International Journal of Pure and Applied Mathematics
Volume 71 No. 3 2011, 375-382

INTERPOLATION INEQUALITIES IN POWER TYPE WEIGHTED LEBESGUE-SOBOLEV SPACES

B. Cekic[1 §], G. Alisoy[2], R.A. Mashiyev[3]
[1,3]Department of Mathematics
Dicle University
21280, Diyarbakir, TURKEY
[2]Department of Mathematical Sciences
Inonu University
44280, Malatya, TURKEY

Abstract: In this paper we prove some interpolation inequalities between functions and their derivatives in the class of power type weighted Lebesgue-Sobolev spaces defined on bounded open subset $\Omega \subset R^N$. As application of our results, we give a compact embedding theorem in power type weighted Sobolev space.

AMS Subject Classification: 46E35, 46B70, 26D10
Key Words: Lebesgue-Sobolev spaces, interpolation, power weight functions, compact embedding

1. Introduction and Preliminaries

Nirenberg [10] and Gagliardo[6] proved the interpolation inequality

$$|u|_{L_{p,m}(\Omega)} \leq C\varepsilon |u|_{L_{p,k}(\Omega)} + CK(\varepsilon) |u|_{L_{p,0}(\Omega)},$$

which holds for $\forall u \in W_p^k$ with a constant $C = C(m, p, \Omega) > 0$ not depending on the function u. This interpolation inequality yields upper bounds for L_p norms of the intermediate derivative of order m, $0 < m \leq k-1$ of functions in W_p^k, in terms of L_p norms of u and its partial derivatives of order k. As an

Received: March 30, 2011
§Correspondence author

application of this inequality can be obtain

$$\|u\|_{W_p^m(\Omega)} \leq C_1 \varepsilon \|u\|_{W_p^k(\Omega)} + C_1 K(\varepsilon) \|u\|_{W_p^0(\Omega)},$$

which holds for $\forall u \in W_p^k$ with a constant $C_1 = C_1(m, p, \Omega) > 0$ not depending on the function u.

Interpolation inequalities are very important tools in the description of properties of Sobolev spaces $W_p^k(\Omega)$ and play an important role in the study of partial differential equations and variational problems.

In the present paper, we prove those interpolation inequalities in weighted Lebesgue $L_{p,\beta}(\Omega)$ and Sobolev space $W_{p,\beta}^m(\Omega)$ where the weight function $\rho(x)$ is the distance of point x from the set $\partial\Omega$. Some variants of this inequality could be derived from [2, 4, 7, 8]. This estimate plays an important role in the study of qualitative properties of solutions for a class of degenerate nonlinear elliptic high-order equations.

First, we introduce some notations and present basic results from the theory of weighted Lebesgue and Sobolev spaces. Second, we prove interpolation inequalities in weighted Lebesgue and Sobolev space. At last, as application of our result, we give a compact embedding theorem in weighted Sobolev space.

Let $\mathbb{R}^N$ be real Euclidean space, $x = (x_{1,\dots,}x_N) \in \mathbb{R}^N$ and $\mathbb{R}_h^N = \{x \in \mathbb{R}^N;\ x_N < h\}$. Denote by $\overline{\mathbb{R}}_h^N$ the closure of $\mathbb{R}_h^N$. Let $s = (s_{1,\dots,}s_N)-$ be an $N-$tuple of nonnegative integers s_j and $D_i = \frac{\partial}{\partial x_i}$, $D^s = D_1^{s_1} \cdot \ldots \cdot D_N^{s_N}$. Denote by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω.

We define the following spaces

$$L_{p,\lambda}(\Omega) = \{u \in \mathcal{U}(\Omega) : [u]_{\lambda,\Omega}^p = \int_\Omega \rho^\lambda(x) |u|^{\,p} \, dx < \infty, 1 \leq p < \infty, \lambda > -1\},$$

$$L_p^k(\Omega) = \{u \in \mathcal{U}(\Omega) : |u|_{k,\Omega}^p = \int_\Omega \sum_{|s|=k} |D^s u|^{\,p} \, dx < \infty, 1 \leq p < \infty\},$$

$$L_{p,\lambda}^k(\Omega) = \{u \in \mathcal{U}(\Omega) : |u|_{k,\lambda,\Omega}^p = \int_\Omega \rho^\lambda(x) \sum_{|s|=k} |D^s u|^{\,p} \, dx < \infty, \lambda > -1\}.$$

where the function $\rho(x)$ stands for the distance of point x from the set $\partial\Omega$.

