International Journal of Pure and Applied Mathematics
Volume 71 No. 3 2011, 427-434

CONTRA $\alpha\psi$-CONTINUOUS FUNCTIONS

Y.K. Kim[1] §, R. Devi[2]

[1]Department of Mathematics
Myongji University
Kyunggi 449-728, SOUTH KOREA

[2]Department of Mathematics
Kongunadu Arts and Science College
Coimabtore, 641029, Tamil Nadu, INDIA

Abstract: The concept of $\alpha\psi$-*closed* sets in a topological space are introdued by R. Devi et al (see [2]). In this paper, we introduce the notion of contra $\alpha\psi$-continuous functions utilizing $\alpha\psi$-*open* sets and study some of the applications of this function.

AMS Subject Classification: 54C08
Key Words: $\alpha\psi$-*closed* sets, contra $\alpha\psi$-*continuous*, $\alpha\psi$-normal

1. Introduction

In 1996, Dontchev (see [3]) introduced the notions of contra continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ is contra continuous if the pre image of every *open* set of Y is *closed* in X. Also a new class of function called contra semi-*continuous* function is introduced and investigated by Dontchev and Noiri (see [4]). The notions of contra super continuous, contra pre continuous and contra α-continuous functions are introduced by Jafari and Noiri (see [6, 7]). Nasef (see [9]) has introduced and studied contra γ-continuous function. In this paper, we introduce the concept of contra $\alpha\psi$-continuous functions via the notion of $\alpha\psi$-*open* set and study some of the applications of this function.

All through this paper, (X, τ) and (Y, σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $cl(A)$ and $int(A)$ respectively. A is regular *open* if $A = int(cl(A))$ and A is regular *closed* if its complement is regular *open*; equivalently A is regular *closed* if $A = cl(int(A))$, (see [11]). Let (X, τ) be a space and let A be a subset of X. A is called $\alpha\psi$-*closed* set (see [2]) if $\psi cl(A) \subseteq U$ whenever $A \subseteq U$ and U is α-*open* set of (X, τ). The complement of an $\alpha\psi$-*closed* set is called $\alpha\psi$-*open*. We set $\alpha\psi O(X, x) = \{U : x \in U \text{ and } U \in \tau_{\alpha\psi}\}$, where $\tau_{\alpha\psi}$ denotes the family of all $\alpha\psi$-*open* subsets of a space (X, τ). The $\alpha\psi$-closure and $\alpha\psi$-*interior*, that can be defined in a manner to $cl(A)$ and $int(A)$ respectively, will be denoted by $cl_{\alpha\psi}(A)$ and $int_{\alpha\psi}(A)$, respectively.

2. Contra $\alpha\psi$-Continuous Functions

Theorem 4. *For a function $f : (X, \tau) \to (Y, \sigma)$ the following conditions are equivalent:*

(1) f is contra $\alpha\psi$-continuous;

(2) for every closed subset F of Y, $f^{-1}(F) \in \alpha\psi O(X)$;

(3) for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq F$;

(4) $f(cl_{\alpha\psi}(A)) \subseteq ker(f(A))$ for every subset A of X;

(5) $cl_{\alpha\psi}(f^{-1}(B)) \subseteq f^{-1}(ker(B))$ for every subset B of Y.

Proof. The implications (1) $\Leftrightarrow$ (2) and (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (2) Let F be any closed set of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \alpha\psi O(X, x)$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \cup\{U_x / x \in f^{-1}(F)\}$ and $f^{-1}(F)$ is *$\alpha\psi$-open*, since $\tau_{\alpha\psi}$ is a topological space.

(2) $\Rightarrow$ (4) Let A be any subset of X. Suppose that $y \notin ker(f(A))$. Then by Lemma 3., there exists $F \in C(Y, f(x))$ such that $f(A) \cap F = \phi$. Thus, we have $A \cap f^{-1}(F) = \phi$. Therefore, we obtain $f(cl_{\alpha\psi}(A)) \cap F = \phi$ and $y \notin f(cl_{\alpha\psi}(A))$. This implies that $f(cl_{\alpha\psi}(A)) \subseteq ker(f(A))$.

