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EXISTENCE OF POTENTIAL FUNCTIONS OF INTRINSIC GRADIENTS IN HEISENBERG GROUPS

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Abstract: We give explicit conditions equivalent to the existence of a potential function $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{R}$ for the intrinsic gradient $\nabla^\phi \phi$, introduced by [2],[9] and studied in [4],[5] to characterize intrinsic regular hypersrfaces in Heisenberg groups.

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1. Introduction and Motivation of Paper

In the last years the study of $\mathbb{H}$-regular intrinsic graphs in the Heisenberg group $\mathbb{H}^n$ has been studied and developed by many authors (see [2], [4], [5], [9], [10]). $\mathbb{H}$-regular intrinsic graphs are a class of *intrinsic regular* hypersurfaces in the setting of the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$, endowed with a left-invariant not euclidean metric d_∞. Here hypersurface simply means a topological codimension 1 surface and by the words "intrinsic" and "regular" we will mean of notions involving respectively the group structure of $\mathbb{H}^n$ and its differential structure. Let us point out that the class of $\mathbb{H}$- regular surfaces is deeply different from the class of Euclidean regular surfaces, in the sense that there are $\mathbb{H}$-regular surfaces in $\mathbb{H}^1 \equiv \mathbb{R}^3$ that are (Euclidean) fractal sets (see [14]), and conversely there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}$- regular hypersurfaces (see [10], Remark 6.2).

As we will explain below, $\mathbb{H}$-regular graphs have been described in [2], [9],

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[10] by an intrinsic viewpoint with a parametric function $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{R}$, where $\mathbb{W}$ is a subgroup of $\mathbb{H}^n$ homeomorphic to $\mathbb{R}^{2n}$. In particular the existence of parametrizations ϕ, which are not continuously differentiable but only continuous, has been showed in [14].

Important characterizations of $\mathbb{H}$-regular graphs have been given in [2], [4], [5], [9] using the intrinsic gradient $\nabla^{\phi}\phi$ (see (7) as definition). The parametrization ϕ has been seen as a "*weak*" solution of the problem $\nabla^{\phi}\phi = w$, where w is a vector-function associated with the horizontal normal of the $\mathbb{H}$-regular graph. The problem $\nabla^{\phi}\phi = w$ looks like a non linear PDE's system, in particular the equation $W^{\phi}\phi = w_{n+1}$, where $W^{\phi}\phi := \phi_{y_1} + \phi\phi_t$, looks like a conservation laws, the traditional Burgers' equation, see [2], [4], [5], [15].

In the present paper we study the problem of the existence of a potential function ϕ for a given vector-function w, i.e. we give explicite conditions on w so that there exists a function ϕ such that $\nabla^{\phi}\phi = w$. In Theorem 10 we consider the problem in $\mathbb{H}^1$. In this case the system $\nabla^{\phi}\phi = w$ consists only in the scalar Burgers' equation $\phi_{y_1} + \phi\phi_t = w$. According to the results and proving strategies of [4], [5], [6], we cannot use the classical PDE's theory for conservation laws, because the solution ϕ of the problem $\nabla^{\phi}\phi = w$ is "a priori" only continuous. Indeed we need to introduce a new notion of weak solution of conservation laws, the broad* solution, see (9) for definition and [4], [5], [6], [7], [15]. Intuitively a broad solution is a continuous function which is a solution of the conservation law along the characteristic lines, in our case the exponential maps of $\nabla^{\phi}\phi$.

In $\mathbb{H}^n$, $n \geq 2$ the problem is more difficult. Given a regular vector function w, Theorem 11 gives explicite conditions among the components w_i's so that there exists a potential function ϕ such that $\nabla^{\phi}\phi = w$. In this case we ask more regularity on w with respect to the case of $\mathbb{H}^1$ because the conditions of Theorem 11 need the existence of the derivative $\dfrac{\partial \phi}{\partial t}$. Indeed the strategy of the proof is to linearize the problem $\nabla^{\phi}\phi = w$ using a function $\psi = \dfrac{\partial \phi}{\partial t}$, so that it is possible to apply classical ODEs' theory.

The structure of the paper is the following: in Section 2 we present notation and a short introduction to the $\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$. In particular we recall in Theorem 4 the problem of the existence of a potential function $f : \Omega \subset \mathbb{H}^n \to \mathbb{R}$ for the problem of the horizontal gradient $\nabla_{\mathbb{H}} f$. In Section 3 we present and prove the main results of the paper in Theorems 10 and 11. At the end of Section 3 we present some examples and remarks to explain clearly the conditions of Theorem 11.

2. Notation and Preliminary Results

We shall denote the points of $\mathbb{H}^n$ by $P = [z, t] = [x + iy, t]$, $z \in \mathbb{C}^n$, $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}$, and also by $P = (x_1, \ldots, x_n, y_1, \ldots, y_n, t) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, t)$. If $P = [z, t]$, $Q = [\zeta, \tau] \in \mathbb{H}^n$ and $r > 0$, following the notations of [18], we define the group operation

$$P \cdot Q := \left[z + \zeta, t + \tau - \frac{1}{2}\Im m(z \cdot \bar{\zeta})\right], \tag{1}$$

the family of non isotropic dilations $\delta_r(P) := [rz, r^2 t]$, for $r > 0$ and the group of left-translations $\tau_P(Q) = P \cdot Q$. We denote as $P^{-1} := [-z, -t]$ the inverse of P and as 0 the origin of $\mathbb{R}^{2n+1}$.

