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TOPOLOGICAL SPECTRUM OF THE HARMONIC OSCILLATOR

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Abstract: We present a derivation of the canonical energy spectrum of the harmonic oscillator by using the approach of topological quantization. The topological spectrum is derived by analyzing the Euler characteristic class of a particular principal fiber bundle in which the base space corresponds to the configuration manifold, equipped with a Jacobi metric, and the standard fiber is represented by the rotation group. The resulting topological spectrum is shown to be equivalent to the energy spectrum following from the application of the standard procedure of canonical quantization.

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1. Introduction

Although the idea of using topology to find quantum properties of classical systems is not new, it has been used very actively only in the context of diverse monopole and instanton configurations [1, 2, 3, 5, 4]. In fact, it is widely

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believed that topological quantization can be applied only in these simple field configurations, because only in these cases it is possible to use the existing infinite number of degrees of freedom to construct the particular topological spaces in which quantum information is encoded. In the case of classical configurations of gauge theories, the underlying fiber bundle structure is well known and has been studied in detail [6], but in the case of classical gravitational fields a strict existence proof was provided only recently [7, 8]. In all these field configurations, the corresponding infinite number of degrees of freedom are contained in a non-degenerate metric structure that essentially suggests the existence of a geometric approach . Consequently, in the case of classical systems with a finite number of degrees of freedom, one would expect that topological quantization cannot be applied. In this work we will show that this is not true.

To this end, we will study one of the simplest mechanical systems, namely the one-dimensional harmonic oscillator which is characterized by one single degree of freedom. To apply the procedure of topological quantization in this case, it is necessary to introduce the analogous of the fiber bundle structure that characterizes classical field configurations. The standard fiber bundle structure of Lagrangian mechanics is not suitable for our purposes. In fact, from the general Lagrangian $L = K - V$ it is possible to extract the kinetic metric structure $K = h_{ij}\dot{q}^i\dot{q}^j/2$ that can be assigned to the corresponding configuration space. The fiber bundle structure of the tangent space follows then in the standard manner [10]. However, this construction does not take into account the information contained in the potential V which is crucial in the process of quantization. For this reason, we propose here to use the equivalent formalism of Maupertuis which assigns a Jacobi metric [10] to each mechanical system, taking into account the explicit form of the potential. We will show that in fact it is possible to derive the energy spectrum of the harmonic oscillator by analyzing the topological invariants of the corresponding Maupertuis principal fiber bundle.

2. The Approach of Topological Quantization

The method of topological quantization can be applied to any field configuration whose geometrical structure allows the existence of a principal fiber bundle with a connection [9]. In the case of gravitational systems with an infinite number of degrees of freedom, a theorem proves the existence and uniqueness of such a bundle [8]. Indeed, the theorem states that any solution of Einstein's equations minimally coupled to any gauge matter field can be represented geometrically as a principal fiber bundle with the spacetime as the base space. The structure group (isomorphic to the standard fiber) follows from the invariance of the metric of the base space with respect to Lorentz transformations, in the case of a vacuum solution, or with respect to a transformation of the gauge group, in the case of a gauge matter field. The topological invariants of the corresponding principal fiber bundle lead to a discretization of the parameters entering the metric of the base space. In the approach of topological quantization this is the kind of discretization we are interested in. For the proof of the above theorem it is very important that the base space is equipped with a metric whose invariance determines the standard fiber. In the search for a similar structure in classical mechanics we found that Maupertuis formalism provides a natural metric which can be used to fix the base space.

Consider a classical conservative system with k degrees of freedom described by the Lagrangian (summation over repeated indices)

$$L = \frac{1}{2} h_{ij} \dot{q}^i \dot{q}^j - V(q) \ , \tag{1}$$

where $i, j = 1, ..., k$. Although the evolution of this system can be completely described within the Lagrangian formalism by varying the action $S = \int L dt$, we will use Maupertuis formalism which is based upon the reduced action

$$S_0 = \int ds \ , \quad \text{with} \quad ds^2 = 2(E - V) h_{ij} dq^i dq^j \ , \tag{2}$$

where E is the total energy. The equations following from the variation $\delta S_0 = 0$, together with the expression for the time parameter in terms of the reduced action, completely describe the evolution of the system [10]. The reduced action (2) defines the natural Jacobi metric

$$g_{ij} = 2(E - V) h_{ij} \ , \tag{3}$$

which we use to specify the base space B. The physical trajectories of the system corresponds then to geodesics on the base space. In general, we can see that to any physical system with k degrees of freedom in classical mechanics corresponds an $k-$dimensional Riemannian space B with metric g_{ij}. The potential $V(q)$ is used to characterize different physical systems.

