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JOINS AND GENERIC UNIQUENESS OF A SECANT SPACE

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Abstract: Here we study two cases (a join and secant varieties to a curve $X \subset \mathbb{P}^r$) in which a general point of the join in contained in a unique secant space.

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Let $\mathbb{K}$ be an algebraically closed base field. Let $X, Y, Y_j \subset \mathbb{P}^r$, $j \geq 1$, be integral and closed subvarieties. Let $[X, Y]$ denote the join of X and Y, i.e. set $[X, Y] = \{P\}$ if $X = Y = \{P\}$ are the same point, while in all other cases let $[X, Y]$ denote the closure in $\mathbb{P}^r$ of the union of all lines $\langle \{P, Q\} \rangle$ with $P \in X$, $Q \in Y$ and $P \neq Q$. For each integer $s \geq 3$ define inductively the join $[Y_1, \dots, Y_s]$ by the formula $[Y_1, \dots, Y_s] := [[Y_1, \dots, Y_{s-1}], Y_s]$. Set $\sigma_1(X) := X$. For any integer $s \geq 2$ let $\sigma_s(X)$ denote the join of s copies of X ($\sigma_s(X)$ is sometimes called the $(s-1)$-secant variety, but we prefer to call it the s-secant variety of X). For any integer e such that $0 \leq e \leq r$ let $G(e, r)$ denote the Grassmannian of all lines of $\mathbb{P}^r$. For any closed subscheme $E \subseteq \mathbb{P}^r$ let $\langle E \rangle$ denote its linear span. For any integer $s \in \{0, \dots, r\}$ let $G(s, r)$ denote the Grassmannian of all s-dimensional linear subspaces of $\mathbb{P}^r$. Let $\mathbb{I}(s, r) \subset \mathbb{P}^r \times G(s-1, r)$ denote the incidence correspondence, i.e. set $\mathbb{I}(s, r) := \{(P, V) \in \mathbb{P}^r \times G(s-1, r) : P \in V\}$. Let $\pi_1 : \mathbb{I}(s, r) \to \mathbb{P}^r$ and $\pi_2 : \mathbb{I}(s, r) \to G(s, r)$ denote the restriction to $\mathbb{I}(s, r)$

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of the projections of $\mathbb{P}^r \times G(s,r)$ onto its factors. If X and Y are positive dimensional, then $[X,Y]$ and $\sigma_s(X)$, $s \ge 2$, may be obtained in the following way. Set $U_{X,Y} := \{(P,Q) \in X \times Y : P \ne Q\}$. Let $J_{X,Y} \subseteq X \times Y \times G(1,r)$ denote the closure of set $\cup_{(P,Q)\in X\times Y, P\ne Q}\{(P,Q,\langle\{P,Q\}\rangle\})$. We have $\dim(J_{X,Y}) = \dim(X) + \dim(Y) + 1$. Composing the inclusions $J_{X,Y} \hookrightarrow X \times Y \times G(1,r)$ with the inclusions $X \hookrightarrow \mathbb{P}^r$ and $Y \hookrightarrow \mathbb{P}^r$ we may see $J_{X,Y}$ as a closed subscheme of $\mathbb{P}^r \times \mathbb{P}^r \times G(1,r)$. See $\mathbb{P}^r \times \mathbb{I}(1,r)$ as a closed subscheme of $\mathbb{P}^r \times \mathbb{P}^r \times G(1,r)$ (call j the inclusion) and set $E_{X,Y} := j^{-1}(\mathbb{P}^r \times \mathbb{I}(r,s)) \cap J_{X,Y}$. Let $\beta_{X,Y} : E_{X,Y} \to \mathbb{P}^r$ denote the projection onto the first factor. We have $[X,Y] = \beta_{X,Y}(E_{X,Y})$ and we see $\beta_{X,Y} : E_{X,Y} \to [X,Y]$ as a proper surjective morphism. Now assume that X is non-degenerate. Fix an integer s such that $2 \le s \le r$ and let $U_{X,s}$ denote the open subsets of X^s formed by all s-ples $(P_1,\dots,P_s)$ such that $\dim(\langle\{P_1,\dots,P_s\}\rangle) = s-1$. Let $F_{X,s} \subseteq \mathbb{P}^r \times G(s-1,r)$ be the closure of the union of all $(P_1,\dots,P_s,\langle\{P_1,\dots,P_s\}\rangle)$ with $(P_1,\dots,P_s) \in U_{X,s}$. Using again the incidence correspondence of $\mathbb{P}^r \times G(s-1,r)$ we get an integral variety $E_{X,s}$ and a proper morphism $\beta_{X,s} : E_{X,s} \to \mathbb{P}^r$ with $\sigma_s(X)$ as its image. Inspired by [2] we prove the following result.

