CONTRA $\alpha\psi$-CONTINUOUS FUNCTIONS

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Abstract: The concept of $\alpha\psi$-closed sets in a topological space are introduced by R. Devi et al (see [2]). In this paper, we introduce the notion of contra $\alpha\psi$-continuous functions utilizing $\alpha\psi$-open sets and study some of the applications of this function.

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1. Introduction

In 1996, Dontchev (see [3]) introduced the notions of contra continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ is contra continuous if the pre image of every open set of $Y$ is closed in $X$. Also a new class of function called contra semi-continuous function is introduced and investigated by Dontchev and Noiri (see [4]). The notions of contra super continuous, contra pre continuous and contra $\alpha$-continuous functions are introduced by Jafari and Noiri (see [6, 7]). Nasef (see [9]) has introduced and studied contra $\gamma$-continuous function. In this paper, we introduce the concept of contra $\alpha\psi$-continuous functions via the notion of $\alpha\psi$-open set and study some of the applications of this function.

All through this paper, $(X, \tau)$ and $(Y, \sigma)$ stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of $A$ and the interior of $A$ will be denoted by $cl(A)$ and $int(A)$ respectively. $A$ is regular open if $A = int(cl(A))$ and $A$ is regular closed if its complement is regular open; equivalently $A$ is regular closed if $A = cl(int(A))$, (see [11]). Let $(X, \tau)$ be a space and let $A$ be a subset of $X$. $A$ is called $\psi$-closed set (see [2]) if $\psi(cl(A)) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\alpha$-open set of $(X, \tau)$. The complement of an $\psi$-closed set is called $\psi$-open. We set $\psi\alpha(O)(X, x) = \{ U : x \in U \text{ and } U \in \tau_\psi \}$, where $\tau_\psi$ denotes the family of all $\psi$-open subsets of a space $(X, \tau)$. The $\psi$-closure and $\psi$-interior, that can be defined in a manner to $cl(A)$ and $int(A)$ respectively, will be denoted by $cl_\psi(A)$ and $int_\psi(A)$ respectively.

2. Contra $\alpha\psi$-Continuous Functions
Theorem 4. For a function \( f : (X, \tau) \to (Y, \sigma) \) the following conditions are equivalent:

(1) \( f \) is contra \( \alpha\psi \)-continuous;

(2) for every closed subset \( F \) of \( Y \), \( f^{-1}(F) \in \alpha\psi O(X) \);

(3) for each \( x \in X \) and each \( F \in C(Y, f(x)) \), there exists \( U \in \alpha\psi O(X, x) \) such that \( f(U) \subseteq F \);

(4) \( f(cl_{\alpha\psi}(A)) \subseteq ker(f(A)) \) for every subset \( A \) of \( X \);

(5) \( cl_{\alpha\psi}(f^{-1}(B)) \subseteq f^{-1}(ker(B)) \) for every subset \( B \) of \( Y \).

Proof. The implications (1) \( \iff \) (2) and (2) \( \implies \) (3) are obvious.

(3) \( \implies \) (2) Let \( F \) be any closed set of \( Y \) and \( x \in f^{-1}(F) \). Then \( f(x) \in F \) and there exists \( U_x \in \alpha\psi O(X, x) \) such that \( f(U_x) \subseteq F \). Therefore, we obtain \( f^{-1}(F) = \cup \{ U_x/x \in f^{-1}(F) \} \) and \( f^{-1}(F) \) is \( \alpha\psi \)-open, since \( \tau_{\alpha\psi} \) is a topological space.

(2) \( \implies \) (4) Let \( A \) be any subset of \( X \). Suppose that \( y \notin ker(f(A)) \). Then by Lemma 3., there exists \( F \in C(Y, f(x)) \) such that \( f(A) \cap F = \emptyset \). Thus, we have \( A \cap f^{-1}(F) = \emptyset \). Therefore, we obtain \( f(cl_{\alpha\psi}(A)) \cap F = \emptyset \) and \( y \notin f(cl_{\alpha\psi}(A)) \). This implies that \( f(cl_{\alpha\psi}(A)) \subseteq ker(f(A)) \).

