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# THEOMOELASTIC PROBLEM OF A THIN ANNULAR DISC DUE TO RADIATION

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**Abstract:** An attempt has been made to study thermoelastic response of a thin annular disc occupying the space $D : a \leq r \leq b,\ -h \leq z \leq h$, with boundary conditions are of radiation type. We apply transformation techniques to find the thermoelastic solution. Numerical calculations are carried out and results are depicted graphically.

**AMS Subject Classification:** 74I25, 74H39, 74D99
**Key Words:** thermoelastic problem, annular disc, thermal stress

## 1. Introduction

Nowacki in [1] has determined steady-state thermal stresses in a thick circular plate subjected to an axisymmetric temperature distribution on the upper face with zero temperature on the lower face and circular edge. Roy Choudhary in [2] has succeeded in determining the quasi-static thermal stresses in a circular plate subjected to transient temperature along the circumference of circular upper face with lower face at zero temperature and the fixed circular edge thermally insulated.

Wankhede in [3] has determined the quasi-static thermal stresses in circular plate subjected to arbitrary initial temperature on the upper face with lower face at zero temperature..

This paper is concerned with transient thermoelastic problem of a thin annular disc occupying the space $D : a \leq r \leq b,\ -h \leq z \leq h$, with boundary conditions of radiation type.

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## 2. Statement of the Problem

Consider thin annular disc of thickness $2h$ occupying the space $D : a \leq r \leq b$, $-h \leq z \leq h$, the material is homogeneous and isotropic. The differential equation governing to the displacement function $U(r, z, t)$ is

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = (1+v)\, a_t T, \tag{1}$$

$$U_r = 0 \text{ at } r = a, b, \tag{2}$$

where $v$ and $a_t$ are Poison's ratio and the linear coefficient of thermal expansion of the material of the plate and $T$ is the temperature of the plate satisfying the differential equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{k}\frac{\partial T}{\partial t}, \tag{3}$$

subject to initial condition

$$M_t(T,\ 1,\ 0,\ 0) \ = \ 0, \text{ for all } a \leq r \leq b,\ -h \leq z \leq h. \tag{4}$$

The boundary conditions are

$$M_r(T,\ 1,\ k_1,\ a) \ = \ 0, \text{ for all } -h \leq z \leq h,\ t > 0, \tag{5}$$

$$M_r(T,\ 1,\ k_2,\ b) \ = \ 0, \text{ for all } -h \leq z \leq h,\ t > 0, \tag{6}$$

$$M_z(T,\ 1,\ k_3,\ h) \ = \ e^{-\omega t} f(z)\, \delta(r - r_0), \tag{7}$$

$$M_z(T,\ 1,\ k_4,\ -h) \ = \ e^{-\omega t} g(z) \delta(r - r_0), \text{ for all } a \leq r \leq b,\ t > 0. \tag{8}$$

The general expression for these conditions can be written as

$$M_v(f, k,\ \bar{\bar{k}}, s) = (\bar{k} f + \bar{\bar{k}} \hat{f})_{v=s},$$

where the prime $\hat{()}$ denotes differentiation with respect to $v$, $\delta(r - r_0)$ is the Dirac Delta function having $a \leq r_0 \leq b$, $\omega > 0$ is constant, $e^{-\omega t}\delta(r - r_0)$ is the additional sectional heat available on its surface at $z = -h$, $h$.

The stress function $\sigma_{rr}$ and $\sigma_{\theta\theta}$ are given by

$$\sigma_{rr} = -2\mu \frac{1}{r}\frac{\partial U}{\partial r}, \tag{9}$$

$$\sigma_{\theta\theta} = -2\mu\, \frac{\partial^2 U}{\partial r^2}, \tag{10}$$

where $\mu$ is the Lame's constant, while each of the stress functions $\sigma_{rr}$, $\sigma_{zz}$ and $\sigma_{\theta z}$ are zero within the plate in the plane state of stress.

The equations (1) to (10) constitute the mathematical formulation of the problem under consideration.