(4) $\Rightarrow$ (5) Let B be any subset of Y. By (4) and Lemma 3., we have $f(cl_{\alpha\psi}(f^{-1}(B))) \subseteq ker(f(f^{-1}(B))) \subseteq ker(B)$ thus $cl_{\alpha\psi}(f^{-1}(B)) \subseteq f^{-1}(ker(B))$.

(5) $\Rightarrow$ (1) Let V be any *open* set of Y. Then, by Lemma 3., we have $cl_{\alpha\psi}(f^{-1}(V)) \subseteq f^{-1}(ker(V)) = f^{-1}(V)$ and $cl_{\alpha\psi}(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is *$\alpha\psi$-closed* in X. □

Corollary 5. *If a function $f : X \to Y$ is contra $\alpha\psi$-continuous and Y is regular, then f is continuous.*

Remark 6. The converse of Corollary 5 is not true. The following example shows that continuity does not necessarily imply contra $\alpha\psi$-continuity even if the range is regular.

Example 7. The identity function on the real line with the usual topology is continuous and hence *$\alpha\psi$-continuous*. The inverse image of $(0, 1)$ is not *$\alpha\psi$-closed* and consequently the function is not contra *$\alpha\psi$-continuous*.

Theorem 8. *If a function $f : X \to Y$ is contra $\alpha\psi$-continuous and Y is regular, then f is $\alpha\psi$-continuous.*

Proof. Let x be an arbitrary point of X and let V be an *open* set of Y containing $f(x)$; since Y is regular, there exists an *open* set W in Y containing $f(x)$ such that $cl(W) \subseteq V$. Since f is contra *$\alpha\psi$-continuous*, so by Theorem 4. (3) there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq cl(W)$. Then $f(U) \subseteq cl(W) \subseteq V$. Hence, f is *$\alpha\psi$-continuous.* □

Definition 9. A space (X, τ) is said to be $\alpha\psi$-space (resp., locally $\alpha\psi$-indiscrete) if every *$\alpha\psi$-open* set is *open* (resp. *closed*) in X.

For any space (X, τ), we have $\tau \subseteq \tau_{\alpha\psi}$. So the following results follows immediately.

Theorem 10. *A function $f : (X, \tau) \to (Y, \sigma)$ is contra $\alpha\psi$-continuous if and only if $f : (X, \tau_{\alpha\psi}) \to (Y, \sigma)$ is contra $\alpha\psi$-continuous.*

Theorem 11. *If a function $f : X \to Y$ is contra $\alpha\psi$-continuous and X is $\alpha\psi$-space, then f is contra-continuous.*

Theorem 12. *Let X be locally $\alpha\psi$-indiscrete. If a function $f : X \to Y$ is contra $\alpha\psi$-continuous, then f is continuous.*

Definition 13. A function $f : X \to Y$ is called almost *$\alpha\psi$-continuous* if for each $x \in X$ and each *open* set V of Y containing $f(x)$, there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq int_{\alpha\psi}(cl(V))$.

Definition 14. A function $f : X \to Y$ is said to be pre *$\alpha\psi$-open* if the image of each *$\alpha\psi$-open* set is *$\alpha\psi$-open.*

Theorem 15. *If a function $f : X \to Y$ is a pre $\alpha\psi$-open and contra $\alpha\psi$-continuous, then f is almost $\alpha\psi$-continuous.*

Proof. Let x be any arbitrary point of X and V be an *open* set containing $f(x)$. Since f is contra *$\alpha\psi$-continuous*, then by Theorem 4.(3)there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq cl(V)$. Since f is pre *$\alpha\psi$-open*, $f(U)$ is *$\alpha\psi$-open* in Y. Therefore, $f(U) = int_{\alpha\psi} f(U) \subseteq int_{\alpha\psi}(cl(f(U))) \subseteq int_{\alpha\psi}(cl(V))$. This shows that f is almost *$\alpha\psi$-continuous.* □

Definition 16. A function $f : X \to Y$ is said to be almost weakly *$\alpha\psi$-continuous* if for each $x \in X$ and each *open* set V of $f(x)$, there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq cl(V)$.