Moreover $\mathbb{H}^n$ can be endowed with the homogeneous norm $\|P\|_\infty := \max\{|z|, |t|^{1/2}\}$ and the distance d_∞ we shall deal with is defined as $d_\infty(P, Q) := \|P^{-1} \cdot Q\|_\infty$. From now on, $U_\infty(P, r)$ will be the open ball with centre P and radius r with respect to the distance d_∞. We notice that $U_\infty(P, r)$ is an Euclidean Lipschitz domain in $\mathbb{R}^n 2n + 1$.

$(\mathbb{H}^n, d_\infty)$ provides the simplest example of a metric space that is not Euclidean, even locally, but is still endowed with a sufficiently rich compatible underlying structure, due to the existence of intrinsic families of left translations and dilations respectively induced from the group law (1) and dilations. Indeed, the geometry of $\mathbb{H}^n$ is noneuclidean at every scale, since it was proved in [17] that there are no bilitschitz maps from $\mathbb{H}^n$ to any Euclidean space.

$\mathbb{H}^n$ is a Carnot group of step 2. Indeed its Lie algebra $\mathfrak{h}_n$ is (linearly) generated by

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \; Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t}, \text{ for } j = 1, \ldots, n; \; T = \frac{\partial}{\partial t}, \tag{2}$$

and the only non-trivial commutator relations are

$$[X_j, Y_j] = T, \qquad \text{for } j = 1, \ldots, n. \tag{3}$$

We could use the notation $W_i = X_i$ if $i \leq n$ and $W_i = Y_i$ if $n + 1 \leq i \leq 2n$, $W_{2n+1} = T$.

We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1, \ldots, X_n$, $Y_1, \ldots, Y_n$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [13]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $\mathrm{H}\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section φ of $H\mathbb{H}^n$ can be identified with a map $\varphi : \mathbb{H}^n \to \mathbb{R}^{2n+1}$. At each point $P \in \mathbb{H}$ the horizontal fiber is indicated as $H\mathbb{H}^n_P$ and each fiber can be endowed with the scalar product $\langle \cdot, \cdot \rangle_P$ and the associated norm $| \cdot |_P$ that make the vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ orthonormal.

If Ω is an open subset of $\mathbb{H}^n$ and $k \geq 0$ is a non negative integer, the symbols $C^k(\Omega)$, $C^\infty(\Omega)$ indicate the usual (Euclidean) spaces of real valued continuously differentiable functions. We denote by $C^k(\Omega; H\mathbb{H}^n)$ the set of all C^k-sections of $H\mathbb{H}^n$ where the C^k regularity is understood as regularity between smooth manifolds.

The similarity among some statements in $\mathbb{H}^n$ with others in $\mathbb{R}^{2n+1}$ is clear using intrinsic notions of gradient for functions $f : \mathbb{H}^n \to \mathbb{R}$ and of divergence for sections of $H\mathbb{H}^n$.

Definition 1. If Ω is an open subset of $\mathbb{H}^n$, $f \in C^1(\Omega)$ and $\varphi = (\varphi_1, \dots, \varphi_{2n}) \in C^1(\Omega; H\mathbb{H}^n)$, define $\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_n f,\ Y_1 f, \dots, Y_n f)$; $\div_{\mathbb{H}} \varphi := \sum_{j=1}^n X_j \varphi_j + Y_j \varphi_{n+j}$.

We shall denote by $C^k_{\mathbb{H}}(\Omega)$ the set of continuous real functions f in Ω such that $\nabla_{\mathbb{H}} f$ in the sense of distribution is of class C^{k-1} in Ω. Moreover, we shall denote by $C^k_{\mathbb{H}}(\Omega; H\mathbb{H}^n)$ the set of all sections φ of $H\mathbb{H}^n$ whose canonical coordinates φ_j belong to $C^k_{\mathbb{H}}(\Omega)$ for $j = 1, \dots, 2n$. We denote by $C^k(\Omega; H\mathbb{H}^n)$ the set of all C^k-sections of $H\mathbb{H}^n$ where the C^k regularity is understood as regularity between smooth manifolds. The notions of $C^k_c(\Omega; H\mathbb{H}^n)$, $C^\infty(\Omega; H\mathbb{H}^n)$ and $C^\infty_c(\Omega; H\mathbb{H}^n)$ are defined analogously.

It is well-know that $\nabla_{\mathbb{H}}$ acts as a gradient operator in $\mathbb{H}^n$. In particular

Lemma 2. *Let $\Omega \subseteq \mathbb{H}^n$ be a connected open set and let $f \in L^1_{loc}(\Omega)$ such that $\nabla_{\mathbb{H}} f = 0$ in the sense of distributions. Then $f \equiv cost$ in Ω.*

Let us now introduce the $\mathrm{curl}_{\mathbb{H}}$ operator. It was been explicitely given in [12] and [3] respectively for $n = 1$ and $n = 2$, using the theory and the language of differential forms in $\mathbb{H}^n$, for detailed calculations see [6], [16] too.

Definition 3. Let $F = (F_1, ..., F_{2n})$ be a smooth section of $H\mathbb{H}^n$, let us define if $n = 1$

$$\mathrm{curl}_{\mathbb{H}} F := (2W_1W_2F_1 - W_2W_1F_1 - W_1^2F_2\, ,\ W_1W_2F_2 - 2W_2W_1F_2 - W_2^2F_1); \quad (4)$$

$$\text{if } n \geq 2 \quad \mathrm{curl}_{\mathbb{H}} F := \left(F_{i,j}, \frac{1}{\sqrt{2}}(F_{h,h+n} - F_{h+1,h+1+n}) \right), \qquad (5)$$

with $1 \leq i < j \leq 2n$, $j \neq i + n$ and $h = 1, ..., n - 1$ and $F_{i,j} := (W_iF_j - W_jF_i)$.