(4) \( \implies \) (5) Let \( B \) be any subset of \( Y \). By (4) and Lemma 3., we have \( f(cl_{\alpha\psi}(f^{-1}(B))) \subseteq ker(f(f^{-1}(B))) \subseteq ker(B) \) thus \( cl_{\alpha\psi}(f^{-1}(B)) \subseteq f^{-1}(ker(B)) \).

(5) \( \implies \) (1) Let \( V \) be any open set of \( Y \). Then, by Lemma 3., we have \( cl_{\alpha\psi}(f^{-1}(V)) \subseteq f^{-1}(ker(V)) = f^{-1}(V) \) and \( cl_{\alpha\psi}(f^{-1}(V)) = f^{-1}(V) \). This shows that \( f^{-1}(V) \) is \( \alpha\psi \)-closed in \( X \).

Corollary 5. If a function \( f : X \to Y \) is contra \( \alpha\psi \)-continuous and \( Y \) is regular, then \( f \) is continuous.

Remark 6. The converse of Corollary 5 is not true. The following example shows that continuity does not necessarily imply contra \( \alpha\psi \)-continuity even if the range is regular.

Example 7. The identity function on the real line with the usual topology is continuous and hence \( \alpha\psi \)-continuous. The inverse image of \((0,1)\) is not \( \alpha\psi \)-closed and consequently the function is not contra \( \alpha\psi \)-continuous.
Theorem 8. If a function $f : X \to Y$ is contra $\alpha\psi$-continuous and $Y$ is regular, then $f$ is $\alpha\psi$-continuous.

Proof. Let $x$ be an arbitrary point of $X$ and let $V$ be an open set of $Y$ containing $f(x)$; since $Y$ is regular, there exists an open set $W$ in $Y$ containing $f(x)$ such that $cl(W) \subseteq V$. Since $f$ is contra $\alpha\psi$-continuous, so by Theorem 4. (3) there exists $U \in \alpha\psi O(X,x)$ such that $f(U) \subseteq cl(W)$. Then $f(U) \subseteq cl(W) \subseteq V$. Hence, $f$ is $\alpha\psi$-continuous.

Definition 9. A space $(X, \tau)$ is said to be $\alpha\psi$-space (resp., locally $\alpha\psi$-indiscrete) if every $\alpha\psi$-open set is open (resp. closed) in $X$.

For any space $(X, \tau)$, we have $\tau \subseteq \tau_{\alpha\psi}$. So the following results follows immediately.

Theorem 10. A function $f : (X, \tau) \to (Y, \sigma)$ is contra $\alpha\psi$-continuous if and only if $f : (X, \tau_{\alpha\psi}) \to (Y, \sigma)$ is contra $\alpha\psi$-continuous.

Theorem 11. If a function $f : X \to Y$ is contra $\alpha\psi$-continuous and $X$ is $\alpha\psi$-space, then $f$ is contra-continuous.

Theorem 12. Let $X$ be locally $\alpha\psi$-indiscrete. If a function $f : X \to Y$ is contra $\alpha\psi$-continuous, then $f$ is continuous.

Definition 13. A function $f : X \to Y$ is called almost $\alpha\psi$-continuous if for each $x \in X$ and each open set $V$ of $Y$ containing $f(x)$, there exists $U \in \alpha\psi O(X,x)$ such that $f(U) \subseteq int_{\alpha\psi}(cl(V))$.

Definition 14. A function $f : X \to Y$ is said to be pre $\alpha\psi$-open if the image of each $\alpha\psi$-open set is $\alpha\psi$-open.

Theorem 15. If a function $f : X \to Y$ is a pre $\alpha\psi$-open and contra $\alpha\psi$-continuous, then $f$ is almost $\alpha\psi$-continuous.

