0 = u(a)\}$, and respectively with domain $D(L^{(q)}) = \{u \in H^2(0, a), u(0) = 0 = u'(a) + \alpha u(a)\}$. The name was suggested by the analogy with the *matrix* case, where the eigenvalues of a square matrix $A \in \mathcal{M}_n(\mathbb{R})$ (thought of as a linear operator on $\mathbb{R}^n$) are the zeros of the characteristic polynomial $p(\lambda) := \det(A - \lambda I)$.

Applying the Analyticity Property (a) of [7, page 10], one has that $S(a;q,\lambda)$ and $S'(a;q,\lambda)$ are entire functions of λ (i.e. $f:\mathbb{C}\to\mathbb{C}$ is said to be an *entire function* if it is continuously differentiable in all of $\mathbb{C}$). Therefore, the two functions in (2.4) are entire functions of λ.

Also, it is known in the theory of direct Sturm-Liouville problems (see for example the beginning of Section 4.5, and Theorem 4.18 and Example 4.19 in [6, pages 154-157]) that $S(\cdot;q,\lambda)$ admits an integral representation:

$$S(x;q,\lambda)=\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}+\int_0^x K(x,t;q)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\ x\in[0,a], \tag{2.5}$$

where $K(x,t;q)$ (called the 'Gelfand-Levitan kernel') is the weak solution (in the sense of Theorem 4.15(b,c) in [6, page 147]) to the Goursat problem in the triangle $\{(x,t)\in\mathbb{R}^2|0\le t\le x\le a\}$:

$$\begin{cases} K_{xx}(x,t)-K_{tt}(x,t)-q(x)K(x,t)=0, & 0<t<x<a\\ K(x,x)=\frac{1}{2}\int_0^x q(s)ds, & 0\le x\le a\\ K(x,0)=0, & 0\le x\le a.\end{cases} \tag{2.6}$$

Taking the derivative with respect to x in (2.5) and using the boundary conditions of (2.6) we write:

$$\begin{aligned} S'(x;q,\lambda)=\cos(\sqrt{\lambda}x)+\left(\frac{1}{2}\int_0^x q(s)ds\right)\frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}}\\ +\int_0^x K_x(x,t;q)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt,\ x\in[0,a]. \end{aligned} \tag{2.7}$$

Taking $x=a$ in (2.5) and (2.7), and writing $[q]:=\frac{1}{a}\int_0^a q(s)ds$ for the mean value of q on $[0,a]$ one has:

$$S(a;q,\lambda)=\frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}+\int_0^a K(a,t;q)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt, \tag{2.8}$$

$$S'(a;q,\lambda)=\cos(\sqrt{\lambda}a)+\left(\frac{a[q]}{2}\right)\frac{\sin(\sqrt{\lambda}a)}{\sqrt{\lambda}}+\int_0^a K_x(a,t;q)\frac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}}dt. \tag{2.9}$$

From the direct Sturm-Liouville theory (see for example Lemma 4.7 in [6, page 135] and formula (4.25) in [6, page 139], and rescale $[0,1]$ to $[0,a]$, or Corollary 1 in [7, page 116] and Theorem 4 in [7, page 35], and rescale $[0,1]$

to $[0, a]$) we learn that the Dirichlet eigenvalues of the Sturm-Liouville operator $L^{(q)}$ with $q \in L^2_{\mathbb{R}}(0, a)$ form a sequence $\{\lambda_n\}_{n\geq 1}$ of real numbers, strictly increasing, and satisfying:

$$\lambda_n = \left(\frac{n\pi}{a}\right)^2 + [q] + \tilde{r}_n, \text{ as } n \to \infty, \tag{2.10}$$

where $\{\tilde{r}_n\}_{n\geq 1} \in l_2$.

Also, from the direct Sturm-Liouville theory (see formulas (4.30a)-(4.30b), (4.32) in [6, pages 144-145], or [2, page 255] and rescale $[0, 1]$ to $[0, a]$, and also Theorem 3 in [2, page 257]), we learn that the Dirichlet-Robin eigenvalues of the Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0, a)$ and boundary parameter $\alpha \in \mathbb{R}$ form a sequence $\{\lambda_n\}_{n\geq 1}$ of real numbers, strictly increasing and satisfying the asymptotics:

$$\lambda_n = \left(\frac{(n - \frac{1}{2})\pi}{a}\right)^2 + 2\frac{\alpha}{a} + [q] + \tilde{r}_n, \text{ as } n \to \infty, \tag{2.11}$$

where $\{\tilde{r}_n\}_{n\geq 1} \in l_2$.

Secondly, we present the definitions of entire functions of exponential/sine type a. An entire function is said to be of *exponential type a* if there exists a constant $C > 0$ such that $|f(z)| \leq Ce^{a|z|}$, for all $z \in \mathbb{C}$. An entire function is said to be of *sine type a* if **(1)** it is of exponential type a, **(2)** all its zeros are *separated* (i.e. there exists a constant $\varepsilon > 0$ such that $|z - w| > \varepsilon$, for all $z \neq w$ with $f(z) = 0 = f(w)$), and **(3)** there exist constants $A, B, H > 0$ such that $Ae^{a|Imz|} \leq |f(z)| \leq Be^{a|Imz|}$, for all $z \in \mathbb{C}$ with $|Imz| \geq H$. These definitions are slight adaptations of those in [9, pages 52, 143].

Thirdly, we recall the definitions of a boundedly invertible linear operator, and of orthonormal/Riesz bases in a Hilbert space. A linear operator $T : H \to H$ is said to be *bounded* if there exists a constant $C > 0$ such that $||T(u)|| \leq C||u||$, for all $u \in H$. An equivalent characterization of a bounded linear operator is: 'T is a bounded linear operator on H *if and only if* T is linear and continuous on H'. A linear operator $T : H \to H$ is said to be *boundedly invertible* if it is invertible, and both T and T^{-1} are bounded operators.