For a better comprehension of the problem of this paper and in view of Theorem 11, let us study study the existence of a potential function $f : \Omega \to \mathbb{R}$

for the problem of the horizontal gradient $\nabla_{\mathbb{H}} f = F$ where $F := (F_1, \dots, F_{2n})$ is a vector-function defined by $F_j : \Omega \to \mathbb{R}$. Like in the euclidean setting, where the problem of the existence of a potential function is related to the De Rham complex theory, in the setting of $\mathbb{H}^n$ this problem is related to the Rumin complex theory, see [3], [16], [12]. The Rumin Theorem yields an exactitudes' result for 1-differential forms in $\mathbb{H}^n$.

Theorem 4. *Let $\Omega \subseteq \mathbb{H}^n$ be a simply connected open set and let $F = (F_1, ..., F_{2n})$ with $F_j \in \mathcal{D}'(\Omega)$ $j = 1, ..., n$. Then the following conditions are equivalent*

i *there exists $f \in \mathcal{D}'(\Omega)$ such that $\nabla_{\mathbb{H}} f = F$ in Ω in the sense of distributions.*

ii $\mathrm{curl}_{\mathbb{H}} F = 0$ *in Ω in the sense of distributions.*

An alternative proof of this result is given in [6], [12] and does not use the differential forms' language: it is obtained by the commutator relations of the vector fields X_j, Y_j, T.

Let us now introduce the notion of $\mathbb{H}$-regular hypersurfaces in $\mathbb{H}^n$, recalling the following definitions and preliminary results.

Definition 5. ([10]). We say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular hypersurface* if, for every $p \in S$, there exist a neighbourhood U of p and a function $f \in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}} f \neq 0$ and $S \cap U = \{q \in U : f(q) = 0\}$. The *horizontal normal* to S at p is $\nu_S(p) := -\dfrac{\nabla_{\mathbb{H}} f(p)}{|\nabla_{\mathbb{H}} f(p)|}$.

In the following we will denote $\mathbb{W} := \{(x, y, t) \in \mathbb{H}^n : x_1 = 0\}$ and we will write $(y_1, x_2, \dots, x_n, y_2, \dots, y_n, t)$ instead of $(0, x_2, \dots, x_n, y_1, \dots, y_n, t)$ if $n \geq 2$ and (y_1, t) instead of $(0, y_1, t)$ if $n = 1$.

We will use the notation $v = (v_2, \dots, v_n, v_{n+2}, \dots, v_{2n}) := (x_2, \dots, x_n, y_2, \dots, y_n) \in \mathbb{R}^{2n-2}$ too. We shall denote

$$I_r(A_0) := \left\{(y_1, t) \in \mathbb{W} : |y_1 - y_1^0| < r,\ |t - t^0| < r\right\}$$

for $A_0 = (y_1^0, t^0) \in \mathbb{W} \equiv \mathbb{R}^2$ if $n = 1$ and

$$\begin{aligned} I_r(A_0) \ &:= \big\{(y_1, x_2, \dots, x_n, y_2, \dots, y_n, t) \in \mathbb{W} : |y_1 - y_1^0| < r, \\ &\textstyle\sum_{i=2}^n [(x_i - x_i^0)^2 + (y_i - y_i^0)^2] < r^2,\ |t - t^0| < r\big\} \end{aligned}$$

for $A_0 = (y_1^0, x_2^0, \dots, x_n^0, y_2^0, \dots, y_n^0, t^0) \in \mathbb{W} \equiv \mathbb{R}^{2n}$ if $n \geq 2$.

The following Implicit Function Theorem for $\mathbb{H}$-regular hypersurfaces is proved in [10], [11].

Theorem 6. (Implicit Function Theorem) *Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in C^1_H(\Omega)$ be such that $X_1 f(0) > 0$, $f(0) = 0$. Let $S := \{[z,t] \in \Omega : f([z,t]) = 0\}$, then there exist a connected open neighbourhood $\mathcal{U}$ of 0 and there exists a unique continuous function $\phi : \overline{I_\delta(0)} \subset \mathbb{W} \to [-h, h]$ such that $S \cap \overline{\mathcal{U}} = \Phi\left(\overline{I_\delta(0)}\right)$, where $\delta, h > 0$ and Φ is defined as*

$$\Phi(y_1, v, t) = \left(\phi(y_1, v, t), v_2, \dots, v_n, y_1, v_{n+2}, \dots, v_{2n}, t - \tfrac{y_1}{2}\phi(y_1, v, t)\right)$$
if $n \geq 2$
$$\Phi(y_1, t) = \left(\phi(y_1, t), y_1, t - \tfrac{y_1}{2}\phi(y_1, t)\right) \qquad \text{if } n = 1.$$

By Theorem 6 we can see an $\mathbb{H}$-regular surfaces as an intrinsic graph. A set $S \subset \mathbb{H}^n$ is called X_1-*graph*, induced by a function $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$, if

$$S = \{A \cdot \phi(A)\, e_1 : A \in \omega\}. \tag{6}$$

Let us recall the following improvement of Theorem 6 contained in [2]:

Theorem 7. *Under the same assumption of Theorem 6, let $\mathfrak{B}\phi$ the distribution $\mathfrak{B}\phi := \dfrac{\partial \phi}{\partial y_1} + \dfrac{1}{2}\dfrac{\partial \phi^2}{\partial t}$ on $I_\delta(0)$, where ϕ and δ are given by Theorem 6. Then if $n = 1$*

$$\mathfrak{B}\phi = -\frac{Y_1 f}{X_1 f} \circ \Phi,$$

$$\text{if } n \geq 2 \qquad X_j\phi = -\frac{X_j f}{X_1 f} \circ \Phi, \quad Y_j\phi = -\frac{Y_j f}{X_1 f} \circ \Phi, \quad \mathfrak{B}\phi = -\frac{Y_1 f}{X_1 f} \circ \Phi$$

where the equalities must be understood in the sense of distributions on $I_\delta(0)$.

