International Journal of Pure and Applied Mathematics
Volume 71 No. 3 2011, 391-401

COMMON COINCIDENCE POINT THEOREMS IN T_1 TOPOLOGICAL SPACES WITH APPLICATION TO THE SOLUTIONS OF FUNCTIONAL EQUATIONS IN DYNAMIC PROGRAMMING

Jinbiao Hao[1], Hongyan Guan[2], Shin Min Kang[3 §]

[1]Department of Mathematics
Liaoning Normal University
Dalian, Liaoning, 116029, P.R. CHINA

[2]Department of Mathematics
Shenyang Normal University
Shenyang, Liaoning, 110000, P.R. CHINA

[3]Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, KOREA

Abstract: In this paper, we prove a few coincidence point theorems for two pairs of mappings in T_1 topological spaces. As application, we discuss the existence of common solutions for a class of system of functional equations arising in dynamic programming.

AMS Subject Classification: 54H25
Key Words: T_1 topological space, coincidence point, proper mapping

1. Introduction

Machuca [5] established a coincidence point theorem involving a pair of mappings in T_1 topological spaces. Khan [2] and Liu [3] extended Machuca's result to three mappings. Liu et al. [4] obtained some coincidence point theorems for two pairs of mappings in T_1 topological spaces. The purpose of this paper is to establish some coincidence point theorems for two pairs of mappings and three mappings in T_1 topological spaces. As application, we use the results presented

Received: May 19, 2011
© 2011 Academic Publications, Ltd.
§Correspondence author

to study the existence problem of common solutions for a class of system of functional equations arising in dynamic programming.

Throughout this paper, R and R^+ denote the sets of all real numbers and nonnegative numbers respectively. Let X and Y be topological spaces, and I denote the identity mapping on X. A mapping $f : X \to Y$ is said to be *proper* if $f^{-1}(A)$ is compact for each compact subset A of Y with $A \subseteq f(X)$. For any subset $A \subseteq Y$, $\overline{A}$ stands for the closure of A.

Define $\Phi = \{\phi : \phi : (R^+)^8 \to R^+$ is upper semicontinuous and nondecreasing in each coordinate variable and satisfying (1.1)$\}$, where

$$\bar{\phi}(t) = \max\{\phi(t,t,t,t,t,0,t,t), \phi(0,0,t,0,0,t,0,t)\} < t, \quad \forall t > 0. \tag{1.1}$$

Lemma 1.1. (see [6]) *Let $\bar{\phi} : R^+ \to R^+$ be nondecreasing and upper semicontinuous. Then for each $t > 0$, $\bar{\phi}(t) < t$ if and only if $\lim_{n\to\infty} \bar{\phi}^n(t) = 0$, where $\bar{\phi}^n$ denotes the composition of $\bar{\phi}$ with itself n-times.*

2. Main results

Theorem 2.1. *Let X be a T_1 topological space satisfying the first axiom of countability. Let (Y, d) be a complete metric space and $A, B, S, T : X \to Y$ satisfy that*

(a1) *$AX \subseteq TX$ and $BX \subseteq SX$ and one of the following conditions:*
(a2) *A and S are continuous, A is proper with AX closed;*
(a3) *A and S are continuous, S is proper with SX closed;*
(a4) *B and T are continuous, B is proper with BX closed;*
(a5) *B and T are continuous, T is proper with TX closed.*

If there exists $\phi \in \Phi$ such that

$$\begin{aligned} d^2(Ax, By) \le \phi(&d(Ax, Sx)d(By, Ty), d(Ax, Sx)d(Sx, Ty), \\ &d(Ax, By)d(Sx, Ty), d(Ax, By)d(Ax, Sx), \\ &d(Ax, By)d(By, Ty), d(Ax, Ty)d(By, Sx), \\ &d^2(By, Ty), d^2(Sx, Ty)) \end{aligned} \tag{2.1}$$

for all $x, y \in X$, then there exist $u, v \in X$ such that $Au = Su = Bv = Tv$.

