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GROUP DIVISIBLE DESIGNS WITH TWO ASSOCIATE CLASSES AND $\lambda_2 = 3$

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Abstract: Necessary and sufficient conditions for the existence of Group divisible designs with two groups of unequal sizes and block size tree with $\lambda_2 = 3$, $\lambda_1 \geq 3$ are here considered. We find that the necessary conditions, derived from graph theoretic conditions, are sufficient as well. We present some constructions to prove sufficiency.

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1. Introduction

A *pairwise balanced design* is an ordered pair $(S, \mathcal{B})$, denoted PBD$(S, \mathcal{B})$, where S is a finite set of symbols and $\mathcal{B}$ is a collection of subsets of S called *blocks*, such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. Here $|S| = v$ is called the *order* of the PBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* S(v, k). A PBD with index λ can be defined similarly: each pair of distinct elements occurs in λ blocks. If all blocks are same size, say k, then we get a balanced incomplete block design BIBD(v, b, r, k, λ). In other words, a BIBD(v, b, r, k, λ) is a set S of v elements together with a collection of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [3], [4], [5]).

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Note that in a BIBD(v, b, r, k, λ) the parameters must satisfy the necessary conditions:

1. $vr = bk$ and

2. $\lambda(v-1) = r(k-1)$.

With these conditions a BIBD(v, b, r, k, λ) is usually written as BIBD(v, k, λ).

A *group divisible design* GDD$(v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2)$ is an ordered triple $(V, G, \mathcal{B})$, where V is a v-set of symbols, G is a partition of V into g sets of size $v_1, v_2, \ldots, v_g$, each set being called *group*, and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V, such that each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [3], [4]). Elements occurring together in the same group are called *first associates*, and elements occurring in different groups we called *second associates*. We say that the GDD is defined on the set V. The existence of such GDDs has been of interest over the years, going back to at least the work of Bose and Shimamoto in 1952 who began classifying such designs [1].

In this paper we consider the problem of determining necessary conditions for an existence of GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient for some infinite families. Since we are dealing on GDDs with two groups and block size 3, we will use GDD$(m, n; \lambda_1, \lambda_2)$ for GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as triples. We denote $(X, Y; \mathcal{B})$ for a GDD$(m, n, \lambda_1, \lambda_2)$ if X and Y are m-set and n-set, respectively. Chaiyasena, Hurd, Punnim and Sarvate [2] have written the first paper in this direction, followed by Pabhapote and Punnim [6]. In particular the first paper [2] completely solved the problem of determining all pairs of integers (n, λ) in which a GDD$(1, n; 1, \lambda)$ exists, while the second paper [6] found all triples of integers $(m, n; \lambda)$ in which a GDD$(m, n; \lambda, 1)$ exists. We continue to investigate in this paper all triples of integers (m, n, λ) in which a GDD$(m, n, \lambda, 3)$ exists, where $\lambda \geq 3$. The case $\lambda = 2$ seems to be difficult, as typical of the cases where $\lambda_1 < \lambda_2$ in constructing the general GDD$(m, n; \lambda_1, \lambda_2)$, since it may involve the construction of very specific designs.

Necessary conditions on the existence of a GDD$(m, n; \lambda_1, \lambda_2)$ can be obtained from a graph theoretic point of view as follows. Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD$(m, n; \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m+n$ and size $\lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies

1. $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, and
2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

2. Preliminary Results

We will review some known results concerning triple designs that will be used in the sequel, most of which are taken from [5].

Theorem 2.1. *Let v be a positive integer. Then there exists a* BIBD$(v, 3, 1)$ *if and only if* $v \equiv 1$ *or* $3 \pmod 6$.

A BIBD$(v, 3, 1)$ is usually called *Steiner triple system* and is denoted by STS(v). Let $(V, \mathcal{B})$ be an STS(v). Then the number of triples $b = |\mathcal{B}| = v(v-1)/6$.

The following results on existence of λ-fold triple systems are well known (see e.g. [5]).