## 3. Solution of the Problem

### 3.1. Transient Heat Conduction Analysis

In order to solve equation (2) under the boundary condition (4), we first introduce the method of Marchi- Zgrablich transform and Marchi- Fasulo transform of order $n$ over the variable $r$. Let $n$ be the parameter of the transform, then the integral transform and its inversion theorems are written

$$\begin{aligned} g^*(\xi_n,\ z,t) &= \int_a^b g(r,z,t)\ S_0(k_1,k_2,\mu_n r)dr\ , \\ g(r,\ z,\ t) &= \sum_{n=1}^{\infty} \frac{g^*(\xi_n,z,t)\ S_0(k_1,k_2,\mu_n r)}{\mu_n}, \end{aligned} \tag{11}$$

and

$$\begin{aligned} f^*(r,\ \lambda_n,t) &= \int_a^b f(r,z,t)\ P_n(z)dz\ , \\ f(r,\ z,\ t) &= \sum_{n=1}^{\infty} \frac{f^*(r,\lambda_n,t)\ P_n(z)}{\lambda_n}, \end{aligned} \tag{12}$$

Applying the transform defined in equation (11) to the equation (3), one obtains

$$k\left[-\mu_m^2\overline{T}(m,\ z,t) + \frac{d^2\overline{T}(m,z,t)}{dz^2}\right] = \frac{d\overline{T}(m,z,t)}{dt} \tag{13}$$

Applying the transform defined in equation (12) to the equation (13), we obtain

$$k\left[-p^2\overline{\overline{T}}(m,\ n,t)\right] = \frac{d\overline{\overline{T}}(m,n,t)}{dt}, \tag{14}$$

where $p^2 = (\mu_m^2 + \lambda_n^2)$

Equation (14) is a first order differential equation whose solution is given by

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{P_n(z)}{\lambda_n^2}\frac{\psi}{p^2}[1 - e^{-p^2kt}]\frac{S_0(k_1,k_2,\mu_m r)}{\mu_m^2}, \tag{15}$$

where $\psi = \frac{P_n(-h)}{k_2}e^{-\omega t}\ \delta(r-r_0) - \frac{P_n(h)}{k_1}e^{-\omega t}\ \delta(r-r_0)$,

$$P_n(z) = Q_n\cos(a_n z) - W_n\sin(a_n z),$$

$$Q_n = a_n(\alpha_1 + \alpha_2)\cos(a_n h) + (\beta_1 - \beta_2)\sin(a_n h),$$

$$W_n = (\beta_1 + \beta_2)\cos(a_n h) + (\alpha_2 - \alpha_1)a_n \sin(a_n h),$$

$$\lambda_n = \int_{-h}^{h} P_n^2(z)\,dz = h\left[Q_n^2 + W_n^2\right] + \frac{\sin(2a_n h)}{2a_n}\left[Q_n^2 - W_n^2\right]$$

The eigen values $a_n$ are the solutions of the equation

$$[\alpha_1 a\cos(ah) + \beta_1\sin(ah)] \times [\beta_2\cos(ah) + \alpha_2 a\sin(ah)] \\ = [\alpha_2 a\cos(ah) - \beta_2\sin(ah)] \times [\beta_1\cos(ah) - \alpha_1 a\sin(ah)],$$

where $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are constants.

The kernel function $S_p(\alpha,\,\beta,\,\mu_m\,r)$ can be defined as

$$S_p(\alpha,\,\beta,\,\mu_m\,r) = J_p(\mu_m r)\,[Y_p(\alpha,\mu_m a) + Y_p(\beta,\mu_m b)] \\ - Y_p(\mu_m r)\,[J_p(\alpha,\mu_m a) + J_p(\beta,\mu_m b)],$$

and $J_p(\mu\,r)$ and $Y_p(\mu\,r)$ are Bessel function of first and second kind respectively.