Proof. Let $x_0 \in X$. Since $AX \subseteq TX$ and $BX \subseteq SX$, we can choose sequences $\{x_n\}_{n\ge0} \subseteq X$ and $\{y_n\}_{n\ge1} \subseteq Y$ such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$

for all $n \geq 0$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for all $n \geq 1$. Put $d_n = d(y_n, y_{n+1})$ for all $n \geq 1$. From (2.1), we have

$$\begin{aligned} d_{2n+1}^2 &= d^2(Ax_{2n}, Bx_{2n+1}) \\ &\leq \phi(d(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Sx_{2n})d(Sx_{2n}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Bx_{2n+1})d(Sx_{2n}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Bx_{2n+1})d(Ax_{2n}, Sx_{2n}), \\ &\qquad d(Ax_{2n}, Bx_{2n+1})d(Bx_{2n+1}, Tx_{2n+1}), \\ &\qquad d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ &\qquad d^2(Bx_{2n+1}, Tx_{2n+1}), d^2(Sx_{2n}, Tx_{2n+1})) \\ &= \phi(d_{2n}d_{2n+1}, d_{2n}^2, d_{2n+1}d_{2n}, d_{2n+1}d_{2n}, d_{2n+1}^2, 0, d_{2n+1}^2, d_{2n}^2) \end{aligned} \tag{2.2}$$

for all $n \geq 1$. Suppose that $d_{2n} < d_{2n+1}$ for some $n \geq 1$. In view of (2.2), we get that

$$\begin{aligned} d_{2n+1}^2 &\leq \phi(d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2, 0, d_{2n+1}^2, d_{2n+1}^2) \\ &\leq \bar{\phi}(d_{2n+1}^2) < d_{2n+1}^2, \end{aligned}$$

which is impossible. Therefore, $d_{2n+1} \leq d_{2n}$ for all $n \geq 1$. It follows from (2.2) that

$$d_{2n+1}^2 \leq \phi(d_{2n}^2, d_{2n}^2, d_{2n}^2, d_{2n}^2, d_{2n}^2, 0, d_{2n}^2, d_{2n}^2) \leq \bar{\phi}(d_{2n}^2), \quad \forall n \geq 1.$$

Similarly, we can prove that $d_{2n}^2 \leq \bar{\phi}(d_{2n-1}^2)$ for all $n \geq 1$. It follows that

$$d_n^2 \leq \bar{\phi}(d_{n-1}^2) \leq \bar{\phi}^2(d_{n-2}^2) \leq \cdots \leq \bar{\phi}^{n-1}(d_1^2), \quad \forall n \geq 1.$$

By Lemma 1.1, we obtain that

$$\lim_{n\to\infty} d_n = 0. \tag{2.3}$$

In order to prove that $\{y_n\}_{n\geq 1}$ is a Cauchy sequence, it is sufficient to prove that $\{y_{2n}\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n\geq 1}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$ such that for each even integer $2k$, there exist even integers $2m(k)$ and $2n(k)$ with

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon, \quad 2m(k) > 2n(k) > 2k. \tag{2.4}$$

For each even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.4), that is,

$$d(y_{2n(k)}, y_{2m(k)-2}) \le \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)}) > \epsilon. \tag{2.5}$$

Note that

$$\epsilon < d(y_{2n(k)}, y_{2m(k)}) \le d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}. \tag{2.6}$$

Using (2.3), (2.5) and (2.6), we conclude that

$$\lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.7}$$

It is easy to verify that

$$\begin{aligned} &|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2m(k)-1};\\ &|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \le d_{2n(k)};\\ &|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \le d_{2n(k)}. \end{aligned} \tag{2.8}$$

According to (2.3), (2.7) and (2.8), we obtain that

$$\begin{aligned} \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1})\\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)})\\ &= \epsilon. \end{aligned} \tag{2.9}$$

In view of (2.1), we have

$$\begin{aligned} &d^2(y_{2n(k)}, y_{2m(k)})\\ &\le d^2_{2n(k)} + 2d(Ax_{2n(k)}, Bx_{2m(k)-1})d_{2n(k)} + d^2(Ax_{2n(k)}, Bx_{2m(k)-1})\\ &\le d^2_{2n(k)} + 2d(y_{2n(k)+1}, y_{2m(k)})d_{2n(k)}\\ &\quad + \phi(d(Ax_{2n(k)}, Sx_{2n(k)})d(Bx_{2m(k)-1}, Tx_{2m(k)-1}),\\ &\qquad d(Ax_{2n(k)}, Sx_{2n(k)})d(Sx_{2n(k)}, Tx_{2m(k)-1}),\\ &\qquad d(Ax_{2n(k)}, Bx_{2m(k)-1})d(Sx_{2n(k)}, Tx_{2m(k)-1}),\\ &\qquad d(Ax_{2n(k)}, Bx_{2m(k)-1})d(Ax_{2n(k)}, Sx_{2n(k)}),\\ &\qquad d(Ax_{2n(k)}, Bx_{2m(k)-1})d(Bx_{2m(k)-1}, Tx_{2m(k)-1}),\\ &\qquad d(Ax_{2n(k)}, Tx_{2m(k)-1})d(Bx_{2m(k)-1}, Sx_{2n(k)}),\\ &\qquad d^2(Bx_{2m(k)-1}, Tx_{2m(k)-1}), d^2(Sx_{2n(k)}, Tx_{2m(k)-1})) \end{aligned}$$