Proof. Let $x$ be any arbitrary point of $X$ and $V$ be an open set containing $f(x)$. Since $f$ is contra $\alpha\psi$-continuous, then by Theorem 4.(3) there exists $U \in \alpha\psi O(X,x)$ such that $f(U) \subseteq cl(V)$. Since $f$ is pre $\alpha\psi$-open, $f(U)$ is $\alpha\psi$-open in $Y$. Therefore, $f(U) = int_{\alpha\psi} f(U) \subseteq int_{\alpha\psi}(cl(f(U))) \subseteq int_{\alpha\psi}(cl(V))$. This shows that $f$ is almost $\alpha\psi$-continuous.

Definition 16. A function $f : X \to Y$ is said to be almost weakly $\alpha\psi$-continuous if for each $x \in X$ and each open set $V$ of $f(x)$, there exists $U \in \alpha\psi O(X,x)$ such that $f(U) \subseteq cl(V)$.

Theorem 17. If a function $f : X \to Y$ is contra $\alpha\psi$-continuous, then $f$ is almost weakly $\alpha\psi$-continuous.
Proof. Let $V$ be any open set of $Y$. Since $\text{cl}(V)$ is closed in $Y$, by Theorem 4. (3) $f^{-1}(\text{cl}(V))$ is $\alpha\psi$-continuous in $X$ and set $U = f^{-1}(\text{cl}(V))$, we have $f(U) \subseteq \text{cl}(V)$. This shows that $f$ is almost $\alpha\psi$-continuous. 

Since the family of all $\alpha\psi$-open subsets of a space $(X, \tau)$, denoted by $\tau_{\alpha\psi}$, forms a topology on $X$ finer than $\tau$, then the $\alpha\psi$-frontier of $A$, where $A \subseteq X$, is denoted by $\text{Fr}_{\alpha\psi}(A) = \text{cl}_{\alpha\psi}(A) \cap \text{cl}_{\alpha\psi}(X - A)$.

**Theorem 18.** The set of all points of $x$ of $X$ at which $f : X \to Y$ is not contra $\alpha\psi$-continuous is identical with the union of the $\alpha\psi$-frontier of the inverse image of closed sets of $Y$ containing $f(x)$.

Proof. Suppose $f$ is not contra $\alpha\psi$-continuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \phi$ for every $U \in \alpha\psi O(X, x)$ by Theorem 4. This implies that $U \cap f^{-1}(Y - F) \neq \phi$. Therefore, we have $x \in \text{cl}_{\alpha\psi}(f^{-1}(Y - F)) = \text{cl}_{\alpha\psi}(X - f^{-1}(F))$. However, since $x \in f^{-1}(F) \subseteq \text{cl}_{\alpha\psi}(f^{-1}(F))$, thus $x \in \text{cl}_{\alpha\psi}(f^{-1}(F)) \cap \text{cl}_{\alpha\psi}(f^{-1}(Y - F))$. Therefore, we obtain $x \in \text{Fr}_{\alpha\psi}(f^{-1}(F))$. Suppose that $x \in \text{Fr}_{\alpha\psi}(f^{-1}(F))$ for some $F \in C(Y, f(x))$ and $f$ is contra $\alpha\psi$-continuous at $x$, then there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq V$. Therefore, we have $x \in U \subseteq f^{-1}(F)$ and hence $x \in \text{int}_{\alpha\psi}(f^{-1}(F)) \subseteq X - \text{Fr}_{\alpha\psi}(f^{-1}(F))$. This is a contradiction. This means that $f$ is not contra $\alpha\psi$-continuous.

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of $f$ and is denoted by $\text{Gr}(f)$.

**Theorem 19.** Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of $f$. If $g$ is contra $\alpha\psi$-continuous, then $f$ is contra $\alpha\psi$-continuous.

Proof. Let $U$ be an open set in $Y$, then $X \times Y$ is an open set in $X \times Y$. Since $g$ is contra $\alpha\psi$-continuous, it follows that $f^{-1}(U) = g^{-1}(X \times Y)$ is an $\alpha\psi$-closed set in $X$. Thus, $f$ is contra $\alpha\psi$-continuous.