A sequence $\{f_n\}_{n\geq 1}$ in a separable Hilbert space H is an *orthonormal basis* of H if the elements are mutually orthogonal (i.e. $< f_n, f_m >= 0$, for all $n \neq m$), of unit length ($||f_n|| = 1$, for all $n \geq 1$), and the sequence is *complete* in H (i.e. if $f \in H$ is such that $< f, f_n >= 0$, for all $n \geq 1$, then $f = 0$ - this is to say that the zero vector alone is orthogonal to all vectors f_n's).

A sequence $\{g_n\}_{n\geq 1}$ is said to be a *Riesz basis* of a Hilbert space H if it is the image of an orthonormal basis of H under a *boundedly invertible linear* *operator*. That is $g_n = T(f_n)$, for all $n \geq 1$, for some orthonormal basis $\{f_n\}_{n\geq 1}$ of H, and some linear, invertible operator $T : H \to H$ such that T and T^{-1} are bounded.

Finally, we remind two results about the Riesz bases: **(1)** a slight adaptation of Theorem 10 in [9, page 144] - 'If $\{\theta_n\}_n$ is the set of zeros of an entire function of sine type a, then $\{e^{i\theta_n t}\}_n$ is a Riesz basis of $L^2(-a, a)$' and **(2)** a characterization of Riesz bases, precisely the equivalence (1) $\leftrightarrow$ (3) of Theorem 9 in [9, page 27] - 'The sequence $\{f_n\}_{n\geq 1}$ is a Riesz basis of the separable Hilbert space H *if and only if* $\{f_n\}_{n\geq 1}$ is complete in H and there exist constants $A, B > 0$ such that for arbitrary $N \geq 1$, and arbitrary scalars $c_1, \ldots, c_N$ the inequalities $A\sum_{n=1}^{N} |c_n|^2 \leq ||\sum_{n=1}^{N} c_n f_n||^2 \leq B\sum_{n=1}^{N} |c_n|^2$ hold'.

Sturm-Liouville problems have also been discussed in [3] and [4], where uniqueness, existence and numerical constructibility of the solutions to some inverse Sturm-Liouville problems have been addressed.

3. The Main Results

We show that the Dirichlet eigenvalues, and respectively, the Dirichlet-Robin eigenvalues of the canonical Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0, a)$ (see Section 2 for definitions and other useful information) are closely related to the zeros of some entire functions of sine type a.

Lemma 1. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of Dirichlet eigenvalues of the canonical Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0, a)$. Then the elements of $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$ are separated.*

Proof. From the discussion in the paragraph of (2.10) we infer that λ_n's are all positive, except possible the first few. So, $\lambda_1 < \lambda_2 < \ldots < \lambda_{\tilde{N}} < 0 < \lambda_{\tilde{N}+1} < \ldots$, for some integer $\tilde{N} \geq 0$. Hence, $\sqrt{\lambda_n}$'s are all real, except the first $\tilde{N}$ of them, which are pure imaginary. Also,

$$\sqrt{\lambda_n} = \frac{n\pi}{a} + \frac{a[q]}{2\pi n} + \frac{c_n}{n^2}, \text{ for } n \geq \tilde{N} + 1, \tag{3.1}$$

for some real sequence $\{c_n\}_{n\geq \tilde{N}+1}$, such that $\{\frac{c_n}{n}\}_{n\geq \tilde{N}+1} \in l_2$. (In this case, $\tilde{r}_n = \mathcal{O}(\frac{1}{n^2}) + \frac{2\pi}{a}\frac{c_n}{n}$ will generate an l_2 sequence, as in (2.10).) This implies that for all $m > n > \tilde{N}$,

$$|(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| \quad \geq \quad ||\pm\sqrt{\lambda_n}| - |\pm\sqrt{\lambda_m}||$$

$$
\begin{aligned}
&= |\sqrt{\lambda_n} - \sqrt{\lambda_m}| \\
&\geq ||\frac{n\pi}{a} - \frac{m\pi}{a}| - |\left(\frac{a[q]}{2\pi n} + \frac{c_n}{n^2}\right) - \left(\frac{a[q]}{2\pi m} + \frac{c_m}{m^2}\right)|| \qquad (3.2)
\end{aligned}
$$

Next,

$$
|\frac{n\pi}{a} - \frac{m\pi}{a}| = \frac{\pi}{a}|n-m| \geq \frac{\pi}{a}, \text{ for all } m > n > \tilde{N}, \qquad (3.3)
$$

and

$$
|\left(\frac{a[q]}{2\pi n} + \frac{c_n}{n^2}\right) - \left(\frac{a[q]}{2\pi m} + \frac{c_m}{m^2}\right)| \leq \frac{a[q]}{2\pi}|\frac{1}{n} - \frac{1}{m}| + |\frac{c_n}{n^2} - \frac{c_m}{m^2}|, \text{ for all } m > n > \tilde{N}. \qquad (3.4)
$$

The sequences $\{\frac{1}{n}\}_{n \geq \tilde{N}+1}$ and $\{\frac{c_n}{n^2}\}_{n \geq \tilde{N}+1}$ are convergent (to 0), so they are Cauchy sequences. Hence, there exists a positive integer $N_1 > \tilde{N}$, such that:

$$
\begin{cases} |\frac{1}{n} - \frac{1}{m}| \leq \frac{\pi^2}{2a^2[q]}, \text{ for all } m > n \geq N_1 \\ |\frac{c_n}{n^2} - \frac{c_m}{m^2}| \leq \frac{\pi}{4a}, \text{ for all } m > n \geq N_1. \end{cases} \qquad (3.5)
$$