Most systems in classical mechanics are invariant with respect to Galilean transformations. For the sake of simplicity, we limit ourselves here to systems which are invariant with respect to rotations only. Then, to each point of the base space B we can associate a standard fiber $SO(k)$. Furthermore, if we identify the structure group G with the Lie group $SO(k)$, we have all the constituents of a $k(k+1)/2-$dimensional principal fiber bundle P. This geometric construction is suggested in a natural manner by the elements of the classical systems and its symmetries.

According to [8, 9], the topological quantization of a classical physical system follows from the investigation of the topological invariants of P. In the present case, the only invariant characteristic class [11] is the Euler class $e(P)$ which is given in terms of the components of the curvature 2-form ${R^i}_j$ of B and whose integration yields the integer characteristic number, say, n

$$\int e(P) = \frac{(-1)^m}{2^{2m}\pi^m m!}\int \epsilon_{i_1 i_2 \ldots i_{2m}} {R^{i_1}}_{i_2}\wedge {R^{i_3}}_{i_4}\wedge\cdots\wedge {R^{i_{2m-1}}}_{i_{2m}} = n, \tag{4}$$

where $2m = n$.

Consider the simple example of a free particle, i.e. $V(q) = 0$. Then, the metric on the base space B is $g_{ij} = Eh_{ij}$ and the corresponding trajectories must be straight lines, i.e. the metric g_{ij} must be flat. This implies that there exists a coordinate system in which $h_{ij} = \delta_{ij}$ is the Euclidean metric. For zero curvature the Euler class vanishes and $n = 0$. We interpret this result as showing that a free particle is not quantized from the point of view of topological quantization. This is in accordance with the results of canonical quantization in quantum mechanics.

In the general case $V(q) \neq 0$ we see that g_{ij} is a conformally flat metric, with conformal factor $2(E-V)$, for which clearly the curvature is non zero, the Euler class does not vanish and, according to Eq.(4), the quantization is not trivial. In classical mechanics, different potentials $V_1(q) \neq V_2(q)$ correspond to different mechanical systems. Since the Jacobi metric (3) contains the explicit form of the potential, the corresponding principal fiber bundles P_1 and P_2 must be different. We conclude that for a given mechanical system we can construct a principal fiber bundle that contains all the information encoded in the kinetic metric h and in the potential $V(q)$ and that differentiates this particular system from any other mechanical system.

3. The Harmonic Oscillator

Consider two harmonic oscillators of the same mass m

$$L = \frac{1}{2}m\left(\dot{q}_1^2 + \dot{q}_2^2\right) - \frac{1}{2}\left(k_1 q_1^2 + k_2 q_2^2\right) \ , \tag{5}$$

where k_1 and k_2 are the corresponding oscillator constants. Then, $h_{ij} = m\delta_{ij}$ and the metric components of the base space B read

$$g_{ij} = 2m\left(E - \frac{1}{2}k_1 q_1^2 - \frac{1}{2}k_2 q_2^2\right)\delta_{ij} := e^{\phi}\delta_{ij} \ . \tag{6}$$

This physical system is invariant with respect to transformations of the group $SO(2)$ which is taken as the structure group and standard fiber of the principal bundle P. The corresponding Euler class can be expressed as

$$e(P) = -\frac{1}{2\pi}R^1{}_2 = \frac{1}{4\pi}\left(\frac{\partial^2\phi}{\partial q_1^2} + \frac{\partial^2\phi}{\partial q_2^2}\right) dq^1 \wedge dq^2 \ . \tag{7}$$

