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APPROXIMATION-SOLVABILITY OF A SYSTEM OF GENERALIZED VARIATIONAL INEQUALITIES IN BANACH SPACES

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Abstract: In this paper we introduce a new system of generalized variational inequalities and two concepts of η-subdifferential and A-η-proximal mappings of a proper functional in Banach spaces and prove the existence and Lipschitz continuity of A-η-proximal mapping of a lower semicontinuous η-subdifferentiable proper functional in reflexive Banach spaces. We suggest a new iterative algorithm for computing the approximate solutions of the system of generalized variational inequalities. Under certain conditions, we establish the existence theorems of solutions and convergence of the iterative algorithm for the system of generalized variational inequalities.

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1. Introduction

Variational inequality theory has become a very effective and powerful tool for studying a wide range of problems arising in pure and applied sciences which include work on mathematical programming, optimization theory, engineering, differential equations, mechanics, contact problems in elasticity, control problems, general equilibrium problems in economics and transportation, etc. Variational inequalities have been extended and generalized in different directions. Some useful and important generalizations of variational inequalities are the generalized set-valued mixed variational inequalities, generalized quasi-variational inclusions including the nonlinear term and system of variational inequalities. (see, for example, [1]-[18]). Hasouni-Moudafi [4] introduced and studied a class of variational inclusions and developed a perturbed algorithm for finding approximate solutions of the variational inclusions. Under Hilbert space setting, there are a substantial number of iterative algorithms for finding the approximate solutions of various variational inequalities. Recently, Ding-Luo [1] introduced two concepts of η-subdifferential and η-proximal mapping for a proper functional in Hilbert spaces, and proved the existence and Lipschitz continuity of η-proximal mappings of a proper functional in Hilbert spaces. By applying these concepts, Ding-Luo [1] investigated a class of general quasi-variational-like inclusions. Ding-Xia [3] introduced a notion of J-proximal mapping for a lower semicontinuous subdifferentiable proper functional in Banach spaces, and discussed the existence and Lipschitz continuity of J-proximal mapping of the functional in reflexive Banach spaces. Nie-Liu-Kim-Kang [9], Rhoades-Verma [10], Verma [11]-[16], Wu-Liu-Shim-Kang [17] and Zhou-Chen [18] discussed the approximation-solvability of several kinds of system of variational inequalities in Euclidean spaces and Hilbert spaces, respectively.

Motivated and inspired by the above research work, in this paper, we introduce a new system of generalized variational inequalities and two concepts of η-subdifferential and A-η-proximal mappings of a proper functional in Banach spaces and prove the existence and Lipschitz continuity of A-η-proximal mapping of a lower semicontinuous η-subdifferentiable proper functional in reflexive Banach spaces. We suggest a new iterative algorithm for computing the approximate solutions of the system of generalized variational inequalities. Under certain conditions, we establish the existence theorems of solutions and convergence of the iterative algorithm for the system of generalized variational inequalities.

2. Preliminaries

Let E be a real Banach space with the dual space E^*, $\langle \cdot, \cdot \rangle$ denote the dual pairing between E^* and E, $J : E \to 2^{E^*}$ be the normalized duality mapping defined by

$$J(x) = \{f^* \in E^* : \langle f^*, x \rangle = \|x\| \|f^*\|, \|x\| = \|f^*\|\}, \quad \forall x \in E.$$

We recall and introduce the following concepts and results.

Definition 2.1. Let $A : E \to E^*$ and $\eta : E \times E \to E$ be two mappings.

(1) A is said to be α-*strongly monotone* with respect to η if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, \eta(x, y) \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in E.$$

(2) η is said to be τ-*Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \quad \forall x, y \in E.$$

Definition 2.2. (see [18]) A functional $f : E \times E \to R \cup \{+\infty\}$ is said to be 0-*diagonally quasi-concave* (in short, DQCV) in x if for any finite set $\{x_1, \cdots, x_n\} \subset E$ and for any $y = \sum_{i=1}^{n} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, $\min_{1 \leq i \leq n} f(x_i, y) \leq 0$.