Using (3.3), (3.4), (3.5) in (3.2), one obtains:

$$
\begin{aligned}
|(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| &\geq |\sqrt{\lambda_n} - \sqrt{\lambda_m}| \\
&\geq ||\frac{n\pi}{a} - \frac{m\pi}{a}| - |\left(\frac{a[q]}{2\pi n} + \frac{c_n}{n^2}\right) - \left(\frac{a[q]}{2\pi m} + \frac{c_m}{m^2}\right)|| \\
&= |\frac{n\pi}{a} - \frac{m\pi}{a}| - |\left(\frac{a[q]}{2\pi n} + \frac{c_n}{n^2}\right) - \left(\frac{a[q]}{2\pi m} + \frac{c_m}{m^2}\right)| \\
&\geq \frac{\pi}{a} - \left(\frac{a[q]}{2\pi}|\frac{1}{n} - \frac{1}{m}| + |\frac{c_n}{n^2} - \frac{c_m}{m^2}|\right) \\
&\geq \frac{\pi}{2a}, \text{ for all } m > n \geq N_1. \qquad (3.6)
\end{aligned}
$$

So far we have that:

$$
|(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| \geq |\sqrt{\lambda_n} - \sqrt{\lambda_m}| \geq \epsilon_1, \text{ for all } m > n > \tilde{N}, \qquad (3.7)
$$

where $\epsilon_1 > 0$ is the smallest of

- $\min\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ \tilde{N}+1 \leq n < m \leq N_1 - 1\} > 0$, because λ_n's are all distinct,
- $\inf\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ \tilde{N}+1 \leq n \leq N_1 - 1 < m\} = \sqrt{\lambda_{N_1}} - \sqrt{\lambda_{N_1 - 1}} > 0$, because $0 < \lambda_{\tilde{N}+1} \leq \lambda_n \leq \lambda_{N_1-1} < \lambda_{N_1} \leq \lambda_m < \ldots$,

- $\inf\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ N_1 \le n < m\} \ge \frac{\pi}{2a} > 0$ (see (3.6)).

[*Note:* If $N_1 = \tilde{N} + 1$, then the sets $\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ \tilde{N} + 1 \le n < m \le N_1 - 1\}$ and $\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ \tilde{N} + 1 \le n \le N_1 - 1 < m\}$ are empty, and the corresponding min and inf are to be ignored.]

Because $\lambda_1 < \lambda_2 < \ldots < \lambda_{\tilde{N}} < 0 < \lambda_{\tilde{N}+1} < \ldots$, it follows that:

$$\begin{aligned} |(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| &= |(\pm i\sqrt{-\lambda_n}) - (\pm\sqrt{\lambda_m})| \\ &= \sqrt{(\pm\sqrt{\lambda_m})^2 + (\pm\sqrt{-\lambda_n})^2}, \end{aligned}$$

using $|x + iy| = \sqrt{x^2 + y^2}$, $x, y \in \mathbb{R}$

$$\begin{aligned} &= \sqrt{\lambda_m - \lambda_n} \\ &\ge \sqrt{\lambda_{\tilde{N}+1} - \lambda_{\tilde{N}}}, \text{ for all } 1 \le n \le \tilde{N} < m, \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} |(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| &\ge ||\pm\sqrt{\lambda_n}| - |\pm\sqrt{\lambda_m})|| \\ &= ||i\sqrt{-\lambda_n}| - |i\sqrt{-\lambda_m}|| \\ &= |\sqrt{-\lambda_n} - \sqrt{-\lambda_m}| \text{ for all } 1 \le n < m \le \tilde{N}. \end{aligned} \tag{3.9}$$

Thus,

$$|(\pm\sqrt{\lambda_n}) - (\pm\sqrt{\lambda_m})| \ge \epsilon_2, \text{ for all } m > n \ge 1, \tag{3.10}$$

where $\epsilon_2 > 0$ is the smallest of

- $\min\{|\sqrt{-\lambda_n} - \sqrt{-\lambda_m}|,\ 1 \le n < m \le \tilde{N}\} > 0$ (see (3.9)),
- $\inf\{|(\pm i\sqrt{-\lambda_n}) - (\pm\sqrt{\lambda_m})|,\ 1 \le n \le \tilde{N} < m\} \ge \sqrt{\lambda_{\tilde{N}+1} - \lambda_{\tilde{N}}} > 0$ (see (3.8)),
- $\inf\{|\sqrt{\lambda_n} - \sqrt{\lambda_m}|,\ m > n > \tilde{N}\} \ge \epsilon_1 > 0$ (see (3.7)).

Since $\lambda_1 < \lambda_2 < \ldots < \lambda_{\tilde{N}} < 0 < \lambda_{\tilde{N}+1} < \ldots$, it follows that $\pm\sqrt{\lambda_1}, \ldots,$ $\pm\sqrt{\lambda_{\tilde{N}}}$ are located on the imaginary axis with $\pm\sqrt{\lambda_{\tilde{N}}} = \pm i\sqrt{-\lambda_{\tilde{N}}}$ closest to 0, and $\pm\sqrt{\lambda_{\tilde{N}+1}}$, $\pm\sqrt{\lambda_{\tilde{N}+2}}$, $\ldots$ are located on the real axis with $\pm\sqrt{\lambda_{\tilde{N}+1}}$ closest to 0. Therefore, using also (3.10) we deduce that any two elements of $\{\pm\sqrt{\lambda_n}\}_{n\ge 1} \cup \{0\}$ are at least a distance $\epsilon := \min\{\epsilon_2, \sqrt{-\lambda_{\tilde{N}}}, \sqrt{\lambda_{\tilde{N}+1}}\} > 0$ apart. □

Theorem 1. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of Dirichlet eigenvalues of the canonical Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0,a)$. Then $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$ are the zeros of an entire function of sine type a.*

Proof. Firstly, we shall secure a function whose zeros are $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$, and then prove that this function is an entire function of *sine type a*. The definitions of entire functions of exponential type, sine type π can be found in [9, pages 52, 143]. To obtain the definitions of an entire function of exponential/sine type a we slightly adjust these definitions. See also the paragraph following (2.9) in Section 2.