Moreover, let us define $W_{p,\lambda}^k(\Omega)$ space

$$W_{p,\lambda}^k(\Omega) := W_\lambda^k(\Omega) = \left\{u \in L_p(\Omega) : \|u\|_{k,\lambda,\Omega}^p = |u|_{p,\Omega}^p + |u|_{k,\lambda,\Omega}^p < \infty\right\},$$

where k is nonnegative integer, $1 \leq p < \infty$, $\lambda > -1$ and $|u|_{p,\Omega}^p = \int_\Omega |u|^p \, dx$.

Now we are ready to state the main results.

2. Main Results

In this section, we give some weighted interpolation inequalities and their proofs.

Theorem 2.1. *Let Ω be a bounded domain with smooth boundary in $\mathbb{R}^N$, $\partial\Omega \in C^k$and $0 \leq m < k$. Let $\alpha, \beta, \gamma > 0$, and $\beta > \frac{m}{k}\alpha + \left(1 - \frac{m}{k}\right)\gamma$. Then for any $\varepsilon > 0$ there exist $K(\varepsilon)$ such that $\forall u \in W^k_{p,\alpha}(\Omega)$ there hold that*

$$i)\ |u|^p_{m,\beta,\Omega} \leq C\varepsilon\, |u|^p_{k,\alpha,\Omega} + CK(\varepsilon)\, [u]^p_{\gamma,\Omega}\,,$$

$$ii)\ |u|_{m,\beta,\Omega} \leq 2C\, |u|^{\frac{m}{k}}_{k,\alpha,\Omega}\, [u]^{\frac{k-m}{k}}_{\gamma,\Omega}\ ,$$

$$iii)\ \|u\|^p_{m,\beta,\Omega} \leq \widetilde{C}\varepsilon\, \|u\|^p_{k,\alpha,\Omega} + \widetilde{C}K(\varepsilon)\, [u]^p_{\gamma,\Omega}\,,$$

$$iv)\ \|u\|_{m,\beta,\Omega} \leq 2\widetilde{C}\, \|u\|^{\frac{m}{k}}_{k,\alpha,\Omega}\, [u]^{\frac{k-m}{k}}_{\gamma,\Omega}\ ,$$

where the constants $C = C(m,p,\Omega) > 0$ and $\widetilde{C} = \widetilde{C}(m,p,\Omega) > 0$ are independent of the function u.

Proof. In order to obtain $ii)$ and $iv)$ it is enough to choose $K(\varepsilon) = \varepsilon^{-\frac{m}{k-m}}$ in $i)$ and $iii)$ respectively. Therefore, it is sufficient to prove $i)$ and $iii)$. Using the elementary inequality

$$(a^p)^{1-\theta}\,(b^p)^{\theta} \leq (1-\theta)\,a^p + \theta b^p,\ \ 0 \leq \theta \leq 1$$

we have

$$\begin{aligned}(a^p)^{1-\theta}\,(b^p)^{\theta} &= \left(\varepsilon^{\frac{1}{1-\theta}}\lambda^{\frac{1}{1-\theta}}a^p\right)^{1-\theta}\left(\varepsilon^{-\frac{1}{\theta}}\lambda^{-\frac{1}{\theta}}b^p\right)^{\theta}, \\ &\leq (1-\theta)\,\varepsilon^{\frac{1}{1-\theta}}\lambda^{\frac{1}{1-\theta}}a^p + \theta\varepsilon^{-\frac{1}{\theta}}\lambda^{-\frac{1}{\theta}}b^p\end{aligned}$$

for any $0 < \theta < 1$. If the term $\varepsilon^{\frac{1}{1-\theta}}$ is replaced by ε in the last inequality, then this yield

$$(a^p)^{1-\theta}\,(b^p)^{\theta} \leq \varepsilon\lambda^{\frac{1}{1-\theta}}a^p + \varepsilon^{-\frac{1-\theta}{\theta}}\lambda^{-\frac{1}{\theta}}b^p. \tag{1}$$