Definition 2.3. Let $\eta : E \times E \to E$ be a mapping. A proper functional $\phi : E \to R \cup \{+\infty\}$ is said to be η-*subdifferentiable* at a point $x \in E$ if there exists a point $f^* \in E^*$ such that

$$\phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \quad \forall y \in E,$$

where f^* is called η-*subgradient* of ϕ at x. The set of all η-subgradients of ϕ at x is denoted by $\Delta\phi(x)$. The mapping $\Delta\phi : E \to 2^E$ defined by

$$\Delta\phi(x) = \{f^* \in E^* : \phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \forall y \in E\}, \quad \forall x \in E$$

is said to be η-*subdifferential* of ϕ.

Remark 2.1. If $\eta(y, x) = y - x$ for all $x, y \in E$, then Definition 2.3 reduces to the usual definition of subdifferential of a functional ϕ.

Definition 2.4. Let $\phi : E \to R \cup \{+\infty\}$be a proper functional. For any given $x^* \in E^*$ and any $\rho > 0$, if there exist mappings $A : E \to E^*$, $\eta : E \times E \to E$ and a unique point $x \in E$ such that

$$\langle Ax - x^*, \eta(y, x) \rangle \geq \rho\phi(x) - \rho\phi(y), \quad \forall y \in E, \tag{2.1}$$

then the mapping $x^* \mapsto x$, denoted by $A_\rho^{\Delta\phi}(x^*)$ is said to be *A-η-proximal mapping* of ϕ.

Note that $A_\rho^{\Delta\phi}(x^*) = (A + \rho\Delta\phi)^{-1}(x^*)$ by $x^* - Ax \in \rho\Delta\phi(x)$.

Remark 2.2. If $\eta(y, x) = y - x$ for all $x, y \in E$, then the A-η-proximal mapping of ϕ reduces to Definition 2.3 in Ding-Xia [3].

Let I denote the identity mapping on E. Let $A, S, T : E \to E^*$, $g : E \to E$ and $\eta : E \times E \to E$ be mappings, $\phi : E \times E \to R \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in E$, $\phi(\cdot, y) : E \to E$ be lower semicontinuous and η-subdifferentiable on E and $g(E) \cap dom\, \Delta\phi(\cdot, y) \neq \emptyset$. Let $\rho > 0$ and $\gamma > 0$ be two constants. We consider the following problem: Find $(x, y) \in E \times E$ such that $g(x), g(y) \in dom\, \Delta\phi(\cdot, x)$ and

$$\begin{aligned} &\langle \rho S(g(y)) + A(g(x)) - A(g(y)), \eta(u, g(x)) \rangle \\ &\quad \geq \rho\phi(g(x), x) - \rho\phi(u, x), \quad \forall u \in E, \\ &\langle \gamma T(g(x)) + A(g(y)) - A(g(x)), \eta(u, g(y)) \rangle \\ &\quad \geq \gamma\phi(g(y), x) - \gamma\phi(u, x), \quad \forall u \in E, \end{aligned} \tag{2.2}$$

which is called the *system of generalized variational inequalities.*

If $A = g = I$, $\eta(x, y) = x - y$ and $\phi(x, y) = I_K(x)$ for all $x, y \in E$, where K is a nonempty closed convex subset of a Hilbert space E and I_K is the indicator function of K, then the problem (2.2) is equivalent to finding $(x, y) \in K \times K$ such that

$$\langle \rho Sy + x - y, u - x \rangle \geq 0, \quad \forall u \in K,$$

$$\langle \gamma Tx + y - x, u - y \rangle \geq 0, \quad \forall u \in K,$$

which is called the *system of nonlinear variational inequalities*, introduced and studied by Rhaodes-Verma [10] and Verma [15].

Definition 2.5. Let $g : E \to E$ and $A : E \to E^*$ be two mappings.