By the discussion in the sixth paragraph of Section 2 we have that:

$$S(a;q,\lambda_n) = 0, \text{ for all } n \geq 1, \text{ and } S(a;q,\lambda) \text{ has no other zeros.} \tag{3.11}$$

Define the function

$$P(z) := zS(a;q,z^2), \ z \in \mathbb{C}. \tag{3.12}$$

By the discussion in the paragraph following the paragraph of (2.4), $P(z)$ is an entire function of z. From (3.12) and (3.11) it is immediate that $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$ are *all* the zeros of $P(z)$. It remains to show that $P(z)$ is of *sine type a*. The arguments follow.

The elements of $\{\pm\sqrt{\lambda_n}\}_{n\geq 1} \cup \{0\}$ are separated (definition in [9, page 83], reminded in the paragraph following formula (2.9) in Section 2) due to (2.10). The details are given in Lemma 1.

Next, using (2.8) in (3.12), and triangle inequality we write:

$$\begin{aligned} |P(z)| &= |\sin(za) + \int_0^a K(a,t;q)\sin(zt)dt| \\ &\leq |\frac{e^{iza} - e^{-iza}}{2i}| + \int_0^a |K(a,t;q)||\frac{e^{izt} - e^{-izt}}{2i}|dt \\ &\leq Ce^{a|Im(z)|}, \text{ using } |e^{\pm izt}| = e^{\mp tIm(z)} \leq e^{t|Im(z)|} \leq e^{a|Im(z)|}, \end{aligned}$$

for all $t \in [0,a]$

$$\leq Ce^{a|z|}, \text{ for all } z \in \mathbb{C},, \tag{3.13}$$

where $C := (1 + \sqrt{a}||K(a,\cdot;q)||_{L^2})$. Thus, $P(z)$ is a function of exponential type a.

Integrating by parts once in the integral of (2.8), possible because $K(a,\cdot;q) \in H^1(0,a)$ (Theorem 4.15(c) in [6, page 147]), and using the boundary conditions of (2.6), we obtain:

$$P(z)=\sin(za)-\frac{a[q]}{2}\frac{\cos(za)}{z}+\int_0^a K_t(a,t;q)\frac{\cos(zt)}{z}dt,\ \text{for } z\neq 0. \tag{3.14}$$

Hence, for z with $|Im(z)|\geq H$ (where $H>0$ is to be chosen as needed later) formula (3.14) holds and:

$$|P(z)|\geq ||\sin(za)|-|\frac{a[q]}{2}\frac{\cos(za)}{z}-\int_0^a K_t(a,t;q)\frac{\cos(zt)}{z}dt||. \tag{3.15}$$

Also

$$\begin{aligned}|\sin(za)|=|\frac{e^{iza}-e^{-iza}}{2i}|\geq\frac{||e^{iza}|-|e^{-iza}||}{2}&=\frac{e^{a|Im(z)|}-e^{-a|Im(z)|}}{2}\\&\geq e^{a|Im(z)|}\frac{1-e^{-2aH}}{2}\end{aligned} \tag{3.16}$$

and using $|z|\geq|Im(z)|\geq H>0$, and $|e^{\pm izt}|=e^{\mp tIm(z)}\leq e^{t|Im(z)|}\leq e^{a|Im(z)|}$, for all $t\in[0,a]$, we obtain:

$$\begin{aligned}|\frac{a[q]}{2}\frac{\cos(za)}{z}-\int_0^a K_t(a,t;q)\frac{\cos(zt)}{z}dt|\leq&\frac{a|[q]|}{2}\frac{|e^{iza}+e^{-iza}|}{2|Im(z)|}\\&+\int_0^a|K_t(a,t;q)|\frac{|e^{izt}+e^{-izt}|}{2|Im(z)|}dt\\\leq&\frac{\tilde{C}}{H}e^{a|Im(z)|},\end{aligned} \tag{3.17}$$

where $\tilde{C}:=\frac{a|[q]|}{2}+\sqrt{a}||K_t(a,\cdot;q)||_{L^2}$. Since $\lim_{H\to\infty}\frac{1-e^{-2aH}}{2}=\frac{1}{2}>0=\lim_{H\to\infty}\frac{\tilde{C}}{H}$, it is possible to choose $H>0$ big enough such that

$$\frac{1-e^{-2aH}}{2}>\frac{1}{3}>\frac{1}{4}>\frac{\tilde{C}}{H}. \tag{3.18}$$

Using (3.16), (3.17) and (3.18) in (3.15), we obtain:

$$\begin{aligned}|P(z)|\geq&||\sin(za)|-|\frac{a[q]}{2}\frac{\cos(za)}{z}-\int_0^a K_t(a,t;q)\frac{\cos(zt)}{z}dt||\\=&|\sin(za)|-|\frac{a[q]}{2}\frac{\cos(za)}{z}-\int_0^a K_t(a,t;q)\frac{\cos(zt)}{z}dt|\end{aligned}$$

$$\geq e^{a|Im(z)|}\left(\frac{1-e^{-2aH}}{2}-\frac{\tilde{C}}{H}\right)\geq\frac{1}{12}e^{a|Im(z)|},$$

$$\text{for all } |Im(z)|\geq H. \qquad (3.19)$$

With (3.13) and (3.19) the arguments that $P(z)$ is a function of sine type a are complete. □

Lemma 2. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of Dirichlet-Robin eigenvalues of the canonical Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0,a)$ and boundary parameter $\alpha \in \mathbb{R}$. Then the elements of $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$ are separated.*