Further, from the well know interpolation inequality [1] we can write

$$\|u\|^p_{m,\Omega} \leq \overline{C}\varepsilon\, \|u\|^p_{k,\Omega} + \overline{C}K(\varepsilon)\, |u|^p_{p,\Omega}\,, 0 \leq m < k$$

and by the choice of ε and $K(\varepsilon)$ we have the following interpolation inequality

$$\|u\|_{m,\Omega}^{p} \leq 2C\,\|u\|_{k,\Omega}^{p(1-\theta)} \cdot |u|_{p,\Omega}^{p\theta} \tag{2}$$

where

$$\theta = 1 - \frac{m}{k},\ \ 0 \leq m < k$$

and the positive constant $C = C(m, p, \Omega)$ is independent of u. Thus, from the inequalities (1) and (2) we get

$$\|u\|_{m,\Omega}^{p} \leq C\varepsilon\lambda^{\frac{1}{1-\theta}}\,\|u\|_{k,\Omega}^{p} + CK(\varepsilon)\lambda^{-\frac{1}{\theta}}\,|u|_{p,\Omega}^{p}\,. \tag{3}$$

Since $\partial\Omega \in C^k$, we can pass to local coordinates. Let

$$\overline{\Omega} = \overline{\mathbb{R}}_1^N = \left\{x \in \overline{\mathbb{R}}^N : x_N \leq 1\right\},$$

where domain $\operatorname{supp}(u) \subset K \subset \overline{\mathbb{R}}_1^N$. If $u \in W_p^k\left(\overline{\mathbb{R}}_1^N\right)$, then $\forall h < 1$, $u \in W_p^k\left(\overline{\mathbb{R}}_h^N\right)$. Therefore, the inequality (3) can be rewrite as the following

$$\|u\|_{m,\overline{\mathbb{R}}_{\frac{1}{2}}^N}^{p} \leq C\varepsilon\lambda^{\frac{1}{1-\theta}}\,\|u\|_{k,\overline{\mathbb{R}}_{\frac{1}{2}}^N}^{p} + CK(\varepsilon)\lambda^{-\frac{1}{\theta}}\,|u|_{p,\overline{\mathbb{R}}_{\frac{1}{2}}^N}^{p} \tag{4}$$

for $\overline{\Omega} = \overline{\mathbb{R}}_{\frac{1}{2}}^N$. Now, let consider the sequence of $h_s = 1 - \frac{1}{2^s}$ $(s = 1, 2, ...)$. Since $u \in W_p^k\left(\overline{\mathbb{R}}_{h_s}^N\right)$, we have

$$\upsilon_s(x) = u\left(2h_s x\right) \in W_p^k\left(\overline{\mathbb{R}}_{\frac{1}{2}}^N\right)$$

for each h_s. Then the inequality (4) is satisfied for $\upsilon_s(x)$. If we change coordinates such that $x \to 2h_s x$, then the space $\overline{\mathbb{R}}_{\frac{1}{2}}^N$ turns into the space $\overline{\mathbb{R}}_{h_s}^N$, and hence we can write

$$\int\limits_{\overline{\mathbb{R}}_{\frac{1}{2}}^N} |D^r \upsilon_s(x)|^p\,dx = (2h_s)^{p|r|-N} \int\limits_{\overline{\mathbb{R}}_{h_s}^N} |D^r u(y)|^p\,dy$$

where $y = 2h_s x$. Rewriting the inequality (4) for the functions $\upsilon_s(x)$ and changing coordinates we obtain

$$|u|_{m,\overline{\mathbb{R}}_{h_s}^N}^{p} \leq C\varepsilon\lambda^{\frac{1}{1-\theta}}\,|u|_{k,\overline{\mathbb{R}}_{h_s}^N}^{p} + CK(\varepsilon)\lambda^{-\frac{1}{\theta}}\,|u|_{p,\overline{\mathbb{R}}_{h_s}^N}^{p}\,,$$

since $1 \leq 2h_s \leq 2$ and the parameter λ is bounded. Let $\lambda = \rho(x)^{\tau}$. Let multiply both sides of the last inequality by $\rho(x)^{\beta}$ and choose α such that $\beta + \frac{\tau}{1-\theta} = \alpha$. Hence, it follows that $\tau = (\alpha - \beta)(1-\theta)$. We now choose the parameter $\gamma = \beta - \frac{\tau}{\theta} = \alpha - \frac{1}{\theta}(\alpha - \beta)$ such that $\alpha - \frac{1}{\theta}(\alpha-\beta) + \gamma > 0$, namely $\beta > \alpha(1-\theta) + \gamma\theta$ or $\beta > \frac{m}{k}\alpha + \left(1 - \frac{m}{k}\right)\gamma$. Then