(1) g is said to be *k-relaxed Lipschitz* if there exists a constant $k > 0$ such that for any $x, y \in E$, there exists $j(x - y) \in J(x - y)$ satisfying

$$\langle j(x - y), g(x) - g(y) \rangle \leq -k\|x - y\|^2;$$

(2) g is said to be *l-Lipschitz continuous* if there exists a constant $l > 0$ such that

$$\|g(x) - g(y)\| \leq l\|x - y\|, \quad \forall x, y \in E;$$

(3) A is said to be *a-Lipschitz continuous* if there exists a constant $a > 0$ such that

$$\|A(x) - A(y)\| \leq a\|x - y\|, \quad \forall x, y \in E.$$

Lemma 2.1. (see [2]) *Let D be a nonempty convex subset of a topological vector space and $f : D \times D \to [-\infty, +\infty]$ be such that*

(a) *for each $x \in D$, $y \mapsto f(x, y)$ is lower semicontinuous on each compact subset of D;*

(b) *for each finite set $\{x_1, \cdots, x_m\} \subset D$ and for each $y = \sum_{i=1}^{m} \lambda_i x_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{m} \lambda_i = 1$, $\min_{1\leq i\leq m} f(x_i, y) \leq 0$;*

(c) *there exist a nonempty compact convex subset D_0 of D and a nonempty compact subset K of D such that for each $y \in D\backslash K$, there is an $x \in co(D_0 \cup \{y\})$ satisfying $f(x, y) > 0$.*

Then there exists $\hat{y} \in D$ such that $f(x, \hat{y}) \leq 0$ for all $x \in D$.

Now we give sufficient conditions that guarantee the existence and Lipschitz continuity of the A-η-proximal mapping of a proper functional in reflexive Banach space.

Lemma 2.2. *Let E be a real reflexive Banach space with the dual space E^* and $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional. Let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in E$ and for any given $x^* \in E^*$, the function $h(y, x) = \langle x^* - Ax, \eta(y, x)\rangle$ is DQCV in y. Let $A : E \to E^*$ be continuous and α-strongly monotone with respect to η. Then for any given $\rho > 0$ and $x^* \in E^*$, there exists a unique $x \in E$ such that*

$$\langle Ax - x^*, \eta(y, x)\rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in E, \tag{2.3}$$

that is, $x = A_\rho^{\Delta\phi}(x^*)$.

Proof. For any given $\eta : E \times E \to E$, $A : E \to E^*, \rho > 0$ and $x^* \in E^*$, define a functional $f : E \times E \to R \cup \{+\infty\}$ by

$$f(y, x) = \langle x^* - Ax, \eta(y, x)\rangle + \rho\phi(x) - \rho\phi(y), \quad \forall x, y \in E.$$

Since A is continuous and ϕ is lower semicnotinuous, we conclude that for any $y \in E$, $x \mapsto f(y, x)$ is lower semicontinuous on E. That is, $f(y, x)$ satisfies the condition (a) of Lemma 2.1. We assert that $f(y, x)$ satisfies the condition (b) of Lemma 2.1. Otherwise, there exists a finite set $\{y_1, \cdots, y_m\} \subset E$ and $x_0 = \sum_{i=1}^{m} \lambda_i y_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^{m} \lambda_i = 1$ such that

$$\langle x^* - Ax_0, \eta(y_i, x_0)\rangle + \rho\phi(x_0) - \rho\phi(y_i) > 0, \quad i = 1, \cdots, m. \tag{2.4}$$

Because ϕ is η-subdifferentiable at x_0, there exists a point $f^* \in E^*$ such that

$$\rho\phi(y_i) - \rho\phi(x_0) \geq \rho\langle f^*, \eta(y_i, x_0)\rangle, \quad i = 1, \cdots, m. \tag{2.5}$$