The eigen values $\mu_m$ are the positive roots of the characteristic equation

$$J_0(k_1,\ \mu a)\,Y_0(k_2,\ \mu b) - J_0(k_2,\ \mu b)\,Y_0(k_1,\ \mu a) = 0.$$

Equation (15) is the desired solution of the given problem.

## 4. Thermoelastic Displacement Function

Substituting value of temperature distribution $T(r,\ z,t)$ from (15) in equation (1), one obtains the thermoelastic displacement function $U(r,z,t)$ as

$$U = -(1+v)\,a_t\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n^2}\frac{\psi}{p^2}[1 - e^{-p^2kt}]\frac{S_0(k_1,k_2,\mu_m r)}{\mu_m^2}. \tag{16}$$

## 5. Determination of Stress Functions

Substituting the value of thermoelastic displacement function $U(r, z, t)$ from equation (16) in equations (9) and (10), one obtain the stress functions

$$\sigma_{rr} = \left(\frac{2\mu\,(1+v)\,a_t}{r}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n^2}\frac{\psi}{p^2}[1-e^{-p^2kt}]\frac{S_0'(k_1,k_2,\mu_m r)}{\mu_m^2}, \tag{17}$$

$$\sigma_{\theta\theta} = 2\mu\,(1+v)\,a_t\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n^2}\frac{\psi}{p^2}[1-e^{-p^2kt}]\frac{S_0^{?}(k_1,k_2,\mu_m r)}{\mu_m}. \tag{18}$$

## 6. Special Case and Numerical Results

Set $T = \delta(r-r_0)(r-a)^2(r-b)^2e^z$.

Take k=0.86, $h = 2$cm, $b = 4$ cm, $t = 1$ sec, $r_0 = 1$ cm $\omega = 1$. Substitute this values in (15), one obtains

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{P_n(z)}{\lambda_n^2}\frac{\psi}{p^2}[1-e^{-0.86p^2}]\frac{S_0(0.25,0.25,\mu_m r)}{\mu_m^2}.$$

## 7. Alternative Solution of the Problem

Applying the transform defined in equation (11) to the equation (3), one obtains

$$k\left[-\mu_m^2\overline{T}(m,\ z,t)+\frac{d^2\overline{T}(m,z,t)}{dz^2}\right] = \frac{d\overline{T}(m,z,t)}{dt}. \tag{19}$$

Applying the Laplace transform to the equation (19), we get

$$\frac{d^2\overline{T}^*(m,n,t)}{dz^2} - \left(\mu_m^2+\frac{s}{k}\right)\overline{T}^* = 0. \tag{20}$$

Equation (20) is a second order differential equation whose solution is given by,

$$\overline{T}^* = Ae^{qz}+Be^{-qz}, \tag{21}$$

where A and B are constants to be determined and

$$q^2 = \left(\mu_m^2+\frac{s}{k}\right).$$

Using boundary conditions we obtain the values of A and B. Substituting these values in equation (21) and then applying inversion of Laplace transform and Marchi- Zgrablich transform, one obtains the expression for temperature distribution T(r,z,t) as

$$T = \sum_{m=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} m \sin\left(\frac{m\pi}{2h}\right)(z-h) S_0(k_1, k_2, \mu_n r)}{\mu_n} \times \int_0^t \overline{g}(r) e^{-k\left(\mu_m^2 + \frac{m^2\pi^2}{4h^2}\right)(t-t')} dt'$$

$$- \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+1} m \sin\left(\frac{m\pi}{2h}\right)(z+h)\, S_0(k_1, k_2, \mu_n r)}{\mu_n} \times \int_0^t \overline{f}(r) e^{-k\left(\mu_m^2 + \frac{m^2\pi^2}{4h^2}\right)(t-t')} dt'. \quad (22)$$