Theorem 17. *If a function $f : X \to Y$ is contra $\alpha\psi$-continuous, then f is almost weakly $\alpha\psi$-continuous.*

Proof. Let V be any open set of Y. Since $cl(V)$ is closed in Y, by Theorem 4. (3) $f^{-1}(cl(V))$ is $\alpha\psi$-*continuous* in X and set $U = f^{-1}(cl(V))$, we have $f(U) \subseteq cl(V)$. This shows that f is almost $\alpha\psi$-*continuous*. □

Since the family of all $\alpha\psi$-*open* subsets of a space (X, τ), denoted by $\tau_{\alpha\psi}$, forms a topology on X finer than τ, then the $\alpha\psi$-frontier of A, where $A \subseteq X$, is denoted by $Fr_{\alpha\psi}(A) = cl_{\alpha\psi}(A) \cap cl_{\alpha\psi}(X - A)$.

Theorem 18. *The set of all points of x of X at which $f : X \to Y$ is not contra $\alpha\psi$-continuous is identical with the union of the $\alpha\psi$-frontier of the inverse image of closed sets of Y containing $f(x)$.*

Proof. Suppose f is not contra $\alpha\psi$-*continuous* at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \phi$ for every $U \in \alpha\psi O(X, x)$ by Theorem 4. This implies that $U \cap f^{-1}(Y - F) \neq \phi$. Therefore, we have $x \in cl_{\alpha\psi}(f^{-1}(Y - F)) = cl_{\alpha\psi}(X - f^{-1}(F))$. However, since $x \in f^{-1}(F) \subseteq cl_{\alpha\psi}(f^{-1}(F))$, thus $x \in cl_{\alpha\psi}(f^{-1}(F)) \cap cl_{\alpha\psi}(f^{-1}(Y - F))$. Therefore, we obtain $x \in Fr_{\alpha\psi}(f^{-1}(F))$. Suppose that $x \in Fr_{\alpha\psi}(f^{-1}(F))$ for some $F \in C(Y, f(x))$ and f is contra $\alpha\psi$-*continuous* at x, then there exists $U \in \alpha\psi O(X, x)$ such that $f(U) \subseteq V$. Therefore, we have $x \in U \subseteq f^{-1}(F)$ and hence $x \in int_{\alpha\psi}(f^{-1}(F)) \subseteq X - Fr_{\alpha\psi}(f^{-1}(F))$. This is a contradiction. This means that f is not contra $\alpha\psi$-*continuous*. □

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $Gr(f)$.

Theorem 19. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f. If g is contra $\alpha\psi$-continuous, then f is contra $\alpha\psi$-continuous.*

Proof. Let U be an *open* set in Y, then $X \times Y$ is an *open* set in $X \times Y$. Since g is contra $\alpha\psi$-*continuous*, it follows that $f^{-1}(U) = g^{-1}(X \times Y)$ is an $\alpha\psi$-*closed* set in X. Thus, f is contra $\alpha\psi$-*continuous*. □

Definition 20. The graph $Gr(f)$ of a function $f : X \to Y$ is said to be contra $\alpha\psi$-*closed* if for each $(x, y) \in (X \times Y) - Gr(f)$, there exists $U \in \alpha\psi O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap Gr(f) = \phi$.

Lemma 21. *The graph $Gr(f)$ of a function $f : X \to Y$ is contra $\alpha\psi$-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - Gr(f)$, there exists $U \in \alpha\psi(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \phi$.*

Theorem 22. *If $f : X \to Y$ is contra $\alpha\psi$-continuous and Y is Urysohn, then $Gr(f)$ is contra $\alpha\psi$-closed in the product space $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - Gr(f)$. Then $y \neq f(x)$ and there exists open sets H_1, H_2 such that $f(x) \in H_1$, $y \in H_2$ and $cl(H_1) \cap cl(H_2) = \phi$. From hypothesis, there exists $V \in \alpha\psi O(X, x)$ such that $f(V) \subset cl(H_1)$. Therefore, we obtain $f(V) \cap cl(H_2) = \phi$. This shows that $Gr(f)$ is contra *$\alpha\psi$-closed*. □