In [2] it has been proved that each $\mathbb{H}$- regular graph $\Phi(\omega)$ admits an intrinsic gradient $\nabla^\phi \phi \in C^0(\omega; \mathbb{R}^{2n})$, in the sense of distributions, which shares a lot of properties with the Euclidean gradient, and it is defined, in distributional sense, by

$$\begin{aligned} &W^\phi \phi := Y_1 \phi + \frac{1}{2} T(\phi^2), \\ &\nabla^\phi \phi := \begin{cases} (X_2\phi, \dots, X_n\phi, W^\phi\phi, Y_2\phi, \dots, Y_n\phi) & \text{if } n \geq 2 \\ W^\phi \phi & \text{if } n = 1 \end{cases}. \end{aligned} \tag{7}$$

We also denote by $\nabla^\phi := (\nabla^\phi_2, \dots, \nabla^\phi_{2n})$ the family of vector fields on $\mathbb{R}^{2n}$, $\nabla^\phi_j := X_j$ for $j \neq n+1$ and $\nabla^\phi_{n+1} = W^\phi := Y_1 + \phi T$. We use the notation $\widetilde{\nabla}_{\mathbb{H}}\phi := (X_2\phi, \dots, X_n\phi, Y_2\phi, \dots, Y_n\phi)$ too. Let us notice that $W^\phi\phi$ looks like the classical Burgers' operator $\dfrac{\partial \phi}{\partial y_1} + \phi\dfrac{\partial \phi}{\partial t}$. Therefore a correct notion of *"weak solution"* becomes fundamental in the study of the intrinsic gradient. Given a continuous vector function $w = (w_2, \dots, w_{2n}) : \omega \subset \mathbb{W} \to \mathbb{R}^{2n-1}$, we introduce the concept of broad* solution of the system

$$\nabla^\phi \phi = w \quad \text{in } \omega, \tag{8}$$

i.e. a continuous function $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ such that for every $A_0 \in \omega$, $\forall\, j = 2, ..., 2n$ there exists an exponential map,

$$\gamma_j^B(s) = \exp(s\nabla_j^\phi)(B) : [-\delta_2, \delta_2] \times \overline{I_{\delta_2}(A_0)} \to \overline{I_{\delta_1}(A_0)} \subset \omega \tag{9}$$

where $0 < \delta_2 < \delta_1$, $s \in [-\delta_2, \delta_2]$, such that $\forall\, B \in I_{\delta_2}(A_0)$

(E.1) $\gamma_j^B \in C^1([-\delta_2, \delta_2])$

(E.2) $\begin{cases} \dot{\gamma}_j^B = \nabla_j^\phi \circ \gamma_j^B \\ \gamma_j^B(0) = B \end{cases}$

(E.3) $\phi\left(\gamma_j^B(s)\right) - \phi\left(\gamma_j^B(0)\right) = \int_0^s w_j\left(\gamma_j^B(r)\right) dr \ \forall s \in [-\delta_2, \delta_2]$

Indeed let us recall the following characterizations of $\mathbb{H}$-regular graphs $\Phi(\omega)$, given respectively in [2] Theorem 1.3, [4] Theorem 1.2 and [5] Theorem 1.2 (see also [9] for a characterization in general Carnot groups).

Theorem 8. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent:*

(i) The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface and $\nu_S^1(p) < 0$ for all $p \in S$, where $\nu_S(p) = \left(\nu_S^1(p), \dots, \nu_S^{2n}(p)\right)$ is the horizontal normal to S at p.

(ii) There exist $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ and a family $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ such that, as $\epsilon \to 0^+$, $\phi_\epsilon \to \phi$ and $\nabla^{\phi_\epsilon}\phi_\epsilon \to w$ in $L^\infty_{loc}(\omega)$ and (8) holds in the sense of distributions.

(iii) There exists $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a broad solution of the system (8).*

Moreover, for all $p \in S$ we have

$$\nu_S(p) = \left(-\frac{1}{\sqrt{1 + |\nabla^\phi \phi|^2}}, \frac{\nabla^\phi \phi}{\sqrt{1 + |\nabla^\phi \phi|^2}}\right)(\Phi^{-1}(p)).$$

(iv) *There exists* $w = (w_2, \dots, w_{2n}) \in C^0(\omega; \mathbb{R}^{2n-1})$ *such that* ϕ *is a distributional solution of the system* $\nabla^\phi \phi = w$

In view of or results, let us recall the following theorem, see [11], where we write

$$\pi_P(x, y, t) := \sum_{j=1}^{n} \big(x_j X_j(P) + y_j Y_j(P)\big) \text{ for } P, (x, y, t) \in \mathbb{H}^n.$$

Theorem 9. (Whitney Extension Theorem) *Let* $F \subset \mathbb{H}^n$ *be a closed set, and let* $f : F \to \mathbb{R}$, $k : F \to H\mathbb{H}^n$ *be two continuous functions. We set*

$$R(Q, P) := \frac{f(Q) - f(P) - \langle k(P), \pi_P(P^{-1} \cdot Q)\rangle_P}{d(P, Q)},$$

and, if $K \subset F$ *is a compact set,* $\rho_K(\delta) := \sup\{|R(Q, P)| : P, Q \in K,\ 0 < d_\infty(P, Q) < \delta\}$. *If* $\rho_K(\delta) \to 0$ *as* $\delta \to 0$ *for every compact set* $K \subset F$, *then there exist* $\tilde{f} : \mathbb{H}^n \to \mathbb{R}$, $\tilde{f} \in C^1_H(\mathbb{H}^n)$ *such that* $\tilde{f}_{|F} \equiv f$ *and* $\nabla_{\mathbb{H}} \tilde{f}_{|F} \equiv k$.