In view of (2.4) and (2.5), we infer that

$$\langle x^* - Ax_0 - \rho f^*, \eta(y_i, x_0)\rangle > 0, \quad i = 1, \cdots, m,$$

which means that

$$\min_{1\le i\le m} \langle x^* - Ax_0 - \rho f^*, \eta(y_i, x_0)\rangle > 0.$$

Notice that, $h(y,x) = \langle x^* - \rho f^* - Ax, \eta(y,x)\rangle$ is DQCV in y. It follows that

$$\min_{1\le i\le m} \langle x^* - Ax_0 - \rho f^*, \eta(y_i, x_0)\rangle \le 0,$$

which is a contradiction. Hence$f(y,x)$ satisfies the condition (b) of Lemma 2.1. Now take a fixed $\hat{y} \in dom\,\phi$. Since ϕ is η-subdifferentiable at $\hat{y}$, there exists a point $f^* \in E^*$ such that

$$\phi(x) - \phi(\hat{y}) \ge \langle f^*, \eta(x, \hat{y})\rangle, \quad \forall x \in E,$$

which implies that

$$\begin{aligned} f(\hat{y}, x) &= \langle x^* - Ax, \eta(\hat{y}, x)\rangle + \rho\phi(x) - \rho\phi(\hat{y}) \\ &\ge \langle A\hat{y} - Ax, \eta(\hat{y}, x)\rangle + \langle x^* - A\hat{y}, \eta(\hat{y}, x)\rangle + \rho\langle f^*, \eta(x, \hat{y})\rangle \\ &\ge \alpha\|\hat{y} - x\|^2 + \langle x^*, \eta(\hat{y}, x)\rangle - \langle A\hat{y}, \eta(\hat{y}, x)\rangle - \rho\langle f^*, \eta(\hat{y}, x)\rangle \\ &\ge \alpha\|\hat{y} - x\|^2 - (\|x^*\| + \|A\hat{y}\| + \rho\|f^*\|)\|\eta(\hat{y}, x)\| \\ &\ge \alpha\|\hat{y} - x\|^2 - \tau(\|x^*\| + \|A\hat{y}\| + \rho\|f^*\|)\|\hat{y} - x\| \\ &= \|\hat{y} - x\|[\alpha\|\hat{y} - x\| - \tau(\|x^*\| + \|A\hat{y}\| + \rho\|f^*\|)]. \end{aligned}$$

Let $r = \alpha^{-1}\tau(\|x^*\| + \|A\hat{y}\| + \rho\|f^*\|)$, $K = \{u \in E : \|\hat{y} - u\| \le r\}$ and $D_0 = \{\hat{y}\}$. Then D_0 and K are both weakly compact convex subsets of E. For each $x \in E\backslash K$, there exists a point $\hat{y} \in co(D_0 \cup \{x\})$ such that $f(\hat{y}, x) > 0$. Thus all conditions of Lemma 2.1 are fulfilled. Lemma 2.1 ensures that there exists an $\hat{x} \in E$ such that $f(y, \hat{x}) \le 0$ for all $y \in E$, that is

$$\langle A\hat{x} - x^*, \eta(y, \hat{x})\rangle + \rho\phi(y) - \rho\phi(\hat{x}) \ge 0.$$

Next we prove that $\hat{x}$ is a unique solution of the auxiliary variational inequality (2.3). Suppose that x_1 and x_2 are two solutions of the auxiliary variational inequality (2.3), that is

$$\langle Ax_1 - x^*, \eta(y, x_1)\rangle + \rho\phi(y) - \rho\phi(x_1) \ge 0, \quad \forall y \in E, \tag{2.6}$$

$$\langle Ax_2 - x^*, \eta(y, x_2)\rangle + \rho\phi(y) - \rho\phi(x_2) \geq 0, \quad \forall y \in E. \tag{2.7}$$

Letting $y = x_2$ in (2.6) and $y = x_1$ in (2.7), and adding these inequalities, we deduce that

$$\langle Ax_2 - Ax_1, \eta(x_2, x_1)\rangle \leq 0.$$

Since is α-strongly monotone with respect to η, it follows that

$$\alpha\|x_2 - x_1\|^2 \leq \langle Ax_2 - Ax_1, \eta(x_2, x_1)\rangle \leq 0,$$

which means that $x_1 = x_2$. This completes the proof. □

Remark 2.3. Lemma 2.2 shows that the existence of the A-η-proximal mapping $A_\rho^{\Delta\phi}$ of a lower semicontinuous η-subdifferentiable proper functional ϕ.