**Definition 20.** The graph $\text{Gr}(f)$ of a function $f : X \to Y$ is said to be contra $\alpha\psi$-closed if for each $(x, y) \in (X \times Y) - \text{Gr}(f)$, there exists $U \in \alpha\psi O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap \text{Gr}(f) = \phi$.

**Lemma 21.** The graph $\text{Gr}(f)$ of a function $f : X \to Y$ is contra $\alpha\psi$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - \text{Gr}(f)$, there exists $U \in \alpha\psi(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \phi$. 

Theorem 22. If $f : X \to Y$ is contra $\alpha\psi$-continuous and $Y$ is Urysohn, then $Gr(f)$ is contra $\alpha\psi$-closed in the product space $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - Gr(f)$. Then $y \neq f(x)$ and there exists open sets $H_1, H_2$ such that $f(x) \in H_1$, $y \in H_2$ and $cl(H_1) \cap cl(H_2) = \phi$. From hypothesis, there exists $V \in \alpha\psi O(X, x)$ such that $f(V) \subset cl(H_1)$. Therefore, we obtain $f(V) \cap cl(H_2) = \phi$. This shows that $Gr(f)$ is contra $\alpha\psi$-closed.

Theorem 23. If $f : X \to Y$ is contra $\alpha\psi$-continuous and $Y$ is $T_1$, then $G(f)$ is contra $\alpha\psi$-closed in $X \times Y$.

Proof. Let $(x, y) \in (X \times Y) - Gr(f)$. Then $f(x) \neq y$ and there exists an open set $V$ of $Y$ such that $f(x) \in V$ and $y \notin V$. Since $f$ is contra $\alpha\psi$-continuous there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subset V$. Therefore, we have $f(U) \cap (Y - V) = \phi$ and $Y - V \in C(Y, y)$. This shows that $G(f)$ is contra $\alpha\psi$-continuous in $X \times Y$.

Theorem 24. Let $(X_\lambda : \lambda \in \Lambda)$ be any family of topological spaces. If $f : X \to \prod X_\lambda$ is a contra $\alpha\psi$-continuous function. Then $P_{r\lambda} \circ f : X \to X_\lambda$ is contra $\alpha\psi$-continuous for each $\lambda \in \Lambda$, where $P_{r\lambda}$ is the projection of $\prod X_\lambda$ onto $X_\lambda$.

Proof. We shall consider a fixed $\lambda \in \Lambda$. Suppose $U_\lambda$ is an arbitrary open set in $X_\lambda$. Then $P_{r\lambda}^{-1}(U_\lambda)$ is open in $\prod X_\lambda$. Since $f$ is contra $\alpha\psi$-continuous, we have by definition $f^{-1}(P_{r\lambda}^{-1}(U_\lambda)) = (P_{r\lambda} \circ f)^{-1}(U_\lambda)$ is $\alpha\psi$-continuous in $X$. Therefore $P_{r\lambda} \circ f$ is contra $\alpha\psi$-continuous.

Theorem 25. If $f : X \to Y$ is a contra $\alpha\psi$-continuous function and $g : Y \to Z$ is a continuous function, then $g \circ f : X \to Z$ is contra $\alpha\psi$-continuous if and only if $g$ is contra $\alpha\psi$-continuous.

Proof. Let $g \circ f : X \to Z$ is contra $\alpha\psi$-continuous and let $F$ be a closed subset of $Z$. Then $(g \circ f)^{-1}(F)$ is a $\alpha\psi$-open of $X$. That is $f^{-1}(g^{-1}(F))$ is $\alpha\psi$-open subset of $Y$. So, $g^{-1}(F)$ is a $\alpha\psi$-open in $Y$. Hence $g$ is contra $\alpha\psi$-continuous.
3. Applications

Definition 26. (a) $\alpha\psi$-normal if each pair of non-empty disjoint closed sets can be separated by disjoint $\alpha\psi$-open sets,

(b) ultranormal (see [10]) if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 27. If $f : X \rightarrow Y$ is a contra $\alpha\psi$-continuous, closed injection and $Y$ is ultranormal, then $X$ is $\alpha\psi$-normal.