3. Main Results and Proofs

Let us consider the problem of the existence of a potential function of the intrinsic gradient $\nabla^\phi \phi$ in $\mathbb{H}^1$. For this pourpose in the following we will write $\phi \in h^{1/2}(D)$ if f is continuous on $D \subset \mathbb{R}^m$ and

$$\lim_{r \to 0} \sup \left\{ \frac{|f(\xi) - f(\zeta)|}{|\xi - \zeta|^{1/2}} : \xi, \zeta \in \overline{D},\ 0 < |\xi - \zeta| < r \right\} = 0.$$

The notion of $h^{1/2}_{loc}(D)$ is defined analogously.

Theorem 10. *Let* $A_0 = (y_1^0, t^0) \in \mathbb{R}^2 = \mathbb{R}_{y_1} \times \mathbb{R}_t$. *Let* $\phi_0 \in h^{\frac{1}{2}}([t^0 - r_0, t^0 + r_0])$, $w_0 \in C^0([t^0 - r_0, t^0 + r_0])$ *be given. Then there exist* ϕ, $w \in C^0\left(\overline{I_{r_0}(A_0)}\right)$ *such that* ϕ *is a broad* solution of the initial value problem*

$$\begin{cases} W^\phi \phi = w & \text{in } I_{r_0}(A_0) \\ \phi(y_1^0, t) = \phi_0(t) & \forall t \in [t^0 - r_0, t^0 + r_0] \end{cases} \qquad \text{if } \ n = 1 \tag{10}$$

for r_0 *small enough and* $w \equiv w_0$ *on* $[t^0 - r_0, t^0 + r_0]$.

Proof. First let us observe without loss of generality we can assume that $A_0 = (0, 0)$. Otherwise let us consider $\phi^*(y_1, t) = \phi(y_1 - y_1^0, t - t^0)$ and the associated initial value problem.

$$\begin{cases} W^{\phi^*}\phi^* = w^* & \text{in } I_{r_0}((0,0)) \\ \phi^*(0,t) = \phi_0^*(t) & \forall t \in [-r_0, r_0] \end{cases} \tag{11}$$

where $w^*(y_1,t) = w(y_1 - y_1^0, t - t^0)$, $\phi_0^*(t) = \phi_0(t - t^0)$, $(y_1,t) \in I_{r_0}((0,0))$, $t \in [-r_0, r_0]$. Then it is easy to see by definition that ϕ is a broad* solution of (10) if and only if ϕ^* is a broad* solution of (11).

Using the notation of Theorem 9 let $F := \{(\phi_0(t), 0, t) : t \in [-r_0, r_0]\}$, $f \equiv 0$, $k : F \to H\mathbb{H}^1 \equiv \mathbb{R}^2$,

$$k(\xi,\eta,\tau) := \left(1, -w_0\left(\eta, \tau + \frac{\xi\eta}{2}\right)\right) \text{ if } (\xi,\eta,\tau) \in F.$$

Let $Q = (\phi_0(t'), 0, t')$, $P = (\phi_0(t), 0, t)$ with $t \neq t' \in [-r_0, r_0]$, then

$$|R(Q,P)| = \frac{|f(Q) - f(P) - \langle k(P), \pi_p(P^{-1}\cdot Q)\rangle_P|}{d_\infty(P,Q)} = \tag{12}$$

$$= \frac{|-(\phi_0(t') - \phi_0(t)) + w_0(t)\cdot 0|}{\max\left\{|\phi_0(t') - \phi_0(t)|, \sqrt{|t'-t|}\right\}} \leq \frac{|\phi_0(t') - \phi_0(t)|}{\sqrt{|t'-t|}}$$

Since $\phi_0 \in h^{\frac{1}{2}}([-r_0, +r_0])$, for compact set $K \subseteq F$, by (12) we get $\lim_{\delta\to 0^+} \rho_K(\delta) = 0$.

Then by Whitney's extension Theorem 9 there exists $\tilde{f} : \mathbb{H}^1 \to \mathbb{R}$, $\tilde{f} \in C^1_{\mathbb{H}}(\mathbb{H}^1)$ such that

$$\tilde{f} = 0 \quad \text{and} \quad \nabla_{\mathbb{H}}\tilde{f} = k \quad \text{in } F. \tag{13}$$

Let $P_0 := (\phi_0(0), 0, 0) \in F$, $g(P) := \tilde{f}(P_0 \cdot P)$ for $P \in \mathbb{H}^1$, $S = \{P \in \mathbb{H}^1 : g(P) = 0\}$. Since $g \in C^1_{\mathbb{H}}(\mathbb{H}^1)$, $0 \in S$, $X_1 g(0) = 1$ by Implicit Function Theorem 6 and Proposition 7 there exists an open neighborhood $\mathcal{U} \subseteq \mathbb{H}^1$ of 0 such that

$$S \cap \mathcal{U} \text{ is } \mathbb{H}\text{-regular}. \tag{14}$$

Moreover there exist $\delta > 0$ and an unique continuous function $\widetilde{\phi} : \widetilde{I} = [-\delta, \delta] \times [-\delta^2, \delta^2] \to \mathbb{R}$ such that

$$\widetilde{\Phi}\left(\widetilde{I}\right) = G^1_{\mathbb{H},\widetilde{\phi}}\left(\widetilde{I}\right) = S \cap \overline{\mathcal{U}} \tag{15}$$

if $\widetilde{\Phi}(\tilde{y_1}, \tilde{t}) = (0, \tilde{y_1}, \tilde{t}) \cdot \widetilde{\phi}(\tilde{y_1}, \tilde{t}) e_1$ with $(\tilde{y_1}, \tilde{t}) \in \widetilde{I}$

$$\mathfrak{B}\widetilde{\phi} = \widetilde{w} \quad \text{in } \widetilde{I} \tag{16}$$

in the sense of distributions, where

$$\widetilde{w}(\tilde{y_1},\tilde{t}) = \left(-\frac{Y_1 g}{X_1 g}\circ\Phi\right)(\tilde{y_1},\tilde{t}) = -\frac{Y_1\tilde{f}}{X_1\tilde{f}}\left(P_0\cdot\widetilde{\Phi}(\tilde{y_1},\tilde{t})\right).$$