Theorem 2.2. *Let n be a positive integer. Then a* BIBD$(n, 3, \lambda)$ *exists if and only if λ and n are in one of the following cases:*

(a) $\lambda \equiv 0 \pmod 6$ *and* $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ *and* $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ *and* $n \equiv 0$ *or* $1 \pmod 3$, *and*

(d) $\lambda \equiv 3 \pmod 6$ *and n is odd.*

The results of Chaiyasena, Hurd, Punnim and Sarvate [2] will be useful and we will state their results as follows:

Theorem 2.3. *Let v be a positive integer with $v \geq 3$. The spectrum of λ, denoted $S_{1,v}$ is defined as*

$$S_{1,v} = \{\lambda : \text{ a GDD}(1, v; 1, \lambda) \text{ exists}\}.$$

Then

(a) $S_{1,v} = \{1, 3, 5, \ldots, v-1\}$ if $v \equiv 0 \pmod 6$,

(b) $S_{1,v} = \{6, 12, 18, \ldots, v-1\}$ if $v \equiv 1 \pmod 6$,

(c) $S_{1,v} = \{1, 7, 13, \ldots, v-1\}$ if $v \equiv 2 \pmod 6$,

(d) $S_{1,v} = \{2, 4, 6, \ldots, v-1\}$ if $v \equiv 3 \pmod 6$,

(e) $S_{1,v} = \{3, 9, 15, \ldots, v-1\}$ if $v \equiv 4 \pmod 6$, and

(f) $S_{1,v} = \{4, 10, 16, \ldots, v-1\}$ if $v \equiv 5 \pmod 6$.

The following notations will be used throughout the paper for our constructions.

1. Let V be a v-set. Let STS(V) be defined as

   $$\text{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an } \text{STS}(v)\}.$$

   BIBD($V, 3, \lambda$) can be defined similarly, That is:

   $$\text{BIBD}(V, 3, \lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \text{BIBD}(v, 3, \lambda)\}.$$

   Let X and Y be disjoint sets of cardinality m and n, respectively. We define GDD($X, Y; \lambda_1, \lambda_2$) as

   $$\text{GDD}(X, Y; \lambda_1, \lambda_2) = \{\mathcal{B} : (X, Y; \mathcal{B}) \text{ is a } \text{GDD}(m, n; \lambda_1, \lambda_2)\}.$$

2. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus "$\cup$" in our construction will be used for the union of multi-sets.

3. Finally, if we have a set X, the number of members or vertices of X shall be denoted by $|X|$.

3. GDD$(m, n; \lambda, 3)$

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers (m, n) in which a GDD$(m, n; \lambda, 3)$ exists. Recall that the existence of GDD$(m, n; \lambda, 3)$ implies

1. $3 \mid \lambda[m(m-1) + n(n-1)]$, and

2. $2 \mid \lambda(m-1)+n$ and $2 \mid \lambda(n-1)+m$.

Let our spectrum be defined as

$$S_3(\lambda) := \{(m,n) : a \ \text{GDD}(m,n;\lambda,3) \ \textit{exists}\}.$$

For the remainder of this paper, our notion of spectrum S_3 for the existence of GDD$(m,n;\lambda,3)$ will be the main focus.

Lemma 3.1. *Let t be a non-negative integer:*

(a) *If $(m,n) \in S_3(6t+7)$, then there exist non-negative integers h and k such that $\{m,n\} \in \{\{6k+1, 6h+4\}, \{6k+1, 6h+6\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}\}$.*

(b) *If $(m,n) \in S_3(6t+8)$, then there exist non-negative integers h and k such that $\{m,n\} \in \{\{6k+4, 6h+4\}, \{6k+4, 6h+6\}, \{6k+6, 6h+6\}\}$.*

(c) *If $(m,n) \in S_3(6t+3)$, then m and n must have different parity, that is, if m is even, n has to be odd, and vice versa. Hence there exist non-negative integers h and k such that $\{m,n\} \in \{\{6k+1, 6h+2\}, \{6k+1, 6h+4\}, \{6k+1, 6h+6), \{6k+2, 6h+3\}, \{6k+2, 6h+5\}, \{6k+3, 6h+4\}, \{6k+3, 6h+6\}, \{6k+4, 6h+5\}, \{6k+5, 6h+6\}\}$.*

(d) *If $(m,n) \in S_3(6t+4)$, then the pairs are the same as those of case* (b).

(e) *If $(m,n) \in S_3(6t+5)$, then the pairs are the same as those of case* (a).