Theorem 23. *If $f : X \to Y$ is contra $\alpha\psi$-continuous and Y is T_1, then $G(f)$ is contra $\alpha\psi$-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - Gr(f)$. Then $f(x) \neq y$ and there exists an *open* set V of Y such that $f(x) \in V$ and $y \notin V$. Since f is contra *$\alpha\psi$-continuous* there exists $U \in \alpha\psi(X, x)$ such that $f(U) \subset V$. Therefore, we have $f(U) \cap (Y - V) = \phi$ and $Y - V \in C(Y, y)$. This shows that $G(f)$ is contra *$\alpha\psi$-continuous* in $X \times Y$. □

Theorem 24. *Let $(X_\lambda : \lambda \in \Lambda)$ be any family of topological spaces. If $f : X \to \prod X_\lambda$ is a contra $\alpha\psi$-continuous function. Then $P_{r\lambda} \circ f : X \to X_\lambda$ is contra $\alpha\psi$-continuous for each $\lambda \in \Lambda$, where $P_{r\lambda}$ is the projection of $\prod X_\lambda$ onto X_λ.*

Proof. We shall consider a fixed $\lambda \in \Lambda$. Suppose U_λ is an arbitrary *open* set in X_λ. Then $P_{r\lambda}^{-1}(U_\lambda)$ is *open* in $\prod X_\lambda$. Since f is contra *$\alpha\psi$-continuous*, we have by definition $f^{-1}(P_{r\lambda}^{-1}(U_\lambda)) = (P_{r\lambda} \circ f)^{-1}(U_\lambda)$ is *$\alpha\psi$-continuous* in X. Therefore $P_{r\lambda} \circ f$ is contra *$\alpha\psi$-continuous*. □

Theorem 25. *If $f : X \to Y$ is a contra $\alpha\psi$-continuous function and $g : Y \to Z$ is a continuous function, then $g \circ f : X \to Z$ is contra $\alpha\psi$-continuous if and only if g is contra $\alpha\psi$-continuous.*

Proof. Let $g \circ f : X \to Z$ is contra *$\alpha\psi$-continuous* and let F be a *closed* subset of Z. Then $(g \circ f)^{-1}(F)$ is a *$\alpha\psi$-open* of X. That is $f^{-1}(g^{-1}(F))$ is *$\alpha\psi$-open* subset of Y. So, $g^{-1}(F)$ is a *$\alpha\psi$-open* in Y. Hence g is contra *$\alpha\psi$-continuous*. □

3. Applications

Definition 26. (a) $\alpha\psi$-normal if each pair of non-empty disjoint *closed* sets can be separated by disjoint $\alpha\psi$-*open* sets,

(b) ultranormal (see [10]) if each pair of non-empty disjoint *closed* sets can be separated by disjoint clopen sets.

Theorem 27. *If $f : X \to Y$ is a contra $\alpha\psi$-continuous, closed injection and Y is ultranormal, then X is $\alpha\psi$-normal.*

Proof. Let F_1 and F_2 be disjoint *closed* subsets of X. Since f is *closed* injective, $f(F_1)$ and $f(F_2)$ are disjoint *closed* subsets of Y. Since Y is ultranormal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Hence $F_i \subset f^{-1}(V_i), f^{-1}(V_i) \in \alpha\psi O(X)$ for $i = 1, 2$ and $f^{-1}(V_1) \cap f^{-1}(V_2) = \phi$. Thus X is $\alpha\psi$-normal. □

Definition 28. A topological space X is said to be $\alpha\psi$-connected if X is not the union of two disjoint non-empty $\alpha\psi$-*open* subsets of X.

Theorem 29. *A contra $\alpha\psi$-continuous image of a $\alpha\psi$-connected space is connected.*

Proof. Let $f : X \to Y$ be a contra $\alpha\psi$-*continuous* function of a $\alpha\psi$-connected space X onto to a topological space Y. If possible, let Y is disconnected. Let A and B form a disconnected of Y. Then A and B are clopen and $Y = A \cup B$ where $A \cap B = \phi$. Since f is contra $\alpha\psi$-*continuous*, $X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$ where $f^{-1}(A)$ and $f^{-1}(B)$ are non-empty $\alpha\psi$-*open* sets in X. Also, $f^{-1}(A) \cap f^{-1}(B) = \phi$. Hence X is non $\alpha\psi$-connected which is a contradiction. Therefore Y is connected. □