Lemma 2.3. *Let E be a real reflexive Banach space with the dual space E^* and $\phi : E \to R \cup \{+\infty\}$ be a lower semicontinuous η-subdifferentiable proper functional. Let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in E$ and for any given $x^* \in E^*$, the function $h(y, x) = \langle x^* - Ax, \eta(y, x)\rangle$ is DQCV in y. Let $A : E \to E^*$ be continuous and α-strongly monotone with respect to η. Then the A-η-proximal mapping $A_\rho^{\Delta\phi}$ of ϕ is $\alpha^{-1}\tau$-Lipschitz continuous.*

Proof. It follows from Lemma 2.2 that the A-η-proximal mapping $A_\rho^{\Delta\phi}$ of ϕ is well defined. For any given $x^*, y^* \in E^*$, we know that $x^* - Ax \in \rho\Delta\phi(x)$, $y^* - Ay \in \rho\Delta\phi(y)$, where $x = A_\rho^{\Delta\phi}(x^*)$, $y = A_\rho^{\Delta\phi}(y^*)$. Hence we get that

$$\rho\phi(u) - \rho\phi(x) \geq \langle x^* - Ax, \eta(u, x)\rangle, \quad \forall u \in E, \tag{2.8}$$

$$\rho\phi(u) - \rho\phi(y) \geq \langle y^* - Ay, \eta(u, y)\rangle, \quad \forall u \in E. \tag{2.9}$$

Take $u = y$ in (2.8) and $u = x$ in (2.9), adding these inequalities, we infer that

$$\langle Ay - Ax, \eta(y, x)\rangle \leq \langle y^* - x^*, \eta(y, x)\rangle.$$

Since A is α-strongly monotone with respect to η and η is τ-Lipschitz continuous, we deduce that

$$\alpha\|y - x\|^2 \leq \tau\|y^* - x^*\|\|y - x\|,$$

which means that $A_\rho^{\Delta\phi}$ is $\alpha^{-1}\tau$-Lipschitz continuous. □

Remark 2.4. If $\eta(y,x) = y - x$ for all $x, y \in E$, then Lemma 2.2 reduces to Theorem 2.2 in Ding-Xia [3].

Lemma 2.4. *Let E be a real Banach space. Then*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle j(x+y), y\rangle, \quad \forall x, y \in E,\ j(x+y) \in J(x+y).$$

3. Existence and Algorithm of Solutions for the System of Generalized Variational Inequalities

Lemma 3.1. *Let $\rho > 0$ and $\gamma > 0$ be two constants. Then $(x, y) \in E \times E$ is a solution of the system of generalized variational inequalities* (2.2) *if and only if*

$$\begin{aligned} g(x) &= A_\rho^{\Delta\phi(\cdot,x)}\big(A(g(y)) - \rho S(g(y))\big),\\ g(y) &= A_\gamma^{\Delta\phi(\cdot,x)}\big(A(g(x)) - \gamma T(g(x))\big), \end{aligned} \tag{3.1}$$

where $A_\rho^{\Delta\phi(\cdot,x)} = (A + \rho\Delta\phi(\cdot,x))^{-1}$ and $A_\gamma^{\Delta\phi(\cdot,x)} = (A + \gamma\Delta\phi(\cdot,x))^{-1}$ are both A-η-proximal mappings of $\phi(\cdot,x)$.