Proof. Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since $f$ is closed injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is ultranormal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets $V_1$ and $V_2$, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in \alpha\psi O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus $X$ is $\alpha\psi$-normal.

Definition 28. A topological space $X$ is said to be $\alpha\psi$-connected if $X$ is not the union of two disjoint non-empty $\alpha\psi$-open subsets of $X$.

Theorem 29. A contra $\alpha\psi$-continuous image of a $\alpha\psi$-connected space is connected.

Proof. Let $F_1$ and $F_2$ be disjoint closed subsets of $X$. Since $f$ is closed injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of $Y$. Since $Y$ is ultranormal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets $V_1$ and $V_2$, respectively. Hence $F_i \subset f^{-1}(V_i)$, $f^{-1}(V_i) \in \alpha\psi O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus $X$ is $\alpha\psi$-normal.

Theorem 30. Let $X$ be $\alpha\psi$-connected and $Y$ be $T_1$. If $f : X \rightarrow Y$ is contra $\alpha\psi$-continuous, then $f$ is constant.

Proof. Since $Y$ is $T_1$ space, $v = \{f^{-1}(y) : y \in Y\}$ is a disjoint $\alpha\psi$-open partition of $X$. If $|v| \geq 2$, then $X$ is the union of two non-empty $\alpha\psi$-open sets. Since $X$ is $\alpha\psi$-connected, $|v| = 1$. Therefore, $f$ is constant.

Theorem 31. If $f : X \rightarrow Y$ is a contra $\alpha\psi$-continuous function from a $\alpha\psi$-connected space $X$ onto any space $Y$, then $Y$ is not a discrete space.
Proof. Suppose that $Y$ is discrete. Let $A$ be a proper non-empty open and closed subset of $Y$. Then $f^{-1}(A)$ is a proper nonempty $\alpha\psi$-clopen subset of $X$, which is a contradiction to the fact $X$ is $\alpha\psi$-connected.

**Definition 32.** A space $X$ is said to be $\alpha\psi$-$T_2$ if for each pair of distinct points $x$ and $y$ in $X$, there exists $U \in \alpha\psi O(X, x)$ and $V \in \alpha\psi O(Y, y)$ such that $U \cap V = \phi$.

**Theorem 33.** Let $X$ and $Y$ be topological spaces. If

1. for each pair of distinct points $x$ and $y$ in $X$ there exists a function $f$ of $X$ into $Y$ such that $f(x) \neq f(y)$,
2. $Y$ is an Urysohn space,
3. $f$ is contra $\alpha\psi$-continuous at $x$ and $y$, then $X$ is $\alpha\psi$-$T_2$.

Proof. Let $x$ and $y$ be any distinct points in $X$. Then, there exists a Urysohn space $Y$ and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and $f$ is contra $\alpha\psi$-continuous at $x$ and $y$. Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since $Y$ is Urysohn space, there exists open sets $V$ and $W$ containing $a$ and $b$, respectively, such that $cl(V) \cap cl(W) = \phi$. Since $f$ is contra $\alpha\psi$-continuous at $x$ and $y$, there exist $\alpha\psi$-open sets $A$ and $B$ containing $a$ and $b$, respectively, such that $f(A) \subseteq cl(V)$ and $f(B) \subseteq cl(W)$. Then $f(A) \cap f(B) = \phi$, so $A \cap B = \phi$. Hence, $X$ is $\alpha\psi$-$T_2$.

**Corollary 34.** Let $f : X \to Y$ be contra $\alpha\psi$-continuous injection. If $Y$ is an Urysohn space, then $X$ is $\alpha\psi$-$T_2$.

**References**