Proof. From the discussion in the paragraph of (2.11) we infer that λ_n's are all positive, except possible the first few. So, $\lambda_1 < \ldots < \lambda_{\tilde{N}} < 0 < \lambda_{\tilde{N}+1} < \lambda_{\tilde{N}+2} < \ldots$, for some integer $\tilde{N} \geq 0$. Hence, $\sqrt{\lambda_n}$ are all real, except the first $\tilde{N}$ of them which are pure imaginary. Also,

$$\sqrt{\lambda_n} = \frac{(n-\frac{1}{2})\pi}{a} + \frac{2\alpha + a[q]}{(2n-1)\pi} + \frac{c_n}{n^2}, \text{ as } n \geq \tilde{N}+1, \qquad (3.20)$$

for some real sequence $\{c_n\}_{n\geq\tilde{N}+1}$ such that $\{\frac{c_n}{n}\}_{n\geq\tilde{N}+1} \in l_2$. (In this case, $\tilde{r}_n = \mathcal{O}(\frac{1}{n^2}) + \frac{2\pi}{a}\frac{n-\frac{1}{2}}{n}\frac{c_n}{n}$ will generate an l_2 sequence, as in (2.11.)

Similar arguments with those in the proof of Lemma 1 apply. □

Theorem 2. *Let $\{\lambda_n\}_{n\geq 1}$ be the sequence of Dirichlet-Robin eigenvalues of the canonical Sturm-Liouville operator $L^{(q)}$, with $q \in L^2_{\mathbb{R}}(0,a)$ and boundary parameter $\alpha \in \mathbb{R}$. Then $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$ are the zeros of an entire function of sine type a.*

Proof. The steps are as in Theorem 1: first, we look for a function whose zeros are $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$, and then we show that the function is an entire function of sine type a.

By the discussion in the seventh paragraph of Section 2, we have that:

$$\begin{cases} S'(a;q,\lambda_n) + \alpha S(a;q,\lambda_n) = 0, \text{ for all } n \geq 1, \\ \text{and } S'(a;q,\lambda) + \alpha S(a;q,\lambda) \text{ has no other zeros.} \end{cases} \qquad (3.21)$$

This suggests a candidate for the function we are looking for.

Define

$$P_1(z) := S'(a;q,z^2) + \alpha S(a;q,z^2), \ z \in \mathbb{C}. \qquad (3.22)$$

The discussion in the paragraph following the paragraph of (2.4) implies that $P_1(z)$ is an entire function of z, and (3.21) implies that $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$ are *all* the zeros of $P_1(z)$. In what follows we argue that $P_1(z)$ is of *sine type a*.

The elements of $\{\pm\sqrt{\lambda_n}\}_{n\geq 1}$ are separated due to (2.11). The details are given in Lemma 2.

Next, using (2.8) and (2.9) in (3.22), we get:

$$P_1(z) = \cos(za) + \left(\frac{a[q]}{2} + \alpha\right)\frac{\sin(za)}{z} + \int_0^a (K_x(a,t;q) + \alpha K(a,t;q))\frac{\sin(zt)}{z}dt, \ z \in \mathbb{C}. \quad (3.23)$$

Observe that for any $z \in \mathbb{C}$, and any $s \geq 0$:

$$|\cos(zs)| = |\frac{e^{isz} + e^{-isz}}{2}| \leq \frac{|e^{isz}| + |e^{-isz}|}{2} = \frac{e^{-sIm(z)} + e^{sIm(z)}}{2} \leq e^{s|Im(z)|}, \quad (3.24)$$

$$|\sin(zs)| = |\frac{e^{isz} - e^{-isz}}{2i}| \leq \frac{|e^{isz}| + |e^{-isz}|}{2} = \frac{e^{-sIm(z)} + e^{sIm(z)}}{2} \leq e^{s|Im(z)|}, \quad (3.25)$$

$$|\frac{\sin(zs)}{z}| = |\int_0^s \frac{d}{d\tau}\left(\frac{\sin(z\tau)}{z}\right)d\tau| = |\int_0^s \cos(z\tau)d\tau| \leq \int_0^s e^{\tau|Im(z)|}d\tau \leq se^{s|Im(z)|}, \quad (3.26)$$

where in the first inequality of (3.26) we used (3.24) for $s = \tau$, and in the second inequality of (3.26) we used the fact that $e^{\tau|Im(z)|} \leq e^{s|Im(z)|}$, since $0 \leq \tau \leq s$. Due to (3.23), (3.24), and (3.26), one has:

$$|P_1(z)| \leq Ce^{a|Im(z)|} \leq Ce^{a|z|}, \text{ for all } z \in \mathbb{C}, \quad (3.27)$$

where $C := 1 + |\frac{a[q]}{2} + \alpha|a + a\sqrt{a}||K_x(a,\cdot;q) + \alpha K(a,\cdot;q)||_{L^2}$. It is clear from (3.27) that $P_1(z)$ is a function of exponential type a. It follows from (3.23) that for any $z \in \mathbb{C}$:

$$|P_1(z)| \geq ||\cos(za)| - |\left(\frac{a[q]}{2} + \alpha\right)\frac{\sin(za)}{z} + \int_0^a (K_x(a,t;q) + \alpha K(a,t;q))\frac{\sin(zt)}{z}dt|| \quad (3.28)$$