Let us perform now the change of variable $\psi:\widetilde{I}\to\mathbb{R}^2$

$$\psi(\tilde{y_1},\tilde{t}) = \left(\tilde{y_1},\tilde{t}+\phi_0(0)\tilde{y_1}\right) = (y_1,t)$$

and let $I:=\psi\left(\widetilde{I}\right)$. Let us define $\phi(y_1,t):=\phi_0(0)+\widetilde{\phi}(y_1,t-\phi_0(0)y_1)$, $(y_1,t)\in I$. Then by (15)

$$S_0 := \tau_{P_0}\left(S\cap\overline{\mathcal{U}}\right) = \tau_{P_0}\left(G^1_{\mathbb{H},\widetilde{\phi}}(\widetilde{I})\right) = G^1_{\mathbb{H},\phi}(I). \tag{17}$$

Let $r_0>0$ so small such that $\overline{I_{r_0}(0,0)}\subset I$. By (13), (14) and (17) we get that

$$\phi(0,t)=\phi_0(t)\qquad \forall t\in[-r_0,r_0], \tag{18}$$

$$G^1_{\mathbb{H},\phi}(I_{r_0}(0,0))\ \text{is}\ \mathbb{H}\text{-regular}. \tag{19}$$

On the other hand it is easy to see that by (16)

$$\mathfrak{B}\phi = w \qquad \text{in}\ I_{r_0}(0,0) \tag{20}$$

in the sense of distributions, where $w(y_1,t)=\widetilde{w}\left(\psi^{-1}(y_1,t)\right)$, $(y_1,t)\in I_{r_0}(0,0)$.

Thus by (19) and (20) and Theorem 8 we get

$$\phi\in h^{\frac{1}{2}}_{loc}\left(I_{r_0}(0,0)\right) \tag{21}$$

$$W^{\phi}\phi = w \qquad \text{in}\ I_{r_0}(0,0). \tag{22}$$

Finally by (18), (19), (22) and Theorem 8 we get that ϕ is a broad* solution of $W^{\phi}\phi=w$ in ω. □

A local uniqueness result for broad* solutions of (10) uniformly bounded in ω is given in [4] (see Theorem 3.8) using the theory of entropy solutions, a class of distributional solutions which are admissible by a physical viewpoint, see [15].

Let us now consider the problem in $\mathbb{H}^n$ with $n\geq 2$. In this case there are regular solutions of the system (8) provided compatibility's conditions among the components w_i's.

Theorem 11. *Let us denote $\omega = (y_1^0 - r_0, y_1^0 + r_0) \times \hat{\omega}$ where $\hat{\omega} \subseteq \mathbb{R}^{2n-2}$ is an open set and $P_0 = (y_1^0, v^0, t^0)$ and $r > 0$. Let $w = (w_2, \dots, w_{2n}) \in C^2(\omega; \mathbb{R}^{2n-1})$, $n \geq 2$. Let us define*

$$\psi(y_1, v, t) := \left(\widetilde{X}_2 w_{2+n} - \widetilde{Y}_2 w_2\right)(y_1, v, t);$$

$$E(y_1, v, t) := e^{\left(-\int_{y_1^0}^{y_1} \psi(y_1', v, t)\, dy_1'\right)}$$

$$I(y_1, v, t) := \int_{y_1^0}^{y_1} \frac{w_{n+1}(y_1', v, t)}{E(y_1', v, t)}\, dy_1';$$

$$E_1(y_1, v, t) := E(y_1, v, t) I(y_1, v, t);$$

$$\underline{a} = (a_2, ..., a_n, a_{n+2}, ..., a_{2n}) \quad a_j := \frac{\widetilde{X}_j E}{E};$$

$$\underline{b} = (b_2, ..., b_n, b_{n+2}, ..., b_{2n}) \quad b_j := \frac{w_j - \widetilde{X}_j E_1}{E};$$

where $y_1^0 \in \mathbb{R}$ is fixed and $\hat{w}_{n+1} := (w_2, ..., w_n, w_{n+2}, ..., w_{2n})$. Then the following statements are equivalent:

i *There exists $\phi \in C^2(\omega)$ such that $\nabla^\phi \phi = w$ in ω, i.e. (8);*

ii *There exists $C \in C^2(\hat{\omega})$ such that*

$$\widetilde{\nabla}_{\mathbb{H}} C(v, t) = \hat{w}_{n+1}(y_1^0, v, t) \qquad \forall\, (v, t) \in \hat{\omega}, \tag{23}$$

$$\underline{a}(y_1, v, t) C(v, t) = \underline{b}(y_1, v, t) - \underline{b}(y_1^0, v, t). \tag{24}$$

$\forall y_1 \in (y_1^0 - r, y_1^0 + r)$, $\forall (v, t) \in \hat{\omega}$. Moreover ϕ and C are linked by the relation

$$\phi(y_1, v, t) = E_1(y_1, v, t) + E(y_1, v, t) C(v, t). \tag{25}$$

Proof. It is not restrictive to assume $y_1^0 = 0$.