## 8. Thermoelastic Displacement Function

Substituting the value of temperature distribution $T(r,\ z,t)$ from (22) in equation (1), one obtains the thermoelastic displacement function $U(r,z,t)$ as

$$U = -(1+v)\, a_t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+1} m \sin\left(\frac{m\pi}{2h}\right)(z-h)\, S_0(k_1, k_2, \mu_n r)}{\mu_n} \times \int_0^t \overline{g}(r) e^{-k\left(\mu_m^2 + \frac{m^2\pi^2}{4h^2}\right)(t-t')} dt'$$

$$+ (1+v)\, a_t \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(-1)^{m+1} m \sin\left(\frac{m\pi}{2h}\right)(z+h)\, S_0(k_1, k_2, \mu_n r)}{\mu_n} \times \int_0^t \overline{f}(r) e^{-k\left(\mu_m^2 + \frac{m^2\pi^2}{4h^2}\right)(t-t')} dt'. \quad (23)$$

## 9. Determination of Stress Functions

Substituting the value of thermoelastic displacement function $U(r,z,t)$ from equation (23) in equations (9) and (10), one obtain the stress functions as

$$\sigma_{rr} = \left(\frac{2\mu\,(1+v)\;a_t}{r}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}m\sin\left(\frac{m\pi}{2h}\right)(z-h)\,S_0'(k_1,k_2,\mu_n r)$$
$$\times\int_0^t \overline{g}(r)e^{-k\left(\mu_m^2+\frac{m^2\pi^2}{4h^2}\right)(t-t')}dt'$$

$$-\left(\frac{2\mu\,(1+v)\;a_t}{r}\right)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}m\sin\left(\frac{m\pi}{2h}\right)(z+h)\,S_0'(k_1,k_2,\mu_n r)$$
$$\times\int_0^t \overline{f}(r)e^{-k\left(\mu_m^2+\frac{m^2\pi^2}{4h^2}\right)(t-t')}dt', \quad (24)$$

$$\sigma_{\theta\theta} = 2\mu\,(1+v)\;a_t\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}m\sin\left(\frac{m\pi}{2h}\right)(z-h)\,\mu_n S_0''(k_1,k_2,\mu_n r)$$
$$\times\int_0^t \overline{g}(r)e^{-k\left(\mu_m^2+\frac{m^2\pi^2}{4h^2}\right)(t-t')}dt'$$

$$-2\mu\,(1+v)\;a_t\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}m\sin\left(\frac{m\pi}{2h}\right)(z+h)\,\mu_n S_0''(k_1,k_2,\mu_n r)$$
$$\times\int_0^t \overline{f}(r)e^{-k\left(\mu_m^2+\frac{m^2\pi^2}{4h^2}\right)(t-t')}dt'. \quad (25)$$

## 10. Special Case and Numerical Results

Set$f(z) = e^{-h}e^{-wt}(1-e^{-t})(r-b)\delta(r-r_0)$ and $g(z) = e^{h}e^{-wt}(1-e^{-t})(r-b)\delta(r-r_0)$.

Take $k = 86$, $h = 2$cm, $b = 4$ cm, $t = 1$ sec, $r_0 = 1$ cm $\omega = 1$. Substitute these values in (22), one obtains

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{(-1)^m m \sin(0.79m)\,(z+2)\,S_0(0.25,0.25,\mu_n r)}{\mu_n}\right] S_0(0.25,0.25,\mu_n) \times \int_0^1 (0.06)e^{-0.86\left(\mu_m^2+0.62m^2\right)(1-t')}dt'$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{(-1)^{m+1} m \sin(0.79m)\,(z-2)\,S_0(0.25,0.25,\mu_n r)}{\mu_n}\right] S_0(0.25,0.25,\mu_n) \times \int_0^1 (1.2)e^{-0.86\left(\mu_m^2+0.62m^2\right)(1-t')}dt'. \quad (26)$$

## 11. Conclusion

In this paper, the temperature distribution, displacement function and thermal stresses have been determined for thin annular disc. The finite Marchi-Fasulo transform, March-Zgrablich transform and Laplace transform techniques have been used to obtain numerical results. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications

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**Appendix A: Figures**

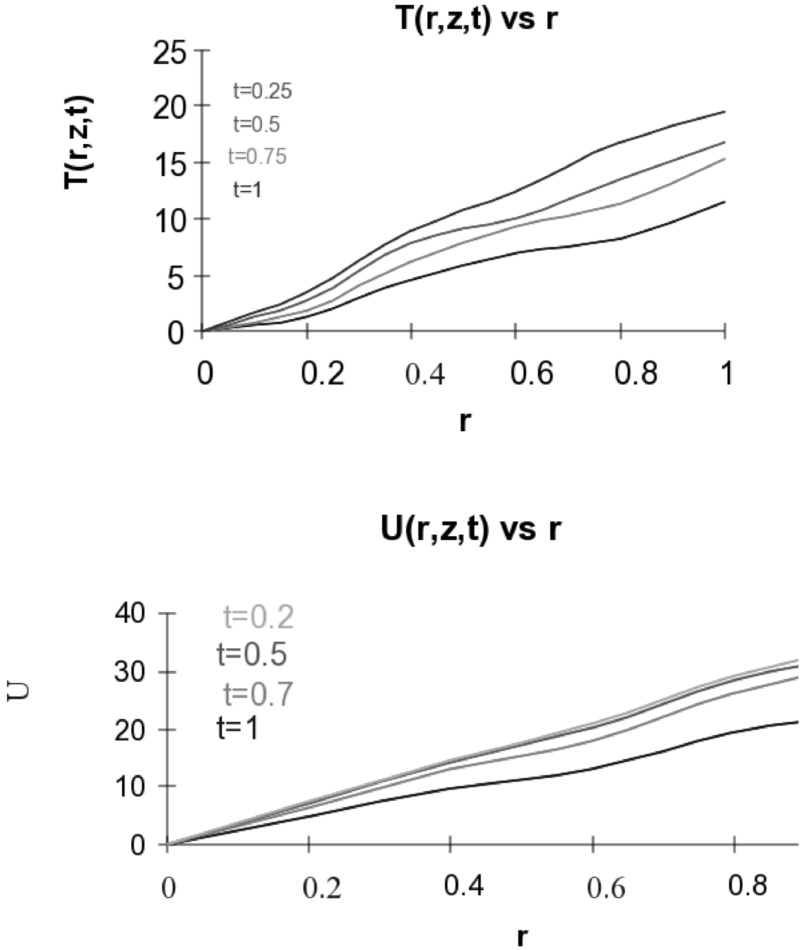


Figure 1: Graph of equation (16)

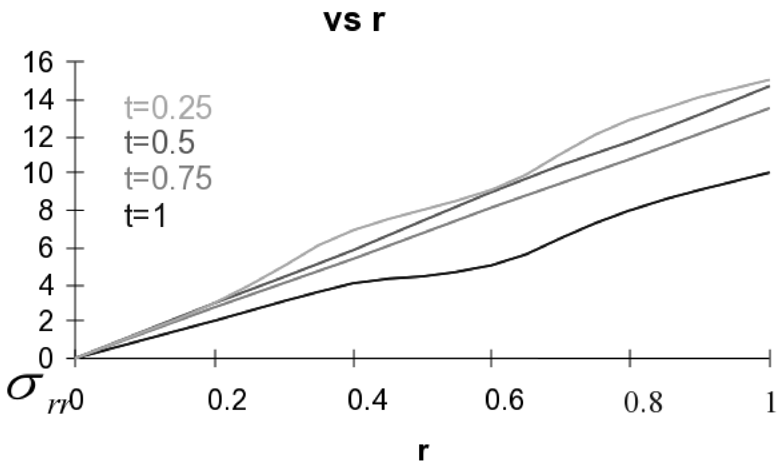


Figure 2: Graph of equation (17)