Let $z \in \mathbb{C}$ be such that $|Im(z)| \geq H$, for some constant $H > 0$ to be chosen conveniently later. The following calculations hold:

$$
\begin{aligned}
|\cos(za)| = |\frac{e^{iza}+e^{-iza}}{2}| &\geq \frac{||e^{iza}|-|e^{-iza}||}{2} \\
&= \frac{e^{a|Im(z)|}-e^{-a|Im(z)|}}{2} \geq e^{a|Im(z)|}\frac{1-e^{-2aH}}{2}
\end{aligned} \tag{3.29}
$$

and also using (3.25) and $|z| \geq |Im(z)| \geq H > 0$ we have:

$$
|\left(\frac{a[q]}{2}+\alpha\right)\frac{\sin(za)}{z} + \int_0^a (K_x(a,t;q)+\alpha K(a,t;q))\frac{\sin(zt)}{z}dt| \leq \frac{\tilde{C}}{H}e^{a|Im(z)|}, \tag{3.30}
$$

where $\tilde{C} := |\frac{a[q]}{2}+\alpha| + \sqrt{a}||K_x(a,\cdot;q)+\alpha K(a,\cdot;q)||_{L^2}$. As explained in the proof of Theorem 1 we can choose $H > 0$ big enough such that (3.18) holds, where now $\tilde{C}$ is as introduced right above. With (3.18) and (3.28) - (3.30) we have:

$$
\begin{aligned}
|P_1(z)| \geq &|\cos(za)| - |\left(\frac{a[q]}{2}+\alpha\right)\frac{\sin(za)}{z} \qquad (3.31)\\
&+ \int_0^a (K_x(a,t;q)+\alpha K(a,t;q))\frac{\sin(zt)}{z}dt| \\
\geq & e^{a|Im(z)|}\left(\frac{1-e^{-2aH}}{2} - \frac{\tilde{C}}{H}\right) \\
\geq & \frac{1}{12}e^{a|Im(z)|}, \text{ for all } z \text{ with } |Im(z)| \geq H. \qquad (3.32)
\end{aligned}
$$

With (3.32) and (3.27) the arguments that $P_1(z)$ is a function of sine type a are complete. □

4. Corollaries

The following results are known (e.g. [5]). However, they also follow as a consequence of our main results.

Corollary 1. *Let* $\{\lambda_n\}_{n\geq 1}$ *be as in the hypothesis of Theorem 1. Then* $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ *is a Riesz basis of* $L^2(0,a)$.

Proof. As argued in the proof of Lemma 1, $\sqrt{\lambda_n}$'s are all real, except the first $\tilde{N} \geq 0$ of them, which are pure imaginary.

By our Theorem 1, and Theorem 10 of [9, page 144] (with the adjustment of 'function of sine type a' in place of 'function of sine type π') we conclude that the set $\{e^{i\theta_n t}\}_{n\geq 1}$ is a Riesz basis of $L^2(-a,a)$, where

$$\theta_n := \begin{cases} -\sqrt{\lambda_{-n}}, & \text{if } n=-1,-2,-3\ldots \\ 0, & \text{if } n=0 \\ \sqrt{\lambda_n}, & \text{if } n=1,2,3,\ldots. \end{cases} \tag{4.1}$$

It follows next, by Theorem 9 of [9, page 27], the implication (1) → (3), that

$$\{e^{i\theta_n t}\}_{n=-\infty}^{\infty} \text{ is complete in } L^2(-a,a), \tag{4.2}$$

and there are positive constants A, B such that:

$$A\sum_{n=-M}^{N}|\tilde{c}_n|^2 \leq \|\sum_{n=-M}^{N}\tilde{c}_n e^{i\theta_n t}\|^2_{L^2(-a,a)} \leq B\sum_{n=-M}^{N}|\tilde{c}_n|^2, \tag{4.3}$$

for arbitrary $M, N \in \mathbb{N}$, and arbitrary scalars $\tilde{c}_{-M},\ldots,\tilde{c}_N$.

We shall use (4.2) and (4.3) to show that the hypotheses of statement (3) of Theorem 9 in [9, page 27] are satisfied for the set $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$, and therefore, by the same theorem but this time the implication (3) → (1), we can conclude that $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ is a Riesz basis of $L^2(0,a)$, as needed.

We prove now that the set $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ is complete in $L^2(0,a)$ (see the beginning of page 6 in [9] for the definition of a complete sequence in a Hilbert space). Let $f \in L^2(0,a)$ be such that

$$\int_0^a f(t)\sin(\sqrt{\lambda_n}t)dt = 0, \text{ for all } n \geq 1. \tag{4.4}$$

And let $\tilde{f}(t)$ be the odd extension of $f(t)$ to $[-a,a]$. That is

$$\tilde{f}(t) = \begin{cases} f(t), & \text{if } t \in [0,a] \\ -f(-t), & \text{if } t \in [-a,0). \end{cases} \tag{4.5}$$

Then the following are easily established, because the integrands are odd or even functions of t on $[-a,a]$:

$$\int_{-a}^{a}\tilde{f}(t)e^{i\theta_0 t}dt = \int_{-a}^{a}\tilde{f}(t)dt = 0, \tag{4.6}$$

$$\int_{-a}^{a}\tilde{f}(t)e^{i\theta_n t}dt = 2i\int_0^a f(t)\sin(\theta_n t)dt = 0, \text{ for all } n = \pm 1, \pm 2, \ldots. \tag{4.7}$$

In the first equality of (4.7) we used $e^{i\theta_n t} = \cos(\theta_n t) + i\sin(\theta_n t)$, which is true for θ_n both real and pure imaginary (see [1, formula 2.15, page 38]), and the fact that $\tilde{f}(t)$, $\sin(\theta_n t)$ are odd functions of t, whereas $\cos(\theta_n t)$ is an even function of $t \in [-a,a]$ (and so are their real and imaginary parts), for θ_n both real and pure imaginary. The property of being an odd or even function of t for these sines and cosines follows from the fact that $\sin z$ and $\cos z$ are odd, respectively even functions of $z \in \mathbb{C}$ (see their power series expansion in [1, page 38]). The last equality of (4.7) is due to (4.1) and (4.4).