$$\rho^{\beta} |u|^{p}_{m,\overline{\mathbb{R}}^N_{h_s}} \leq C\varepsilon\rho^{\alpha} |u|^{p}_{k,\overline{\mathbb{R}}^N_{h_s}} + CK(\varepsilon)\rho^{\gamma} |u|^{p}_{p,\overline{\mathbb{R}}^N_{h_s}}. \tag{5}$$

Let $\rho = \rho(s) = \frac{1}{2^s}$, $s = 1, 2, ...$in the inequality (5) where the function $\rho(s)$ is distance from $\mathbb{R}^N_{h_s}$ to hyperplane $x_N = 1$. Let sum from 1 to M in the inequality (5) :

$$\sum_{s=1}^{M} \left(\frac{1}{2^s}\right)^{\beta} |u|^{p}_{m,\overline{\mathbb{R}}^N_{h_s}}$$

$$\leq C\left(\varepsilon\sum_{s=1}^{M}\left(\frac{1}{2^s}\right)^{\alpha} |u|^{p}_{k,\overline{\mathbb{R}}^N_{h_s}} + K(\varepsilon)\sum_{s=1}^{M}\left(\frac{1}{2^s}\right)^{\gamma} |u|^{p}_{p,\overline{\mathbb{R}}^N_{h_s}}\right). \tag{6}$$

Taking into account the equality

$$\begin{aligned}\sum_{s=1}^{M}\left(\frac{1}{2^s}\right)^{\mu} |u|^{p}_{l,\overline{\mathbb{R}}^N_{h_s}} &= \left(\frac{1}{2^M}\right)^{\mu} |u|^{p}_{l,\left(\overline{\mathbb{R}}^N_{h_M}\setminus\overline{\mathbb{R}}^N_{h_{M-1}}\right)} \\ &+ \left[\left(\frac{1}{2^M}\right)^{\mu} + \left(\frac{1}{2^{M-1}}\right)^{\mu}\right] |u|^{p}_{l,\left(\overline{\mathbb{R}}^N_{h_{M-1}}\setminus\overline{\mathbb{R}}^N_{h_{M-2}}\right)} + ... \\ &+ \left[\left(\frac{1}{2^M}\right)^{\mu} + ... + \left(\frac{1}{2}\right)^{\mu}\right] |u|^{p}_{l,\left(\overline{\mathbb{R}}^N_{h_1}\right)},\end{aligned}$$

we can obtain an equivalent expression for each sum in the inequality (6). Since

$$\left(\frac{1}{2^k}\right)^{\mu} \leq \sum_{s=k}^{M}\left(\frac{1}{2^s}\right)^{\mu} < \frac{1}{1-\left(\frac{1}{2}\right)^{\mu}}\left(\frac{1}{2^k}\right)^{\mu},\ \mu > 0$$

we can write

$$\left[1 - \left(\frac{1}{2}\right)^{\mu}\right]\left(\frac{1}{2^k}\right)^{\mu} \leq \left[1 - \left(\frac{1}{2}\right)^{\mu}\right]\sum_{s=k}^{M}\left(\frac{1}{2^s}\right)^{\mu} < \left(\frac{1}{2^k}\right)^{\mu}. \tag{7}$$

From (7) we have

$$\left[1 - \left(\frac{1}{2}\right)^{\mu}\right]\sum_{s=1}^{M}\left(\frac{1}{2^s}\right)^{\mu} |u|^{p}_{l,\left(\overline{\mathbb{R}}^N_{h_s}\setminus\overline{\mathbb{R}}^N_{h_{s-1}}\right)}$$

$$\leq \left[1-\left(\frac{1}{2}\right)^{\mu}\right]\sum_{s=1}^{M}\left(\frac{1}{2^{s}}\right)^{\mu}|u|^{p}_{l,\left(\overline{\mathbb{R}}^{N}_{h_s}\right)} \leq \sum_{s=1}^{M}\left(\frac{1}{2^{s}}\right)^{\mu}|u|^{p}_{l,\left(\overline{\mathbb{R}}^{N}_{h_s}\setminus\overline{\mathbb{R}}^{N}_{h_{s-1}}\right)}.$$