i $\Rightarrow$ ii Let us assume that there exists $\phi \in C^2(\omega)$ such that $W^\phi \phi = w$ in ω. Let us observe that

$$\frac{\partial \phi}{\partial t} = T\phi = [X_2 Y_2 - Y_2 X_2]\phi = X_2 w_{2+n} - Y_2 w_2 =: \psi. \tag{26}$$

Thus we can linearize the system getting

$$\widetilde{\nabla}_{\mathbb{H}} \phi = \hat{w}_{n+1} \qquad \text{in}\, \omega \tag{27}$$

$$\frac{\partial \phi}{\partial y_1} + \phi\psi = w_{n+1} \qquad \text{in}\, \omega \tag{28}$$

For fixed $(v,t) \in \hat{\omega}$, by the uniqueness of linear ODE (28), we can represent ϕ as

$$\phi(y_1, v, t) = E_1(y_1, v, t) + E(y_1, v, t)\phi(0, v, t) \tag{29}$$

Let us denote

$$C(v,t) := \phi(0, v, t) \qquad (v,t) \in \hat{\omega}$$

and let us prove (24). By (27) and (29) we get that

$$\hat{w}_{n+1} = \widetilde{\nabla}_{\mathbb{H}}(E_1 + EC) = \widetilde{\nabla}_{\mathbb{H}} E_1 + \widetilde{\nabla}_{\mathbb{H}} E \cdot C + E\widetilde{\nabla}_{\mathbb{H}} C$$

and then $\forall\, (y_1, v, t) \in \omega$

$$\widetilde{\nabla}_{\mathbb{H}} C(v,t) + \underline{a}(y_1, v, t)C(v,t) = \underline{b}(y_1, v, t). \tag{30}$$

By choosing $y_1 = 0$, since $\underline{b}(0, v, t) = \hat{w}_{n+1}(0, v, t)$ and $\underline{a}(0, v, t) \equiv 0$ we get at once (23) and (24).

ii $\Rightarrow$ **i** Let us assume that there exists $C \in C^2(\hat{\omega})$ such that (23) and (24) hold. Let us define ϕ as in (29) with $C(y_1, t) \equiv \phi(0, v, t)$, then it is easy to verify that $W^\phi \phi = w$ in ω. □

Remark 12. We need the hypotesis $\phi \in C^2(\omega)$ for the existence of $T\phi$. In general we cannot use the proof' strategy of Theorem 4. Indeed "a priori" if ϕ is only continuous there is not a good definition of commutator for the fields X_j, W^ϕ, Y_j, because ϕ could be not Lipschitz continuous.

Remark 13. Let us explicitly point out the system (8) differs from system $\nabla_{\mathbb{H}} \phi = V$. For instance, let us assume that $w \in C^2(\mathbb{R}^{2n}, \mathbb{R}^{2n-1})$ such that

$$\hat{w}_{n+1}(y_1, v, t) \equiv 0 \qquad \text{in } \omega := \mathbb{R}^{2n} \qquad \text{and} \tag{31}$$

$$w_{n+1}(y_1, v, t) = w_{n+1}(v, t) \tag{32}$$

with

$$\widetilde{\nabla}_{\mathbb{H}} w_{n+1} \not\equiv 0 \qquad \text{in } \omega \tag{33}$$

Then compatibility's condition (23) is satisfied with $C \equiv$ cost in $\hat{\omega} := \mathbb{R}^{2n-1}$ by Lemma 2. On the other hand since $\psi \equiv 0$ we have
$E \equiv 1$, $E_1(y_1, v, t) = I(y_1, v, t) = (y_1 - y_1^0)w_{n+1}(v, t)$, $\underline{a} \equiv 0$,
$\underline{b}(y_1, v, t) = -(y_1 - y_1^0)\widetilde{\nabla}_{\mathbb{H}} w_{n+1}(v, t)$. Then by (33) $\underline{b}(y_1, v, t) - \underline{b}(y_1^0, v, t) = -\widetilde{\nabla}_{\mathbb{H}} E_1(y_1, v, t) = -(y_1 - y_1^0)\widetilde{\nabla}_{\mathbb{H}} w_{n+1}(v, t) \not\equiv 0$.
Therefore compatibility's condition (24) is not satisfied and by Theorem 11 there are not C^2 solutions of the system (8).

We are going now to give some explicit regular solutions of the system (8) in $\mathbb{H}^2$ by means of Theorem 11. We will assume in the examples below that $\phi \in C^2(\omega)$ is a solution of system (8), $\omega = (y_1^0 - r_0, y_1^0 + r_0) \times \hat{\omega} = (y_1^0 - r_0, y_1^0 + r_0) \times U(v^0, r_0) \times (t^0 - r_0, t^0 + r_0)$, where $U(v^0, r_0)$ is the euclidean open ball in $\mathbb{R}^{2n-2}$ with centre v_0 and radius r_0. We will use the same notations of Theorem 11.

Remark 14. Let us assume that $\exists \phi \in C^2(\omega)$ solution of (8). If $C(v,t) \equiv 0$ in $\hat{\omega}$ then $\underline{b}(y_1, v, t) \equiv 0$ in ω.

Indeed let us notice that by (24) we have

$$\underline{b}(y_1, v, t) - \underline{b}(0, v, t) = a(y_1, v, t)C(v, t) = 0 \quad \forall (y_1, v, t) \in \omega, \tag{34}$$

then by (23)

$$\widetilde{\nabla}_{\mathbb{H}} C(v, t) = \hat{w}_3(y_1^0, v, t) \equiv 0 \quad \forall (y_1, v, t) \in U(v_0, r_0) \times (t^0 - r_0, t^0 + r_0)$$

and by (25) $\phi(y_1, v, t) = E_1(y_1, v, t)$ in ω. Let us observe that by definition

$$E(y_1^0, v, t) \equiv 1 \qquad E_1(y_1^0, v, t) \equiv 0,$$

therefore $\underline{b}(y_1^0, v, t) \equiv 0$ and by (34) we conclude $\underline{b}(y_1, v, t) \equiv 0$ in ω. □

Remark 15. Let us assume that $\underline{a}(y_1, v, t) \equiv 0$ in ω and that $\exists \phi \in C^2(\omega)$ solution of (8), then it is of the type $\phi(y_1, v, t) = \psi(y_1)t + k(y_1, v)$.