From (4.6), (4.7), and (4.2) we have that $\tilde{f} = 0$, hence $f = 0$. Thus we showed that:

$$\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1} \text{ is a complete sequence in } L^2(0,a). \tag{4.8}$$

Let $N \in \mathbb{N} - \{0\}$, and let $c_1, \ldots, c_N$ be arbitrary scalars. Define

$$\tilde{c}_n := \begin{cases} -c_{-n}, & \text{if } n = -1,-2,-3,\ldots,-N \\ 0, & \text{if } n = 0 \\ c_n, & \text{if } n = 1,2,3,\ldots,N. \end{cases} \tag{4.9}$$

With (4.9) and (4.1) we have the following calculations ($||\cdot||$ on the left hand side is $||\cdot||_{L^2(-a,a)}$):

$$\begin{aligned} ||\sum_{n=-N}^{N} \tilde{c}_n e^{i\theta_n t}||^2 &= < -\sum_{n=1}^{N} c_n e^{-i\sqrt{\lambda_n}t} + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t}, -\sum_{m=1}^{N} c_m e^{-i\sqrt{\lambda_m}t} \\ &\quad + \sum_{m=1}^{N} c_m e^{i\sqrt{\lambda_m}t} > \\ &= \sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} e^{-i\sqrt{\lambda_n}t} e^{i\sqrt{\lambda_m}t} dt \\ &\quad - \sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} e^{i\sqrt{\lambda_n}t} e^{i\sqrt{\lambda_m}t} dt \\ &\quad - \sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} e^{-i\sqrt{\lambda_n}t} e^{-i\sqrt{\lambda_m}t} dt \\ &\quad + \sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} e^{i\sqrt{\lambda_n}t} e^{-i\sqrt{\lambda_m}t} dt \\ &= \sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} \left(e^{-i\sqrt{\lambda_n}t} - e^{i\sqrt{\lambda_n}t}\right)\left(e^{i\sqrt{\lambda_m}t} - e^{-i\sqrt{\lambda_m}t}\right) dt \end{aligned}$$

$$=4\sum_{n,m=1}^{N} c_n\bar{c}_m \int_{-a}^{a} \sin(\sqrt{\lambda_n}t)\sin(\sqrt{\lambda_m}t)dt$$

$$=8||\sum_{n=1}^{N} c_n \sin(\sqrt{\lambda_n}t)||^2_{L^2(0,a)}. \tag{4.10}$$

The third equality in (4.10) was obtained by grouping together the first and the third summation terms, and respectively the second and fourth summation terms, and factoring out. The fourth equality in (4.10) is due to $\frac{e^{iz}-e^{-iz}}{2i} = \sin z$, for all $z \in \mathbb{C}$, and the last equality in (4.10) is due to the fact that the integrand is an even function of t on $[-a, a]$.

Using (4.10) and (4.9) in (4.3) with $M = N$, one has:

$$\frac{A}{4}\sum_{n=1}^{N}|c_n|^2 \le ||\sum_{n=1}^{N} c_n \sin(\sqrt{\lambda_n}t)||^2_{L^2(0,a)} \le \frac{B}{4}\sum_{n=1}^{N}|c_n|^2. \tag{4.11}$$

With (4.8) and (4.11), and the implication (3) → (1) of Theorem 9 in [9, page 27], it follows that $\{\sin(\sqrt{\lambda_n}t)\}_{n\ge1}$ is a Riesz basis in $L^2(0, a)$. □

Corollary 2. *Let $\{\lambda_n\}_{n\ge1}$ be as in the hypothesis of Theorem 2. Then $\{\sin(\sqrt{\lambda_n}t)\}_{n\ge1}$ is a Riesz basis of $L^2(0, a)$.*

Proof. As argued in the proof of Lemma 2, $\sqrt{\lambda_n}$'s are all real except the first $\tilde{N} \ge 0$ of them which are pure imaginary.

Applying again Theorem 10 of [9, page 144] to the result of Theorem 2 we have:

$$\{e^{\pm i\sqrt{\lambda_n}t}\}_{n\ge1} \text{ is a Riesz basis of } L^2(-a, a). \tag{4.12}$$

From (4.12) and the implication (1) → (3) of Theorem 9 in [9, page 27] we infer that:

$$\{e^{\pm i\sqrt{\lambda_n}t}\}_{n\ge1} \text{ is a complete set of } L^2(-a, a), \tag{4.13}$$

and there are constants $A, B > 0$, such that for arbitrary $M, N \in \mathbb{N}$, and arbitrary scalars $\tilde{c}_{-M}, \ldots, \tilde{c}_N$:

$$A\sum_{n=-M}^{N}|\tilde{c}_n|^2 \le ||\sum_{n=-M}^{N} \tilde{c}_n e^{i\theta_n t}||^2_{L^2(-a,a)} \le B\sum_{n=-M}^{N}|\tilde{c}_n|^2, \tag{4.14}$$

where

$$\theta_n := \begin{cases} \sqrt{\lambda_n}, & \text{if } n = 1, 2, 3, \ldots \\ -\sqrt{\lambda_{-n+1}}, & \text{if } n = 0, -1, -2, \ldots. \end{cases} \tag{4.15}$$

To show completeness of $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ in $L^2(0,a)$, let $f \in L^2(0,a)$ be such that

$$\int_0^a f(t)\sin(\sqrt{\lambda_n}t)dt = 0, \text{ for all } n \geq 1. \tag{4.16}$$

Let $\tilde{f}(t)$ be as in (4.5). Then it follows with the same arguments as for (4.7) that:

$$\int_{-a}^a \tilde{f}(t)e^{i\theta_n t}dt = 2i\int_0^a f(t)\sin(\theta_n t)dt = 0, \text{ for all } n \in \mathbb{Z},$$

which due to (4.13) produces $\tilde{f} = 0$, and therefore $f = 0$. Thus $\{\sin(\sqrt{\lambda_n}t)\}_{n\geq 1}$ is complete in $L^2(0,a)$.