(f) *If $(m,n) \in S_3(6t+6)$, then there exist non-negative integers h and k such that $\{m,n\} \in \{\{6k+2, 6h+2\}, \{6k+2, 6h+4\}, \{6k+2, 6h+6\}, \{6k+4, 6h+4\}, \{6k+4, 6h+6\}, \{6k+6, 6h+6\}\}$.*

Proof. The proof follows from solving the corresponding systems of congruences. □

We now proceed with sufficiency for m and n not equal to 2. We note that for simplicity, we only prove sufficiency for say, GDD$(m,n;\lambda,3)$, since the case of GDD$(n,m;\lambda,3)$ can be dealt in an identical manner, simply by switching the sets involved. For the sake of economy of space, we will prove sufficiency for λ being the minimal value for the case involved. Once we have a GDD$(m,n;\lambda,3)$, we can readily extend to any $6t+\lambda$ by the following Lemma.

Lemma 3.2. *Let m and n be positive integers with $m \neq 2$ and $n \neq 2$. If there exists a* GDD$(m,n;\lambda,3)$ *with $\lambda \geq 3$, then a* GDD$(m,n;6t+\lambda,1), t \geq 0$, *exists.*

Proof. We let X be an m-set and Y be an n-set. We consider $(X, Y; \mathcal{B}_1)$ being a GDD$(m, n; \lambda, 3)$ as given. Since m and n are not equal to 2, by Theorem 2.2 (a) there exist $\mathcal{B}_2 \in$ BIBD$(X, 3, 6t)$ and $\mathcal{B}_3 \in$ BIBD$(Y, 3, 6t)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X, Y; \mathcal{B})$ forms a GDD$(m, n; 6t + \lambda, 3)$ as required. □

Lemma 3.3. *Let h and k be non-negative integers. Then*

$$(6k+1, 6h+4), (6k+1, 6h+6), (6k+3, 6h+4), (6k+3, 6h+6) \in S_3(7).$$

Proof. Let (m, n) be such a pair from the list above. We want to construct a GDD$(m, n; 7, 3)$. Let X be an $m-$set and Y be an $n-$set. Since $|X \cup Y| = m+n$ is odd, it follows by Theorem 2.2(d), that there exists $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 3)$. By Theorem 2.2(b) we have that BIBD$(X, 3, 4) \neq \emptyset$ and BIBD$(Y, 3, 4) \neq \emptyset$. So we let $\mathcal{B}_2 \in$ BIBD$(X, 3, 4)$ and $\mathcal{B}_3 \in$ BIBD$(Y, 3, 4)$.

We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X, Y; \mathcal{B})$ forms a GDD$(m, n; 7, 3)$ as desired. □

Lemma 3.4. *Let h and k be non-negative integers. Then*

$$(6k+4, 6h+4), (6k+4, 6h+6)(6k+6, 6h+6) \in S_3(8).$$

Proof. Let $(m, n) \in \{(6k+4, 6h+4), (6k+4, 6h+6), (6k+6, 6h+6)\}$. A GDD$(m, n; 8, 3)$ can be constructed as follows. Let X be an $m-$set and Y be an $n-$set containing the element a. Let $Y' = Y - \{a\}$. There exists $\mathcal{B}_1 \in$ BIBD$(X \cup Y'; 3, 3)$ since $|X \cup Y'| = m + n$ is an odd number (see Theorem 2.2(d)). Let $\mathcal{B}_2 \in$ BIBD$(X \cup \{a\}, 3, 3)$, which exists for the same reason. By Theorem 2.2(c) and (d), there exist $\mathcal{B}_3 \in$ BIBD$(X, 3, 2)$ and $\mathcal{B}_4 \in$ BIBD$(Y', 3, 3)$. Since $|Y'| \equiv 3$ or $5 \pmod 6$, it follows by Theorem 2.3(d) or (f), that there exists $\mathcal{B}_5 \in$ GDD$(\{a\}, Y'; 1, 4)$. Now we let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_5$. Note the double multiset union of $\mathcal{B}_5$ here. Then $(X, Y; \mathcal{B})$ forms a GDD$(m, n; 8, 3)$ as desired. □

Lemma 3.5. *Let h and k be non-negative integers. Then*

$$(6k+1, 6h+2), (6k+1, 6h+4), (6k+1, 6h+6), (6k+2, 6h+3), (6k+2, 6h+5)$$
$$(6k+3, 6h+4), (6k+3, 6h+6), (6k+4, 6h+5), (6k+5, 6h+6) \in S_3(3).$$

Proof. Here (m, n) involved are of different parity: if one is odd, the other is even, and vice versa. We would like to construct a GDD$(m, n, 3, 3)$, where m and n are the integers in the statement of this lemma. Let X be an m-set and Y an n-set. Since $|X \cup Y| = m + n$ is odd, it follows by Theorem 2.2(d), that there exists $\mathcal{B} \in \text{BIBD}(X \cup Y, 3, 3)$. Then $(X, Y; \mathcal{B})$ forms a $\text{GDD}(m, n; 3, 3)$ as required. □

Lemma 3.6. *Let h and k be non-negative integers. Then*

(a) $(6k+6, 6h+4), (6k+6, 6h+6) \in S_3(4)$, *and*

(b) $(6k+4, 6h+4) \in S_3(4)$.