The calculation of the characteristic number is straightforward, but the resulting expression is quite cumbersome. To simplify the analysis we consider the special case $k_2 = 0$, and let $k_1 = k$ and $q_1 = q$. Then

$$\int e(P) = -\frac{kb}{4}\int \frac{E + \frac{1}{2}kq^2}{(E - \frac{1}{2}kq^2)^2} dq = n \ , \tag{8}$$

where we choose as $b\pi$ the constant resulting from the integration over q_2. Integrating over q within the interval $[-q_0, q_0]$, we obtain

$$\frac{bq_0}{q_0^2 - a^2} = n \ , \qquad a^2 = \frac{2E}{k} \ , \tag{9}$$

where a represents the classical turning point. This represents a relationship among the parameters describing the harmonic oscillator, i.e., the energy E, the constant k and q_0 that depends on the former two. This relationship is unique and gives us information about the discrete nature of the system from the point of view of topological quantization. We call this the topological spectrum of the harmonic oscillator. On the other hand, the canonical formalism provides us with a unique canonical spectrum for the energy of the system, and we aim for a direct relation between this and the topological spectrum. This, however, requires an exact definition of quantum states in the context of topological quantization, a concept which is still under construction and beyond the scope of the present work. Nevertheless, a simple way to show the equivalence is to choose the limit of integration q_0 as

$$q_0 = \frac{1}{C} - \sqrt{\frac{1}{C^2} + a^2} \ , \quad C = \frac{2}{b}\left(\frac{E}{\hbar\omega} - \frac{1}{2}\right) \ , \quad \omega = \sqrt{\frac{k}{m}} \tag{10}$$

which for any positive finite value of C reduces the topological spectrum (9) to the canonical spectrum

$$E = \hbar\omega\left(n + \frac{1}{2}\right) . \tag{11}$$

This means that there exists an $1-1$ relationship between the canonical and the topological spectra. An analysis of the limiting cases shows that the choice (10) is physically meaningful. Indeed, when $E >> \hbar\omega$ then $\frac{1}{C} \rightarrow 0$, and $q_0 \rightarrow a$, that is to say q_0 tends to the turning point and we recover the classical limit. Moreover, in the limit $E \rightarrow 0$, the turning point goes to zero and we have that $q_0 \rightarrow 0$, as expected.

The above results show that it is possible to obtain the canonical energy spectrum of the harmonic oscillator by using the approach of topological quantization. Moreover, in the case of a free particle the calculation of the corresponding characteristic number shows that no discrete topological spectrum exists, a result which, again, is in accordance with the procedure of canonical quantization. The examples analyzed in this work show that in fact topological quantization can be applied even in cases where the number of independent degrees of freedom is finite. For a given mechanical system, characterized by an explicit potential function, we saw that it is possible to construct a principal fiber bundle which contains all the information encoded in the potential. Although we use only classical concepts of mechanical systems for our geometric construction, we have shown that it is possible to derive a discrete spectrum which is usually associated with quantization. We expect to explore in the future the possibility of deriving further quantum aspects by using only the underlying fiber bundle structure.

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References

[1] P.A.M. Dirac, Quantized singularities in the electromagnetic field, *Proc. Roy. Soc. A*, **133** (1931), 60.

[2] A.S. Schwarz, On regular solutions of Euclidean Yang-Mills equations, *Phys. Lett. B*, **67** (1977), 172.

[3] A.S. Schwarz, On symmetric gauge fields, *Comm. Math. Phys.*, **56** (1977), 79.

[4] T. Frankel, *The Geometry of Physics: An Introduction*, Cambridge University Press, Cambridge, UK (1997).

[5] T. Eguchi, P.B. Galkey, A.J. Hanson, Gravitation, gauge fields and differential geometry, *Phys. Rep.*, **66** (1980), 213.

[6] G.L. Naber, *Topology, Geometry, and Gauge Fields*, Springer Verlag, New York (1997).

[7] L. Patiño, H. Quevedo, Bosonic and fermionic behavior in gravitational configurations, *Mod. Phys. Lett. A*, **18** (2003), 1331.

[8] L. Patiño, H. Quevedo, Topological quantization of gravitational fields, *J. Math. Phys.*, **46** (2005), 22502.

[9] F. Nettel, H. Quevedo, Topological spectrum of classical configurations, *AIP Conf. Proc.*, **956** (2007), 9.

[10] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer Verlag, New York (1980).

[11] C. Nash, S. Sen, *Topology and Geometry for Physicists*, Academic Press, London (1983).

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