Next choose arbitrarily $N \in \mathbb{N} - \{0\}$, and the scalars $c_1, \ldots, c_N$. Take $M = N-1$ and introduce

$$\tilde{c}_n := \begin{cases} c_n, & \text{if } n = 1,2,3,\ldots,N \\ -c_{-n+1}, & \text{if } n = 0,-1,-2,\ldots,-M. \end{cases} \tag{4.17}$$

Then one writes:

$$\sum_{n=-M}^{N} |\tilde{c}_n|^2 = \sum_{n=-M}^{0} |-c_{-n+1}|^2 + \sum_{n=1}^{N} |c_n|^2 = \sum_{m=1}^{M+1=N} |-c_m|^2 + \sum_{n=1}^{N} |c_n|^2 = 2\sum_{n=1}^{N} |c_n|^2, \tag{4.18}$$

and, since $M+1 = N$ and $||\cdot||$ on the left hand side being $||\cdot||_{L^2(-a,a)}$ one obtains:

$$\begin{aligned}
||\sum_{n=-M}^{N} \tilde{c}_n e^{i\theta_n t}||^2 &= < \sum_{n=-M}^{0} (-c_{-n+1})e^{-i\sqrt{\lambda_{-n+1}}t} + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t}, \\
&\quad \sum_{n=-M}^{0} (-c_{-n+1})e^{-i\sqrt{\lambda_{-n+1}}t} + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t} > \\
&= < \sum_{m=1}^{M+1} (-c_m)e^{-i\sqrt{\lambda_m}t} + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t}, \sum_{m=1}^{M+1} (-c_m)e^{-i\sqrt{\lambda_m}t} \\
&\quad + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t} > \\
&= < -\sum_{n=1}^{N} c_n e^{-i\sqrt{\lambda_n}t} + \sum_{n=1}^{N} c_n e^{i\sqrt{\lambda_n}t}, -\sum_{m=1}^{N} c_m e^{-i\sqrt{\lambda_m}t} \\
&\quad + \sum_{m=1}^{N} c_m e^{i\sqrt{\lambda_m}t} >
\end{aligned}$$

$$=4 \sum_{n,m=1}^{N} c_n \bar{c}_m \int_{-a}^{a} \sin(\sqrt{\lambda_n} t) \sin(\sqrt{\lambda_m} t) dt$$

$$=8 || \sum_{n=1}^{N} c_n \sin(\sqrt{\lambda_n} t) ||^2_{L^2(0,a)}. \tag{4.19}$$

The second identity in (4.19) is due to letting $m = -n + 1$, the third identity in (4.19) is simply due to a change of the summation indices, and the fourth identity in (4.19) follows identically with the calculations in (4.10).

With (4.18), (4.19) in (4.14) with $M = N-1$ one obtains a formula identical with (4.11).

These show that the hypotheses of statement (3) of Theorem 9 in [9, page 27] are satisfied, and so $\{\sin(\sqrt{\lambda_n} t)\}_{n\geq 1}$ is a Riesz basis of $L^2(0, a)$, by implication (3) → (1) of this theorem. □

5. A Cautionary Note

One might be tempted to think that the fact that $\{\sin(\sqrt{\lambda_n} t)\}_{n\geq 1}$ is a Riesz basis of $L^2(0, a)$ follows directly from (2.5) and the definition of a Riesz basis in a Hilbert space H(i.e. the image of an orthonormal basis of H under a boundedly invertible linear operator on H). However, this is not the case. Our arguments are as follows:

- The set of functions $\{S(\cdot; q, \lambda_n)\}_{n\geq 1}$ is the set of Dirichlet, or respectively the Dirichlet-Robin eigenfunctions, as explained in the sixth and seventh paragraphs of Section 2.

- We know from the direct theory of Sturm-Liouville problems (see for instance the beginning of Section 4.3 and Lemma 4.7(b) in [6, pages 135-136], and the counterpart for the Dirichlet-Robin boundary conditions) that the normalized eigenfunctions form an orthonormal basis. That is,

$$\{\frac{S(\cdot; q, \lambda_n)}{||S(\cdot; q, \lambda_n)||_{L^2(0,a)}}\}_{n\geq 1}$$

  is an orthonormal basis of $L^2(0, a)$.

- Next, we observe that equation (2.5) can be written equivalently in the operator form:

$$S(\cdot; q, \lambda_n) = T[\frac{\sin(\sqrt{\lambda_n} \cdot)}{\sqrt{\lambda_n}}], \tag{5.1}$$

if we define the operator $v \in L^2(0,a) \to T[v] \in L^2(0,a)$ by:

$$T[v](x) := v(x) + \int_0^x K(x,t;q)v(t)dt, \ 0 < x < a,$$

with $K(x,t;q)$ as described in Section 2. This operator T is a boundedly invertible linear operator on $L^2(0,a)$, as a second kind Volterra integral operator (see Example A.30(a) in [6, page 229]).

- Now equation (5.1) and the discussion above about the operator T clear out any doubt: while the set $\{\sin(\sqrt{\lambda_n}\cdot)\}_{n\geq 1}$ is the image under a boundedly invertible linear operator (T^{-1}) of the set $\{\sqrt{\lambda_n}S(\cdot;q,\lambda_n)\}_{n\geq 1}$, one cannot conclude from this that $\{\sin(\sqrt{\lambda_n}\cdot)\}_{n\geq 1}$ is a Riesz basis of $L^2(0,a)$, because $\{\sqrt{\lambda_n}S(\cdot;q,\lambda_n)\}_{n\geq 1}$ was not proved to be an orthonormal basis of $L^2(0,a)$. We only know that $\{\frac{S(\cdot;q,\lambda_n)}{||S(\cdot;q,\lambda_n)||_{L^2(0,a)}}\}_{n\geq 1}$ is an orthonormal basis of $L^2(0,a)$.

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