$$\begin{aligned}
&= d^2_{2n(k)} + 2d(y_{2n(k)+1}, y_{2m(k)})d_{2n(k)} \\
&\quad + \phi(d_{2n(k)}d_{2m(k)-1}, d_{2n(k)}d(y_{2n(k)}, y_{2m(k)-1}), \\
&\qquad d(y_{2n(k)+1}, y_{2m(k)})d(y_{2n(k)}, y_{2m(k)-1}), \\
&\qquad d(y_{2n(k)+1}, y_{2m(k)})d_{2n(k)}, d(y_{2n(k)+1}, y_{2m(k)})d_{2m(k)-1}, \\
&\quad d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)}), d^2_{2m(k)-1}, d^2(y_{2n(k)}, y_{2m(k)-1})).
\end{aligned}$$

Letting $k \to \infty$ in the above inequalities, we get that

$$\epsilon^2 \le \phi(0, 0, \epsilon^2, 0, 0, \epsilon^2, 0, \epsilon^2) \le \bar{\phi}(\epsilon^2) < \epsilon^2,$$

which is impossible. Then $\{y_n\}_{n\ge 1}$ is a Cauchy sequence. Since(Y, d) is complete, there exists some $z \in Y$ with $\lim_{n\to\infty} y_n = z$.

Assume that (a2) holds. Put $C = \{Ax_{2n} : n \ge 0\} \cup \{z\}$. Clearly, $C = \overline{C} \subseteq \overline{AX} = AX \subseteq Y$ and C is compact. Since A is proper, $A^{-1}(C)$is compact in X. Hence there exists $\{x_{2n_k}\}_{k\ge 1} \subseteq \{x_{2n}\}_{n\ge 0}$ such that it converges to some point $u \in X$. The continuity of A and S ensure that

$$\lim_{k\to\infty} Ax_{2n_k} = Au = z = \lim_{k\to\infty} Sx_{2n_k} = Su.$$

Since $AX \subseteq TX$, there exists $v \in X$ such that $Au = Tv$. Now we can claim that $Au = Bv$. Otherwise $Au \ne Bv$. In view of (2.1) we infer that

$$\begin{aligned}
&d^2(Au, Bv) \\
&\le \phi(d(Au, Su)d(Bv, Tv), d(Au, Su)d(Su, Tv), d(Au, Bv)d(Su, Tv), \\
&\qquad d(Au, Bv)d(Au, Su), d(Au, Bv)d(Bv, Tv), d(Au, Tv)d(Bv, Su), \\
&\qquad d^2(Bv, Tv), d^2(Su, Tv)) \\
&= \phi(0, 0, 0, 0, d^2(Au, Bv), 0, d^2(Au, Bv), 0) \\
&\le \bar{\phi}(d^2(Au, Bv)) < d^2(Au, Bv),
\end{aligned}$$

which is a contradiction. Hence $Au = Bv$. Then $Au = Su = Bv = Tv$.

Assume that (a4) holds. Set $C = \{Bx_{2n-1} : n \ge 1\} \cup \{z\}$. It is easy to show that $B^{-1}(C)$ is also compact. Clearly, there exists a subsequence $\{x_{2n_k-1}\}_{k\ge 1}$ of $\{x_{2n-1}\}_{n\ge 1}$ such that it converges to some point $v \in X$. It follows from the continuity of B and T that

$$\lim_{k\to\infty} Bx_{2n_k-1} = Bv = z = \lim_{k\to\infty} Tx_{2n_k-1} = Tv.$$

Notice that $Bv \in BX \subseteq SX$, there exists some $u \in X$ with $Bv = Su$. Suppose that $Au \neq Bv$. From (2.1), we obtain that

$$\begin{aligned} & d^2(Au, Bv) \\ & \le \phi(d(Au, Su)d(Bv, Tv), d(Au, Su)d(Su, Tv), d(Au, Bv)d(Su, Tv), \\ & \quad d(Au, Bv)d(Au, Su), d(Au, Bv)d(Bv, Tv), d(Au, Tv)d(Bv, Su), \\ & \quad d^2(Bv, Tv), d^2(Su, Tv)) \\ & = \phi(0, 0, 0, d^2(Au, Bv), 0, 0, 0, 0) \\ & \le \bar{\phi}(d^2(Au, Bv)) \\ & < d^2(Au, Bv), \end{aligned}$$

which is a contradiction. Therefore $Au = Bv$. That is, $Au = Su = Bv = Tv$. This completes the proof. □