Proof. Assume that $(x, y) \in E \times E$ is a solution of the system of generalized variational inequalities (2.2), that is

$$\begin{aligned} &\langle \rho S(g(y)) + A(g(x)) - A(g(y)), \eta(u, g(x))\rangle \\ &\quad \geq \rho\phi(g(x), x) - \rho\phi(u, x), \quad \forall u \in E, \end{aligned}$$

$$\begin{aligned} &\langle \gamma T(g(x)) + A(g(y)) - A(g(x)), \eta(u, g(y))\rangle \\ &\quad \geq \gamma\phi(g(y), x) - \gamma\phi(u, x), \quad \forall u \in E. \end{aligned}$$

The above relations hold if and only if

$$A(g(y)) - \rho S(g(y)) - A(g(x)) \in \rho\Delta\phi(g(x), x),$$

$$A(g(x)) - \gamma T(g(x)) - A(g(y)) \in \rho\Delta\phi(g(y), x),$$

which are equivalent to

$$g(x) = A_\rho^{\Delta\phi(\cdot,x)}\big(A(g(y)) - \rho S(g(y))\big),$$

$$g(y) = A_\gamma^{\Delta\phi(\cdot,x)}\big(A(g(x)) - \gamma T(g(x))\big).$$

This completes the proof. □

Based on Lemma 3.1, we construct the following iterative algorithm.

Algorithm 3.1. *Let $A, S, T : E \to E^*$, $g : E \to E$ and $\eta : E \times E \to E$ be mappings and $g(E) = E$. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in E, \phi(\cdot, y) : E \to E$ be lower semicontinuous and η-subdifferentiable on E and $g(E) \cap dom\, \Delta\phi(\cdot, y) \neq \emptyset$. For any given $x_0 \in E$, compute $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ by the following iterative scheme:*

$$\begin{aligned} g(x_{n+1}) &= A_\rho^{\Delta\phi(\cdot,x_n)}\big(A(g(y_n)) - \rho S(g(y_n))\big), \\ g(y_n) &= A_\gamma^{\Delta(\cdot,x_n)}\big(A(g(x_n)) - \gamma T(g(x_n))\big), \quad \forall n \geq 0, \end{aligned} \tag{3.2}$$

where $\rho > 0$ and $\gamma > 0$ are two constants.

Next we prove the existence theorems of solutions and convergence of Algorithm 3.1 for the system of generalized variational inequalities (2.2).

Theorem 3.2. *Let E be a real reflexive Banach space. Let $S, T : E \to E^*$ be s-Lipschitz continuous and t-Lipschitz continuous, respectively, $g : E \to E$ be k-relaxed Lipschitz and l-Lipschitz continuous and $g(E) = E$. Let $\eta : E \times E \to E$ be τ-Lipschitz continuous such that $\eta(x, y) = -\eta(y, x)$ for all $x, y \in E$ and for any given $x^* \in E^*$, the function $h(y, x) = \langle x^* - Ax, \eta(y, x)\rangle$ is DQCV in y, $A : E \to E^*$ be a-Lipschitz continuous and α-strongly monotone with respect to η. Let $\phi : E \times E \to R \cup \{+\infty\}$ be a proper functional such that for each fixed $y \in E$, $\phi(\cdot, y) : E \to E$ be lower semicontinuous and η-subdifferentiable on E and $g(E) \cap dom\, \Delta\phi(\cdot, y) \neq \emptyset$. Suppose that there exist $\rho > 0$, $\gamma > 0$, $\mu > 0$ and $\nu > 0$ such that*

$$\big\|A_\rho^{\Delta\phi(\cdot,x)}(z) - A_\rho^{\Delta\phi(\cdot,y)}(z)\big\| \leq \mu\|x - y\|, \quad \forall x, y \in E,\ z \in E^*, \tag{3.3}$$

$$\big\|A_\gamma^{\Delta\phi(\cdot,x)}(z) - A_\gamma^{\Delta\phi(\cdot,y)}(z)\big\| \leq \nu\|x - y\|, \quad \forall x, y \in E,\ z \in E^*, \tag{3.4}$$

$$0 < \rho < \frac{\alpha^2\sqrt{2k-1} - \tau a(l\tau a - l\tau\gamma t - \alpha\nu) - \mu\alpha^2}{\tau s(l\tau a + l\tau\gamma t + \alpha\nu)}, \tag{3.5}$$

$$k > \frac{1}{2} + \frac{(\alpha\tau\nu a + l\tau^2 a^2 + \mu\alpha^2)^2}{2\alpha^4}. \tag{3.6}$$

Then the system of generalized variational inequalities (2.2) has a solution (x, y) in $E \times E$, and the sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ generated by Algorithm 3.1 satisfy that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} g(y_n) = g(y)$.