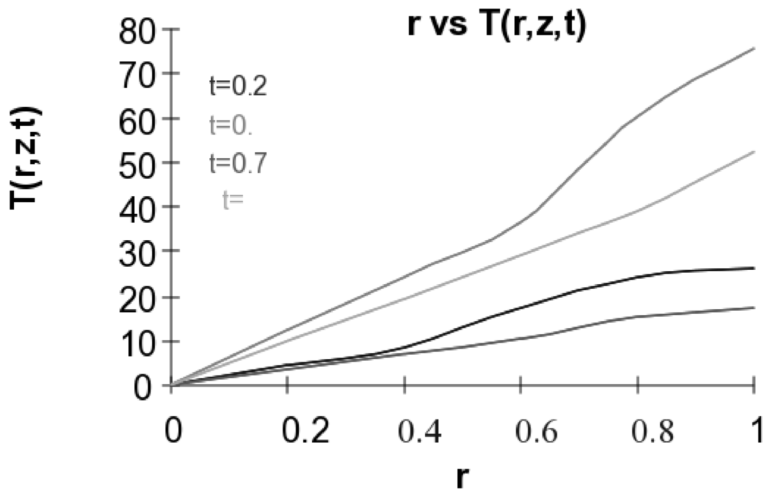


Figure 3: Graph of equation (22)

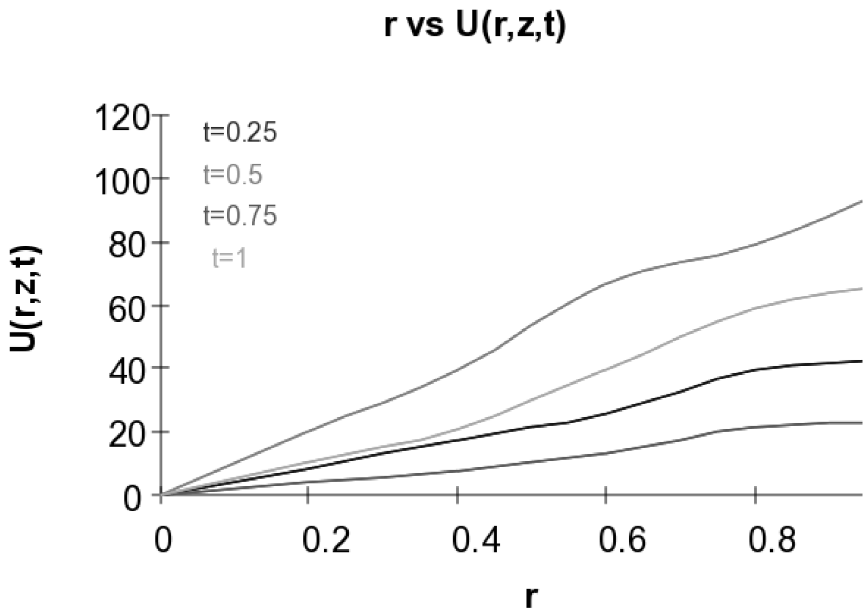


Figure 4: Graph of equation (23)

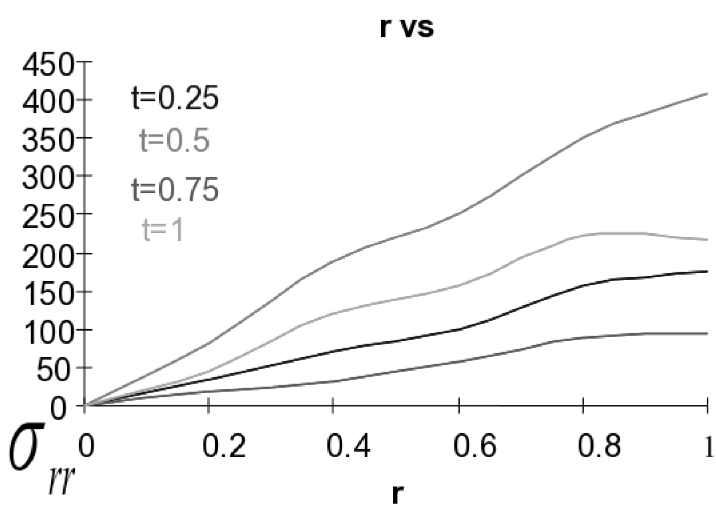


Figure 5: Graph of equation (24)