Corollary 2.1. *Let X be a T_1 topological space satisfying the first axiom of countability. Let (Y, d) be a complete metric space and $A, B, S, T : X \to Y$ satisfy* (a1) *and one of* (a2)-(a5) *in Theorem 2.1. If there exists some $r \in (0, 1)$ satisfying*

$$\begin{aligned} & d^2(Ax, By) \\ & \le r \max\{d(Ax, Sx)d(By, Ty), d(Ax, Sx)d(Sx, Ty), \\ & \quad d(Ax, By)d(Sx, Ty), d(Ax, By)d(Ax, Sx), d(Ax, By)d(By, Ty), \\ & \quad d(Ax, Ty)d(By, Sx), d^2(By, Ty), d^2(Sx, Ty)\} \end{aligned} \tag{2.10}$$

for all $x, y \in X$, then there exist $u, v \in X$ such that $Au = Su = Bv = Tv$.

Theorem 2.2. *Let X be a T_1 topological space satisfying the first axiom of countability. Let (Y, d) be a complete metric space and $A, B, S : X \to Y$ satisfy that*

(a6) *$AX \cup BX \subseteq SX$,*

and one of the following conditions:

(a7) *A and S are continuous, A is proper with AX closed;*

(a8) *A and S are continuous, S is proper with SX closed;*

(a9) *B and S are continuous, B is proper with BX closed;*

(a10) *B and S are continuous, S is proper with SX closed.*

If there exists $\phi \in \Phi$ such that

$$\begin{aligned} &d^2(Ax, By) \\ &\leq \phi(d(Ax, Sx)d(By, Sy), d(Ax, Sx)d(Sx, Sy), d(Ax, By)d(Sx, Sy), \\ &\quad d(Ax, By)d(Ax, Sx), d(Ax, By)d(By, Sy), d(Ax, Sy)d(By, Sx), \\ &\quad d^2(By, Sy), d^2(Sx, Sy)) \end{aligned} \tag{2.11}$$

for all $x, y \in X$, then there exists $u \in X$ such that $Au = Bu = Su$.

Proof. Let $x_0 \in X$. It follows from (a6) that there exist sequences $\{x_n\}_{n\geq 0} \subseteq X$ and $\{y_n\}_{n\geq 1} \subseteq Y$ such that $y_{2n+1} = Sx_{2n+1} = Ax_{2n}$ for all $n \geq 0$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$ for all $n \geq 1$. As in the proof of Theorem 2.1, we conclude that $\{y_n\}_{n\geq 1}$ converges to a point $z \in Y$. Assume that (a7) holds. Let $C = \{Ax_{2n} : n \geq 0\} \cup \{z\}$. It is clear that $C = \overline{C} = \overline{AX} = AX \subseteq Y$ and C is compact. It follows that $A^{-1}(C)$ is compact since A is proper. Then there exists a subsequence $\{x_{2n_k}\}_{k\geq 1}$ of $\{x_{2n}\}_{n\geq 0}$ such that it converges to some point $u \in X$. The continuity of A and S ensure that

$$\lim_{k\to\infty} Ax_{2n_k} = Au = z = \lim_{k\to\infty} Sx_{2n_k} = Su.$$

Suppose that $Au \neq Bu$. In light of (2.11) we deduce that

$$\begin{aligned} &d^2(Au, Bu) \\ &\leq \phi(d(Au, Su)d(Bu, Su), d(Au, Su)d(Su, Su), d(Au, Bu)d(Su, Su), \\ &\quad d(Au, Bu)d(Au, Su), d(Au, Bu)d(Bu, Su), d(Au, Su)d(Bu, Su), \\ &\quad d^2(Bu, Su), d^2(Su, Su)) \\ &= \phi(0, 0, 0, 0, d^2(Au, Bu), 0, d^2(Au, Bu), 0) \\ &\leq \bar{\phi}(d^2(Au, Bu)) \\ &< d^2(Au, Bu), \end{aligned}$$

which is a contradiction. Hence $Au = Bu$. Consequently we get that $Au = Bu = Su$. Similarly, we can complete the proof if one of (a8), (a9) and (a10) holds. This completes the proof. □