Theorem 30. *Let X be $\alpha\psi$-connected and Y be T_1. If $f : X \to Y$ is contra $\alpha\psi$-continuous, then f is constant.*

Proof. Since Y is T_1 space, $v = \{f^{-1}(y) : y \in Y\}$ is a disjoint $\alpha\psi$-*open* partition of X. If $|v| \geq 2$, then X is the union of two non-empty $\alpha\psi$-*open* sets. Since X is $\alpha\psi$-connected, $|v| = 1$. Therefore, f is constant. □

Theorem 31. *If $f : X \to Y$ is a contra $\alpha\psi$-continuous function from a $\alpha\psi$-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper non-empty *open* and *closed* subset of Y. Then $f^{-1}(A)$ is a proper nonempty $\alpha\psi$-clopen subset of X, which is a contradiction to the fact X is $\alpha\psi$-connected. □

Definition 32. A space X is said to be $\alpha\psi$-T_2 if for each pair of distinct points x and y in X, there exists $U \in \alpha\psi O(X, x)$ and $V \in \alpha\psi O(Y, y)$ such that $U \cap V = \phi$.

Theorem 33. *Let X and Y be topological spaces. If*

(1) for each pair of distinct points x and y in X there exists a function f of X into Y such that $f(x) \neq f(y)$,

(2) Y is an Urysohn space,

(3) f is contra $\alpha\psi$-continuous at x and y, then X is $\alpha\psi$-T_2.

Proof. Let x and y be any distinct points in X. Then, there exists a Urysohn space Y and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and f is contra $\alpha\psi$-*continuous* at x and y. Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since Y is Urysohn space, there exists *open* sets V and W containing a and b, respectively, such that $cl(V) \cap cl(W) = \phi$. Since f is contra $\alpha\psi$-*continuous* at x and y, there exist $\alpha\psi$-*open* sets A and B containing a and b, respectively, such that $f(A) \subseteq cl(V)$ and $f(B) \subseteq cl(W)$. Then $f(A) \cap f(B) = \phi$, so $A \cap B = \phi$. Hence, X is $\alpha\psi$-T_2. □

Corollary 34. *Let $f : X \to Y$ be contra $\alpha\psi$-continuous injection. If Y is an Urysohn space, then X is $\alpha\psi$-T_2.*

References

[1] Ahmad Al-Omari, Mohd Salmi Md Noorani, Contra-ω-*continuous* and almost contra-ω-*continuous*, *International Journal of Mathematics and Mathematical Sciences* (2007), 1-13.

[2] R. Devi, A. Selvakumar, M. Parimala, $\alpha\psi$-*closed* sets in topological spaces, Submitted.

[3] J. Dontchev, Contra-*continuous* functions and strongly *S-closed* spaces, *International Journal of Mathematics and Mathematical Sciences*, **19**, No. 2 (1996), 303-310.

[4] J. Dontchev, T. Noiri, Contra-*semicontinuous* functions, *Mathematica Pannonica*, **10**, No. 2 (1999), 159-168.

[5] S. Jafari, T. Noiri, Contra-super-*continuous* functions, *Annales Univ. Sci. Budapest*, **42** (1999), 27-34.

[6] S. Jafari, T. Noiri, Contra-α-*continuous* functions between topological spaces, *Iranian International Journal of Sciences*, **2**, No. 2 (2001), 153-167.

[7] S. Jafari, T. Noiri, On contra-*precontinuous* functions, *Bulletin of the Malaysian Mathematical Sciences Society*, **25**, No. 2 (2002), 115-128.

[8] M. Mrsevic, On pairwise R_0 and pairwise R_1 bitopological spaces, *Bull. Math. Soc. Sci. Math. RS Roumami*, **30**, No. 78 (1986), 141-148.

[9] A.A. Nasef, Some properties of contra-γ-*continuous* functions, *Choas, Solitons and Fractals*, **24**, No. 2 (2005), 471-477.

[10] R. Staum, The algebra of bounded *continuous* functions into nonarchimedean field, *Pacific J. Math.*, **50** (1974), 169-185.

[11] S. Willard, *General Topology*, Addison-Wesley, Reading, Mass, USA (1970).