It is easy to see that any $u \in W^{l}_{p,\mu}\left(\overline{\mathbb{R}}^{N}_{1}\right)$ has a compact support and satisfies the following inequality

$$\left[1-\left(\frac{1}{2}\right)^{\mu}\right]\int\limits_{\overline{\mathbb{R}}^{N}_{h_M}}(1-x_N)^{\mu}\sum_{|s|=l}|D^{s}u|^{p}\,dx$$

$$\leq \left[1-\left(\frac{1}{2}\right)^{\mu}\right]\sum_{s=1}^{M}\left(\frac{1}{2^{s}}\right)^{\mu}|u|^{p}_{l,\left(\overline{\mathbb{R}}^{N}_{h_s}\setminus\overline{\mathbb{R}}^{N}_{h_{s-1}}\right)} \leq \int\limits_{\overline{\mathbb{R}}^{N}_{h_M}}(1-x_N)^{\mu}\sum_{|s|=l}|D^{s}u|^{p}\,dx$$

From the last inequality above and (6) we have

$$\int\limits_{\overline{\mathbb{R}}^{N}_{h_M}}(1-x_N)^{\beta}\sum_{|s|=m}|D^{s}u|^{p}\,dx$$

$$\leq C\varepsilon\int\limits_{\overline{\mathbb{R}}^{N}_{h_M}}(1-x_N)^{\alpha}\sum_{|s|=k}|D^{s}u|^{p}\,dx + CK(\varepsilon)\int\limits_{\overline{\mathbb{R}}^{N}_{h_M}}(1-x_N)^{\gamma}|u|^{p}\,dx.$$

and by letting $M\to\infty$ we get

$$|u|^{p}_{m,\beta,\Omega} \leq C\varepsilon\,|u|^{p}_{k,\alpha,\Omega} + CK(\varepsilon)\,[u]^{p}_{\gamma,\Omega}$$

and

$$\|u\|^{p}_{m,\beta,\Omega} \leq \widetilde{C}\varepsilon\,\|u\|^{p}_{k,\alpha,\Omega} + \widetilde{C}K(\varepsilon)\,[u]^{p}_{\gamma,\Omega}\,.$$

Hence the proof is complete. $\square$

Remark 2.2. m and k may not be nonnegative integers in Theorem 2.1. This theorem is valid for Sobolev-Slobodinsky spaces[11].

3. Compact Embedding Theorems for the Space $W^{k}_{p,\alpha}(\Omega)$

In this section, we give a weighted embedding theorem which generalizes Lemma 13 of chapter 4 in [1, 3].

Theorem 3.1. *Let $k \in \mathbb{Z}_+$, $m \in \mathbb{Z}_+ \cup \{0\}$, $0 \leq m < k$, $1 < p, q < \infty$ and $\alpha, \beta, \gamma > 0$ such that $\beta > \frac{m}{k}\alpha + \left(1 - \frac{m}{k}\right)\gamma$. Let Ω be an open subset of $\mathbb{R}^N$.*

$i)$ If the embedding

$$W^k_{p,\alpha}(\Omega) \hookrightarrow\hookrightarrow W^m_{q,\beta}(\Omega) \tag{8}$$

is compact, then for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that $\forall u \in W^k_{p,\alpha}(\Omega)$.

$$\|u\|^q_{m,\beta,\Omega} \leq \varepsilon \|u\|^p_{k,\alpha,\Omega} + C(\varepsilon)[u]^p_{\gamma,\Omega}. \tag{9}$$

$ii)$ If for any $\varepsilon > 0$, (9) holds and the embedding $W^k_{p,\alpha}(\Omega) \hookrightarrow L_{p,\gamma}(\Omega)$ is compact, then embedding (8) is also compact.