Corollary 2.2. *Let X be a T_1 topological space satisfying the first axiom of countability. Let (Y, d) be a complete metric space and $A, B, S : X \to Y$ satisfy* (a6) *and one of* (a7)-(a10) *in Theorem 2.2. If there exists some $r \in (0, 1)$*

satisfying

$$\begin{aligned} &d^2(Ax,By) \\ &\le r\max\{(d(Ax,Sx)d(By,Sy),d(Ax,Sx)d(Sx,Sy), \\ &\quad d(Ax,By)d(Sx,Sy),d(Ax,By)d(Ax,Sx),d(Ax,By)d(By,Sy), \\ &\quad d(Ax,Sy)d(By,Sx),d^2(By,Sy),d^2(Sx,Sy)\} \end{aligned} \tag{2.12}$$

for all $x, y \in X$, then there exists $u \in X$ such that $Au = Bu = Su$.

In case $X = Y$ and $S = T = I$ in Theorems 2.1 and 2.2, we conclude immediately the following result:

Corollary 2.3. *Let (X, d) be a complete metric space and $A, B : X \to X$ satisfy one of the following conditions:*
(a11) *A is continuous and proper with AX closed;*
(a12) *B is continuous and proper with BX closed.*
If there exists $\phi \in \Phi$ such that

$$\begin{aligned} &d^2(Ax,By) \\ &\le \phi(d(Ax,x)d(By,y),d(Ax,x)d(x,y),d(Ax,By)d(x,y), \\ &\quad d(Ax,By)d(Ax,x),d(Ax,By)d(By,y),d(Ax,y)d(By,x), \\ &\quad d^2(By,y),d^2(x,y)) \end{aligned} \tag{2.13}$$

for all $x, y \in X$, then A and B have a common fixed point in X.

3. Application

Throughout this section, let X and Y be Banach Spaces, $S \subseteq X$ be the state space, $D \subseteq Y$ be the decision space and $B(S)$ denote the set of all bounded real-valued functions on S. Define

$$d(f,g) = \sup\{|f(x) - g(x)| : x \in S\}, \quad \forall f, g \in B(S).$$

It is clear that $(B(S), d)$ is a complete metric space. Following Bellman and Lee [1], the basic form of the functional equation in dynamic programming is as follows:

$$f(x) = \operatorname*{opt}_{y \in D}\{H(x, y, f(T(x, y)))\},$$

where x, y denote the state and decision vectors, respectively, T denotes the transformation of the process and $f(x)$ denotes the optimal return with the initial state x. In this section, we shall study existence of the solutions of the following functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y\in D}\{u(x,y) + H_i(x,y,f_i(T(x,y)))\}, \quad x\in S,\ i=1,2, \tag{3.1}$$

where $u : S\times D\to R$, $T : S\times D\to S$ and $H_i : S\times D\times R\to R$ for $i=1,2$.

Theorem 3.1. *Suppose that the following conditions are satisfied:*
(a13) *u and H_i are bounded for $i=1,2$;*
(a14)

$$\begin{aligned}&|H_1(x,y,g(t)) - H_2(x,y,h(t))|\\ &\le [\phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h),\\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), d(A_1g,h)d(A_2h,g),\\ &\qquad d^2(A_2h,h), d^2(g,h))]^{\frac{1}{2}}\end{aligned}$$

for all $(x,y)\in S\times D$, $g,h\in B(S)$ and $t\in S$, where $\phi\in\Phi$, A_1 and A_2 are defined as follows:

$$A_ig_i(x) = \sup_{y\in D}\{u(x,y) + H_i(x,y,g_i(T(x,y)))\}, \quad x\in S,\ g_i\in B(S),\ i=1,2.$$

If there exists some $A_i\in\{A_1,A_2\}$ satisfies the following conditions:
(a15) *for any compact subset $C\subseteq A_i(B(S))$, $A_i^{-1}(C)$ is compact in $B(S)$;*
(a16) *for any sequence $\{h_n\}_{n\ge1}\subseteq B(S)$ and $h\in B(S)$*

$$\lim_{n\to\infty}\sup_{x\in S}|h_n(x)-h(x)| = 0 \quad\Longrightarrow\quad \lim_{n\to\infty}\sup_{x\in S}|A_ih_n(x)-A_ih(x)| = 0;$$

(a17) *for any sequence $\{h_n\}_{n\ge1}\subseteq B(S)$ and $h\in B(S)$*

$$\lim_{n\to\infty}\sup_{x\in S}|A_ih_n(x)-h(x)| = 0 \quad\Longrightarrow\quad h\in A_i(B(S)),$$

then the system of functional equations (3.1) *possesses a common solution in* $B(S)$.