Proof. In view of Lemma 2.4, we have for any $n \geq 0$

$$\begin{aligned} &\|x_{n+1} - x_n\|^2 \\ &= \|g(x_{n+1}) - g(x_n) - g(x_{n+1}) + g(x_n) - x_{n+1} + x_n\|^2 \\ &\leq \|g(x_{n+1}) - g(x_n)\|^2 + 2\langle j(x_{n+1} - x_n), g(x_{n+1}) - g(x_n)\rangle \\ &\quad + 2\langle j(x_{n+1} - x_n), x_{n+1} - x_n\rangle \\ &\leq \|g(x_{n+1}) - g(x_n)\|^2 - (2k-2)\|x_{n+1} - x_n\|^2. \end{aligned} \tag{3.7}$$

Using (3.2) and the Lipschitz continuity of the mappings, we know that for any $n \geq 0$

$$\begin{aligned} &\|g(x_{n+1}) - g(x_n)\| \\ &\leq \big\|A_\rho^{\Delta\phi(\cdot,x_n)}\big(A(g(y_n)) \\ &\qquad - \rho S(g(y_n))\big) - A_\rho^{\Delta\phi(\cdot,x_n)}\big(A(g(y_{n-1})) - \rho S(g(y_{n-1}))\big)\big\| \\ &\quad + \big\|A_\rho^{\Delta\phi(\cdot,x_n)}\big(A(g(y_{n-1})) - \rho S(g(y_{n-1}))\big) \\ &\qquad - A_\rho^{\Delta\phi(\cdot,x_{n-1})}\big(A(g(y_{n-1})) - \rho S(g(y_{n-1}))\big)\big\| \\ &\leq \frac{\tau}{\alpha}\|A(g(y_n)) - A(g(y_{n-1}))\| + \frac{\tau\rho}{\alpha}\|S(g(y_n)) - S(g(y_{n-1}))\| \\ &\quad + \mu\|x_n - x_{n-1}\| \\ &\leq \mu\|x_n - x_{n-1}\| + \frac{\tau}{\alpha}(a + \rho s)\|g(y_n) - g(y_{n-1})\| \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &\|g(y_n) - g(y_{n-1})\| \\ &\leq \big\|A_\gamma^{\Delta\phi(\cdot,x_n)}\big(A(g(x_n)) - \gamma T(g(x_n))\big) \\ &\qquad - A_\gamma^{\Delta\phi(\cdot,x_n)}\big(A(g(x_{n-1})) - \gamma T(g(x_{n-1}))\big)\big\| \\ &\quad + \big\|A_\gamma^{\Delta\phi(\cdot,x_n)}\big(A(g(x_{n-1})) - \gamma T(g(x_{n-1}))\big) \\ &\qquad - A_\gamma^{\Delta\phi(\cdot,x_{n-1})}\big(A(g(x_{n-1})) - \gamma T(g(x_{n-1}))\big)\big\| \\ &\leq \nu\|x_n - x_{n-1}\| + \frac{\tau}{\alpha}\|A(g(x_n)) - A(g(x_{n-1}))\| \\ &\quad + \frac{\tau}{\alpha}\gamma\|T(g(x_n)) - T(g(x_{n-1}))\| \\ &\leq \Big(\nu + \frac{\tau}{\alpha}(a + \gamma t)l\Big)\|x_n - x_{n-1}\|. \end{aligned} \tag{3.9}$$