Proof. $i)$ Suppose that inequality (9) does not hold for all $\varepsilon > 0$, i.e. there exist $\varepsilon_0 > 0$ and functions $u_s \in W^k_{p,\alpha}(\Omega)$, $s \in \mathbb{N}$, such that

$$\|u_s\|^p_{k,\alpha,\Omega} \leq C \tag{10}$$

and

$$\|u_s\|^q_{m,\beta,\Omega} > \varepsilon_0 \|u_s\|^p_{k,\alpha,\Omega} + s[u_s]^p_{\gamma,\Omega}. \tag{11}$$

Since (11), by (8) it follows that

$$\|u_s\|_{m,\beta,\Omega} \leq A$$

where A is independent of s. Consequently, by (11) we have

$$[u_s]^p_{\gamma,\Omega} < \frac{A^q}{s}.$$

Thus $\lim_{s\to\infty} [u_s]^p_{\gamma,\Omega} = 0$. Employing (10) again we have

$$\lim_{s\to\infty}\inf \|u_s\|^q_{m,\beta,\Omega} \geq \varepsilon_0 \tag{12}$$

Since embedding (8) is compact, there exists a subsequence $\{u_{s_j}\}$ converging to a function u in $W^m_{q,\beta}(\Omega)$. Since $u_{s_j} \to 0$ in $L_{p,\gamma}(\Omega)$ as $j \to \infty$. Thus $u = 0$ a.e. in Ω. This contradicts the inequality (12).

$ii)$ Let $A > 0$ and $S = \left\{u \in W^k_{p,\alpha}(\Omega) : \|u\|^p_{k,\alpha,\Omega} \leq A\right\}$. Since the embedding $W^k_{p,\alpha}(\Omega) \hookrightarrow L_{p,\gamma}(\Omega)$ is compact, there exists a sequence $u_s \in S$, $s \in \mathbb{N}$, which is a Cauchy sequence in $L_{p,\gamma}(\Omega)$. Furthermore from (10) for any $\varepsilon > 0$,

$$\|u_s - u_j\|^q_{m,\beta,\Omega} \leq \varepsilon \|u_s - u_j\|^p_{k,\alpha,\Omega} + C(\varepsilon)[u_s - u_j]^p_{\gamma,\Omega}. \tag{13}$$

From (13), we have $\lim\limits_{s,j\longrightarrow\infty} \|u_s - u_j\|^q_{m,\beta,\Omega} \leq \varepsilon A$. Since ε is arbitrary, the sequence $\{u_s\}$ is a Cauchy sequence in $W^m_{q,\beta}(\Omega)$. Therefore there exists a function $u \in W^m_{p,\beta}(\Omega)$ such that $u_s \longrightarrow u$ in $W^m_{p,\beta}(\Omega)$ as $s \longrightarrow \infty$, which completes the proof. $\square$

References

[1] R.A. Adams, J.F. Fournier, *Sobolev Space*, Second Edition, Academic Press, New York (2003).

[2] S. Boccia, L. Caso, Interpolation inequalities in weighted Sobolev spaces, *J. Math. Inequal.* **2** (2008), 309-322.

[3] V.I. Burenkov, *Sobolev Spaces on Domain*, B.G. Teubner Verlag (1998).

[4] S.K. Chua, On weighted Sobolev interpolation inequalities, *Proc. Amer. Math. Soc.*, **121**, No. 2 (1994), 441-449.

[5] G. Erling, On a type of eigenvalue problem for certain elliptic diferantial operators, *Math. Scand.*, **2** (1954), 267-285.

[6] E. Gagliardo, Properietà di alcune classi di funzioni in piu variabili, *Ricerche Mat.*, **8** (1959), 24-51.

[7] C.E. Gutierrez, R.L. Wheeden, Sobolev interpolation inequalities with weights, *Trans. Amer. Math. Soc.*, **323**, No. 1 (1991), 263-281.

[8] A. Kalamajska, Pointwise multicative inequalities and Nirenberg type estimates in weighted Sobolev spaces, *Studia Math.*, **108** (1994), 275-290.

[9] C.-S. Lin, Interpolation inequalities with weight, *Commun. Part. Diff. Eq.*, **11** (1986), 1515-1538.

[10] L. Nirenberg, An extended interpolation inequality, *Ann. Scu. Norm. Sup. Pisa Cl. Sci.*, **20** (1996), 733-737.

[11] M.I. Vishik, G.I. Eskin, Sobolev-Slobodinsky spaces of variable order with weighted norm and their applications to mixed boundary value problems, *Sibirsk. Mat. Zh.*, **9**, No. 5 (1968), 973-997, In Russian.