Proof. It follows from (a13)-(a17) that A_1 and A_2 are self mappings of $B(S)$ and satisfy one of (a11) and (a12). For any $g,h\in B(S)$, $x\in S$ and $\epsilon>0$, there exist $y,z\in D$ such that

$$A_1g(x) < u(x,y) + H_1(x,y,g(T(x,y))) + \epsilon, \tag{3.2}$$

$$A_2h(x) < u(x,z) + H_2(x,z,h(T(x,z))) + \epsilon. \tag{3.3}$$

Note that

$$A_1g(x) \geq u(x,z) + H_1(x,z,g(T(x,z))), \tag{3.4}$$

$$A_2h(x) \geq u(x,y) + H_2(x,y,h(T(x,y))). \tag{3.5}$$

From (3.2), (3.5) and (a14), we have

$$\begin{aligned} &A_1g(x) - A_2h(x) \\ &< H_1(x,y,g(T(x,y))) - H_2(x,y,h(T(x,y))) + \epsilon \\ &\leq [\phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h), \\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), d(A_1g,h)d(A_2h,g), \\ &\qquad d^2(A_2h,h), d^2(g,h))]^{\frac{1}{2}} + \epsilon. \end{aligned} \tag{3.6}$$

By virtue of (3.3), (3.4) and (a14), we know that

$$\begin{aligned} &A_1g(x) - A_2h(x) \\ &> H_1(x,z,g(T(x,z))) - H_2(x,z,h(T(x,z))) - \epsilon \\ &\geq -[\phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h), \\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), d(A_1g,h)d(A_2h,g), \\ &\qquad d^2(A_2h,h), d^2(g,h))]^{\frac{1}{2}} - \epsilon. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} d(A_1g,A_2h) &= \sup_{x\in S} |A_1g(x) - A_2h(x)| \\ &\leq [\phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h), \\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), \\ &\qquad d(A_1g,h)d(A_2h,g), d^2(A_2h,h), d^2(g,h))]^{\frac{1}{2}} + \epsilon, \end{aligned}$$

letting ϵ tend to zero in the above inequality, we arrive at

$$\begin{aligned} d(A_1g,A_2h) &\leq [\phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h), \\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), \\ &\qquad d(A_1g,h)d(A_2h,g), d^2(A_2h,h), d^2(g,h))]^{\frac{1}{2}}. \end{aligned}$$

This implies that

$$\begin{aligned} d^2(A_1g,A_2h) &\leq \phi(d(A_1g,g)d(A_2h,h), d(A_1g,g)d(g,h), d(A_1g,A_2h)d(g,h), \\ &\qquad d(A_1g,A_2h)d(A_1g,g), d(A_1g,A_2h)d(A_2h,h), \\ &\qquad d(A_1g,h)d(A_2h,g), d^2(A_2h,h), d^2(g,h)). \end{aligned}$$

It follows from Corollary 2.3 that A_1 and A_2 have a common fixed point $v \in B(S)$, that is, v is a common solution of the functional equations (3.1). This completes the proof. □

References

[1] R. Bellman, R.S. Lee, Functional equations arising in dynamic programming, *Aequations Math.*, **17** (1978), 1-18.

[2] M.S. Khan, A note on a nonlinear functional equation, *Rend. Sem. Fac. Sci. Univ. Cagliari*, **49** (1979), 87-89.

[3] Z. Liu, On coincidence point theorems in T_1 topological spaces, *Bull. Calcutta Math. Soc.*, **85** (1993), 531-534.

[4] Z. Liu, S.M. Kang, Y.S. Kim, Coincidence points in T_1 topological spaces, *East. Asian Math. J.*, **18** (2002), 147-154.

[5] R. Machuca, A coincidence theorem, *Amer. Math. Monthly*, **74** (1976), 569.

[6] J. Matkowski, Fixed point theorems for mappings with a contractive iterate at a point, *Proc. Amer. Math. Soc.*, **62** (1977), 344-348. 80.