Substituting (3.9) into (3.8), we infer that for any $n \geq 0$

$$\begin{aligned} &\|g(x_{n+1}) - g(x_n)\| \\ &\leq \Big\{\mu + \frac{\tau}{\alpha}(a + \rho s)\Big(\nu + \frac{\tau}{\alpha}(a + \gamma t)l\Big)\Big\}\|x_n - x_{n-1}\|. \end{aligned} \tag{3.10}$$

It follows from (3.7) and (3.10) that

$$\|x_{n+1} - x_n\| \le \theta \|x_n - x_{n-1}\|, \quad \forall n \ge 0, \tag{3.11}$$

where

$$\theta = \frac{\tau^2 (a + \rho s)(a + \gamma t) l + \tau \alpha (a + \rho s) \nu + \alpha^2 \mu}{\alpha^2 \sqrt{2k - 1}} < 1 \tag{3.12}$$

by (3.5) and (3.6). In view of (3.9), (3.11) and (3.12), we deduce that $\{x_n\}_{n \ge 0}$ and $\{g(y_n)\}_{n \ge 0}$ are Cauchy sequences in E. Let $x_n \to x$ and $g(y_n) \to u$ as $n \to \infty$. Since $g(E) = E$, there exists $y \in E$ with $u = g(y)$. Note that g, S, T, A and $A_\rho^{\Delta\phi(\cdot,x)}$ are Lipschitz continuous. It is easy to verify that

$$g(x) = A_\rho^{\Delta\phi(\cdot,x)} \big(A(g(y)) - \rho S(g(y)) \big),$$
$$g(y) = A_\gamma^{\Delta\phi(\cdot,x)} \big(A(g(x)) - \gamma T(g(x)) \big).$$

By virtue of Lemma 3.1, we know that (x, y) is a solution of the system of generalized variational inequalities (2.2). This completes the proof. □

Theorem 3.3. *Let S, T, N, η, A and ϕ be the same as in Theorem 3.2. Suppose that g is l-Lipschitz continuous and $g + I$ is k-relaxed Lipschitz and there exist $\rho > 0$ and $\mu > 0$ satisfying* (3.3), (3.4) *and the following conditions:*

$$0 < \rho < \frac{\alpha^2 \sqrt{2k + 1} - l \tau^2 a^2 - l \tau^2 a \gamma t - \alpha \tau \nu a - \mu \alpha^2}{\tau s (l \tau a + l \tau \gamma t + \alpha \nu)}; \tag{3.13}$$

$$k > \frac{(\alpha \tau \nu a + l \tau^2 a^2 + \mu \alpha^2)^2}{2 \alpha^4} - \frac{1}{2}. \tag{3.14}$$

Then the system of generalized variational inequalities (2.2) *has a solution (x, y) in $E \times E$, and the sequences $\{x_n\}_{n \ge 0}$ and $\{y_n\}_{n \ge 0}$ generated by Algorithm 3.1 satisfy that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} g(y_n) = g(y)$.*

Proof. It follows from Lemma 2.4 that

$$\begin{aligned}
&\|x_{n+1} - x_n\|^2 \\
&= \|g(x_{n+1}) - g(x_n) - g(x_{n+1}) + g(x_n) - x_{n+1} + x_n\|^2 \\
&\le \|g(x_{n+1}) - g(x_n)\|^2 \\
&\quad + 2\langle j(x_{n+1} - x_n), g(x_{n+1}) - g(x_n) + x_{n+1} - x_n \rangle \\
&\le \|g(x_{n+1}) - g(x_n)\|^2 - 2k \|x_{n+1} - x_n\|^2.
\end{aligned} \tag{3.15}$$

Similarly, substituting (3.10) into (3.15), we have

$$\|x_{n+1} - x_n\| \leq \theta \|x_n - x_{n-1}\|, \tag{3.16}$$

where

$$\theta = \frac{\tau^2(a+\rho s)(a+\gamma t)l + \tau\alpha(a+\rho s)\nu + \alpha^2\mu}{\alpha^2\sqrt{2k+1}} < 1 \tag{3.17}$$

by (3.13) and (3.14). The rest of the proof is similar to that of Theorem 3.2, and is omitted. This completes the proof. □

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