Indeed by the definition of $\underline{a}$

$$0 = \widetilde{\nabla}_{\mathbb{H}} E(y_1, v, t) = -E(y_1, v, t) \int_{y_1^0}^{y_1} \widetilde{\nabla}_{\mathbb{H}} \psi(y_1, v, t)\, dy_1'$$

$\forall y_1 \in (y_1^0 - r_0, y_1^0 + r_0)$, $\forall (v, t) \in \hat{\omega}$. Since for fixed $(v, t) \in \hat{\omega}$ $\widetilde{\nabla}_{\mathbb{H}} \psi(\cdot, v, t) \in C^0((y_1^0 - r_0, y_1^0 + r_0); \mathbb{R}^2)$ we can conclude that

$$\widetilde{\nabla}_{\mathbb{H}} \psi \equiv 0 \quad \text{in } \omega \tag{35}$$

By (35) and Lemma 2 we get that $\psi = \psi(y_1)$ $y_1 \in (y_1^0 - r_0, y_1^0 + r_0)$. Therefore by Theorem 11 and Theorem 4 there exists $\phi \in C^2(\omega)$ solution of the system (8) of the type

$$\phi(y_1, v, t) = \psi(y_1)t + k(y_1, v) \quad \forall (y_1, v, t) \in \omega.$$

□

Example 16. Let us assume now $w = w(y_1, v)$. Let us observe that in this case $\psi = \psi(y_1, v)$, $E = E(y_1, v)$, $E_1 = E_1(y_1, v)$, $\underline{a} = \underline{a}(y_1, v)$ and $\underline{b} = \underline{b}(y_1, v)$. By Theorem 11 each solution $\phi \in C^2(\omega)$ of the system (8) is of the type

$$\phi(y_1, v, t) = E_1(y_1, v) + E(y_1, v)C(v, t) \tag{36}$$

and we have

$$\widetilde{\nabla}_{\mathbb{H}} C(v, t) = \hat{w}_3(y_1^0, v) \tag{37}$$

$$\underline{a}(y_1, v)C(v, t) = \underline{b}(y_1, v) - \underline{b}(y_1^0, v) \tag{38}$$

$\forall y_1 \in (y_1^0 - r_0, y_1^0 + r_0)$, $\forall v \in U(v_0, r_0)$, $\forall t \in (t^0 - r_0, t^0 + r_0)$. Recalling that $v = (x_2, y_2)$, by Theorem 4 the condition (37) is equivalent to

$$0 = \left(\frac{\partial^2 w_4}{\partial x_2^2} - \frac{\partial^2 w_2}{\partial y_2 \partial x_2} \right)(y_1^0, v) = \left(\frac{\partial^2 w_4}{\partial x_2 \partial y_2} - \frac{\partial^2 w_2}{\partial y_2^2} \right)(y_1^0, v) \tag{39}$$

Let us assume now that $\underline{a}(y_1, v) \not\equiv 0$ in ω. Then by (38) we get that $C(v, t) = C(v)$. Thus by (36) $\phi(y_1, v, t) = \phi(y_1, v)$ provided (39) holds.
On the other hand let $\underline{a}(y_1, v) \equiv 0$ in ω. In this case ϕ could depend on t. For instance, it is immediate to see that

$$\phi(y_1, v, t) = \frac{t}{y_1 + 2} \qquad (y_1, v, t) \in (-1, 1) \times U(0, 1) \times (-1, 1)$$

is a solution of the system (8) with

$$w(y_1, v, t) := w(y_1, v) = \left(-\frac{y_2}{2(y_1 + 2)}, 0, \frac{x_2}{2(y_1 + 2)} \right) \ \forall (y_1, v, t) \in \omega.$$

Example 17. In the same assumptions of example 16, if $w = w(y_1)$ then a solution ϕ of $\nabla^\phi \phi = w$ is such that $\dfrac{\partial \phi}{\partial t} = 0$. Indeed let us observe that, since $w = w(y_1)$, $\psi = X_2 w_{2+n} - Y_2 w_2 = 0$. We conclude so $\dfrac{\partial \phi}{\partial t} = \psi = 0$.

Example 18. In the case $w = w(v, t)$ we can find $\phi(y_1, v, t)$ solutions of $\nabla^\phi \phi = w$ such that $\dfrac{\partial \phi}{\partial y_1} \neq 0$. Let us assume in $\mathbb{H}^2$ $w = \left(-\frac{y_2}{2}, t, \frac{x_2}{2} \right)$, $\omega = (-1, 1)^4$. Then $\phi(y_1, v, t) = t + e^{-y_1}$ is a solution of the problem $\nabla^\phi \phi = w$ in ω.

Example 19. In the case $w = w(y_1, t)$ we can find $\phi(y_1, v, t)$ solutions of $\nabla^\phi \phi = w$ such that $\dfrac{\partial \phi}{\partial v_i} \neq 0$ for some $i \in \{2, ..., 2n\}$. Let us assume in $\mathbb{H}^2$ $w = (1, 2y_1, 0)$, $\omega = (-1, 1)^4$. Then $\phi(y_1, v, t) = x_2 + y_1^2$ is a solution of the problem $\nabla^\phi \phi = w$ in ω.

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