Proof. (a) Let $(m, n) \in \{(6k+6, 6h+4), (6k+6, 6h+6)\}$. Let X be an $m-$ set and Y be an $n-$set containing the element a. Let $Y' = Y - \{a\}$. Since $|X \cup Y'| = m + n - 1$ is odd, it follows by Theorem 2.2(d), that there exists $\mathcal{B}_1 \in \text{BIBD}(X \cup Y', 3, 3)$. Also there exists $\mathcal{B}_2 \in \text{GDD}(\{a\}, X; 1, 3)$ since $|X| = m \equiv 0 \pmod 6$ (see Theorem 2.3 (a)). Since $|Y'| \equiv 3$ or $5 \pmod 6$, it follows by Theorem 2.3(d) or (f), that there exists $\mathcal{B}_3 \in \text{GDD}(\{a\}, Y'; 1, 4)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup B_3$. Then $(X, Y; \mathcal{B})$ forms a $\text{GDD}(m, n; 4, 3)$ as required.

(b) Let X_k be a $6k+4$-set and Y_h be a $6h+4$-set containing the element a. Furthermore, let $Y_h' = Y_h - \{a\}$. Since $|X_k \cup Y_h'| = 6k + 4 + 6h + 3$ is odd, it follows by Theorem 2.2(d), that there exists $\mathcal{B}_1 \in \text{BIBD}(X_k \cup Y_h', 3, 3)$. Also there exists $\mathcal{B}_2 \in \text{GDD}(\{a\}, X_k; 1, 3)$. since $|X_k| = 6k + 4 \equiv 4 \pmod 6$ (see Theorem 2.3 (e)). Finally there exists $B_3 \in \text{GDD}(\{a\}, Y_h'; 1, 4)$ since $|Y_h'| = 6k+3 \equiv 3 \pmod 6$ (see Theorem 2.3 (d)). We now let $\mathcal{B} = B_1 \cup B_2 \cup B_3$. Thus $(X_k, Y_h; \mathcal{B})$ forms a $\text{GDD}(6k+4, 6h+4; 4, 3)$ as required. □

Lemma 3.7. *Let h and k be non-negative integers. Then*

$$(6k+1, 6h+4), (6k+1, 6h+6), (6k+3, 6h+4), (6k+3, 6h+6) \in S_3(5).$$

Proof. Let (m, n) be a pair from above. A $\text{GDD}(m, n; 5, 3)$ can be constructed as follows. Let X be an $m-$set and Y be an $n-$set. Since $|X \cup Y| = m + n$ is an odd number, it follows by Theorem 2.2(d), that there exist $\mathcal{B}_1 \in \text{BIBD}(X \cup Y, 3, 3)$. We also have the existence of $\text{BIBD}(m, 3, 2)$ since $|X| = m \equiv 0$ or $1 \pmod 3$ (see Theorem 2.2(c)). So let $\mathcal{B}_2 \in \text{BIBD}(X, 3, 2)$. Also there exists $\mathcal{B}_3 \in \text{BIBD}(Y, 3, 2)$ since the order of $|Y| = n \equiv 0$ or $1 \pmod 3$. We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X, Y; \mathcal{B})$ forms a $\text{GDD}(m, n; 5, 3)$ as desired. □

Lemma 3.8. *Let h and k be non-negative integers. Then*

(a) $(6k+6, 6h+6) \in S_3(6)$,

(b) $(6k+2, 6h+2)$ *with h, k not both zero*, $(6k+4, 6h+2), (6k+6, 6h+2) \in S_3(6)$, *and*

(c) $(6k+4, 6h+4), (6k+6, 6h+4), \in S_3(6)$.