Theorem 1. *Assume X,Y non-degenerate,* $\dim(X) = 1$ *and* $\dim(Y) \le r-3$. *Assume that for a general $(a,b,y) \in X \times X \times Y$ we have $\langle\{a,b,y\}\rangle \cap Y = \{y\}$ (set-theoretically). Then a general $P \in [X,Y]$ is contained in a unique line D spanned by a point of X and a point of Y.*

For any varieties $V \subset \mathbb{P}^r$ and any $x \in V_{reg}$ let $T_{V,x} \subset \mathbb{P}^r$ denote the Zariski tangent space. The " if " part of the proof of [2], Corollary 3.4 (with all its prerequisites), gives the following result.

Theorem 2. *Assume* $\dim(X) = 1$ *and* $\dim(T_{X,x} \cap T_{Y,y}) = \dim(Y) + 2 - \dim([X,Y])$ *for a general $(x,y) \in X \times Y$. Then $\beta_{X,Y}$ is separable.*

Proof of Theorem 1. We have $\dim([X,Y]) = \dim(Y) + 2$ (see [1], Part II of Proposition 1.3). Hence there are only finitely many lines (say s lines $D_1,\dots,D_s$) spanned by a point of X and a point of Y and containing P. Assume $s \ge 2$. Since P is general in $[X,Y]$, all these lines are transverse to X and to Y. Set $S_i := X \cap D_i$ and $S_i' := Y \cap D_i$. For general P we have $\sharp(S_i) = \sharp(S_1)$ and $\sharp(S_i') = \sharp(S_1')$ for all i. Fix a general $Q \in D_1$. Since $P \in D_1$ and Q is general in D_1, Q may be considered a general point of $[X,Y]$. Fix $a \in S_1$ and $y \in S_1'$. Since Q is general in $[X,Y]$, it is contained in s lines spanned by a point of X and a point of Y, one of them being D_1. Call $D_2(Q)$ another one and fix $b(Q) \in D_2(Q) \cap X$ and $y(Q) \in D_2(Q) \cap Y$. Since $D_2(Q) \ne D_1$, we have $b(Q) \ne a$ and $y(Q) \ne y$. Now we vary $Q \in D_1$. Assume for the moment $b(Q) = b(Q')$ for a general $(Q,Q') \in D_1 \times D_1$. Since $D_2(Q) \ne D_1$

and $D_2(Q) \neq D_1$, we have $D_1 \cap D_2(Q) = \{Q\}$ and $D_1 \cap D_2(Q') = \{Q'\}$. Since $Q \neq Q'$, we get $y(Q) \neq y(Q')$. Hence the plane $A := \langle D_1 \cup \{b(Q)\}\rangle$ contains infinitely many points of Y. Even in this case for fixed a, but varying y in Y the point $b(Q)$ moves, Hence $(a, b(Q))$ may be considered as a general element of $X \times X$. Notice that the assumption on X implies that a general secant line of X does not intersect Y. Thus $\langle\{a, b(Q)\}\rangle \cap Y = \emptyset$. Since $A \cap Y$ contains a curve and $\langle\{a, b(Q)\}\rangle$ is a line of the plane A, we get $\langle\{a, b(Q)\}\rangle \cap Y \neq \emptyset$, contradiction. Hence $b(Q) \neq b(Q')$ for a general $(Q, Q'$. Since $\dim(X) = 1 = \dim(D_1)$, for general $Q \in D_1$ the triple $(a, b(Q), y)$ may be considered as a general element of $X \times X \times Y$. Our assumption on X implies $y(Q) = y$. Since $Q \in D_1 \setminus \{y\}$, we get $D_2(Q) = D_1$, contradiction. □

Iterating the previous proof (or working directly with $(s-1)$-dimensional linear subspaces and the map $\beta_{X,s}$) we also get the following result.

Corollary 1. *Let $X \subset \mathbb{P}^r$ be an integral projective curve. Fix an integer $s \geq 2$ such that $2s \leq r-2$. Assume that $\{P_1, \dots, P_s\} = X \cap \langle\{P_1, \dots, P_{2s}\}\rangle$ for a general $(P_1, \dots, P_{2s}) \in X^{2s}$. Then for a general $P \in \sigma_s(X)$ there is a unique $V \in G(s, r)$ such that V is s-secant to X and $P \in V$.*

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References

[1] B. Ådlandsvik, Joins and higher secant varieties, *Math. Scand.*, **61** (1987), 213-222.

[2] M. Dale, Terracini's lemma and the secant variety of a curve, *Proc. London Math. Soc.*, **49**, No. 3 (1984), 329-339.

642