Proof. (a) We first consider an existence of $\mathrm{GDD}(6k+6, 6h+6; 6, 3)$. Let X_k be a $(6k+6)-$set and Y_h be a $(6h+6)-$set containing a. Let $Y_h' = Y_h - \{a\}$. Since $|X_k \cup Y_h'| = 6k + 6h - 1$ is an odd number, it follows by Theorem 2.2(d), that there exists $\mathcal{B}_1 \in \mathrm{BIBD}(X_k \cup Y_h', 3, 3)$. Also $\mathrm{BIBD}(X_k \cup \{a\}, 3, 3) \neq \emptyset$. We let $\mathcal{B}_2 \in \mathrm{BIBD}((X_k \cup \{a\}, 3, 3)$. Since $|Y_h'| = 6h + 5 \equiv 5 \pmod 6$, it follows by Theorem 2.3(f), that there exists $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, Y_h'; 1, 4)$. Finally, by Theorem 2.2(c) we have that $\mathrm{BIBD}(Y, 3, 2) \neq \emptyset$ and so we let $\mathcal{B}_4 \in \mathrm{BIBD}(Y_h, 3, 2)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Then $(X_k, Y_h; \mathcal{B})$ forms a $\mathrm{GDD}(6k+6, 6h+6; 6, 3)$ as required.

(b) Let $(m, n) \in \{(6k+2, 6h+2), (6k+4, 6h+2), (6k+6, 6h+2)\}$. We wish to construct a $\mathrm{GDD}(m, n; 6, 3)$. Let X be an $m-$set and Y be an $n-$set containing a. Let $Y' = Y - \{a\}$. Since $|X \cup Y'| = m + n - 1$ is an odd number, it follows by Theorem 2.2(d), that there exists $\mathcal{B}_1 \in \mathrm{BIBD}(X \cup Y', 3, 3)$. Also $\mathrm{BIBD}(X \cup \{a\}, 3, 3) \neq \emptyset$ since $|X \cup \{a\}| = m + 1$ is odd as well. We let $\mathcal{B}_2 \in \mathrm{BIBD}(X \cup \{a\}, 3, 3)$. Since $|Y'| = n - 1 = 6h + 1 \equiv 1 \pmod 3$, it follows by Theorem 2.2(c), that there exists $B_3 \in \mathrm{BIBD}(Y', 3, 2)$. Finally we have $\mathrm{GDD}(\{a\}, Y'; 1, 6) \neq \emptyset$ since $|Y'| = n - 1 = 6h + 1 \equiv 1 \pmod 6$ (see Theorem 2.3(b)). Choose $\mathcal{B}_4 \in \mathrm{GDD}(\{a\}, Y'; 1, 6)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Then $(X, Y; \mathcal{B})$ forms a $\mathrm{GDD}(m, n; 6, 3)$ as desired.

(c) Let $(m, n) \in \{(6k+4, 6h+4), (6k+6, 6h+4)\}$. Let X be an $m-$set and Y be an $n-$set containing a. Let $Y' = Y - \{a\}$. Then $\mathrm{BIBD}(X \cup Y', 3, 3) \neq \emptyset$ since $|X \cup Y'| = m + n - 1$ is an odd number (see Theorem 2.2(d)). Let $\mathcal{B}_1 \in \mathrm{BIBD}(X \cup Y', 3, 3)$. Also $\mathrm{BIBD}(X \cup \{a\}, 3, 3) \neq \emptyset$ since $|X \cup \{a\}| = m + 1$ is also an odd number. We let $\mathcal{B}_2 \in \mathrm{BIBD}((X \cup \{a\}, 3, 3)$. Since $|Y'| = 6h + 3 \equiv 3 \pmod 6$, it follows by Theorem 2.3(d), that there exists $\mathcal{B}_3 \in \mathrm{GDD}(\{a\}, Y'; 1, 6)$. Finally we have $\mathrm{BIBD}(\{Y', 3, 2) \neq \emptyset$ since $|Y'| = n - 1 = 6h + 3 \equiv 3 \pmod 3$ (see Theorem 2.2(c)). Choose $\mathcal{B}_4 \in \mathrm{BIBD}(Y', 3, 2)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup B_4$. Then $(X, Y; \mathcal{B})$ forms a $\mathrm{GDD}(m, n; 6, 3)$ as required. □

4. Conclusions

We can now present our main result:

Theorem 4.1. *Let m and n be positive integers with $m \neq 2$ and $n \neq 2$. There exists a $\mathrm{GDD}(m, n, \lambda, 3), \lambda \geq 3$ if and only if*

1. $3 \mid \lambda[m(m-1) + n(n-1)]$, *and*

2. $2 \mid \lambda(m-1) + n$ *and* $2 \mid \lambda(n-1) + m$.

Proof. The proof follows from Lemmas 3.1-3.8.

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