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ON THE SECANT VARIETIES OF SEGRE-VERONESE EMBEDDINGS OF $(\mathbb{P}^1)^n$ IN POSITIVE CHARACTERISTIC

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Abstract: We explain how to extends a recent result of Laface and Postinghel on the secant varieties of the Segre-Veronese varieties to the case of an algebraically closed field with large characteristic.

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Set $X := (\mathbb{P}^1)^n$, $n \geq 2$, and $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$. We have $\mathrm{Pic}(X) \cong \mathbb{Z}^n$. For any $(a_1, \dots, a_n) \in \mathbb{Z}^n$ let $\mathcal{O}_X(a_1, \dots, a_n)$ denote the line bundle on X with multidegree $(a_1, \dots, a_n)$. The line bundle $\mathcal{O}_X(a_1, \dots, a_n)$ is very ample if and only if $a_i > 0$ for all i. If this is the case, then $h^0(X, \mathcal{O}_X(a_1, \dots, a_n)) = \prod_{i=1}^n (a_i + 1)$. Set $r(a_1, \dots, a_r) = -1 + \prod_{i=1}^n (a_i + 1)$ and call $X(a_1, \dots, a_n)$ the image of X in $\mathbb{P}^{r(a_1,\dots,a_n)}$. For any integral and non-degenerate n-dimensional variety $Y \subset \mathbb{P}^r$ and any integer $s > 0$ let $\sigma_s(Y) \subseteq \mathbb{P}^r$ denote the closure in $\mathbb{P}^r$ of the union of all $(s-1)$-dimensional linear subspaces spanned by s points of Y. The integral variety $\sigma_s(Y)$ has dimension at most $\min\{r, s(n+1) - 1\}$ and the integer $\min\{r, s(n+1) - 1\}$ is called the expected dimension of $\sigma_s(Y)$. We say that (Y, s) is defective or that $\sigma_s(Y)$ is defective if $\sigma_s(Y)$ has not the expected dimension. A. Laface and E. Postinghel gave the full list of all $(n; a_1, \dots, a_n; s)$,

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$n \geq 2$, such that $X(a_1, \dots, a_s)$ is s-defective (in characteristic zero) (see [15]). This is an important result, proved in a very clever way and that culminates several years of efforts by several people (the most important previous paper being [12] which takes the case $a_i = 1$ for all i); among the other ones, se for instance [1], [2], [3], [8], [9], [10], [11] and references therein; everything started from the corresponding problem on $\mathbb{P}^n$ solved by Alexander and Hirschowitz (see [5], [6], [16]). Assuming $a_1 \leq \cdots a_n$ the list is the following one:

(a) $(2; 2, 2a; 2a+1)$, $a \geq 1$;

(b) $(3; 1, 1, 2a; 2a+1)$, $a \geq 1$;

(c) $(3; 2, 2, 2; 7)$;

(d) $(4; 1, 1, 1, 1; 3)$.

The case $a_i = 1$ for all i proved by Catalisano, Geramita and Gimigliano (see [12]) has important consequences for tensors and the notion of tensor rank. Similarly, for arbitrary $a_i > 0$ this problem is related to partially symmetric tensors (it gives the dimension of the set of partially symmetric tensors with prescribed border rank). However, if V is a finite-dimensional vector space over a field of characteristic p the vector spaces $S^c(V)^\vee$ and $S^c(V^\vee)$ are canonically isomorphic only if $p > c$. Hence if $p \leq a_i$ for some i, then one needs to adapt the theory of tensor rank for partially symmetric tensors to use Theorem 1 below in the applications.

Theorem 1. *There is a positive integer γ with the following property. Fix any prime $p \geq \gamma$ and any algebraically closed field $\mathbb{K}$ such that* $\mathrm{char}(\mathbb{K}) = p$. *Take X over $\mathbb{K}$ and fix $0 < a_1 \leq \cdots \leq a_n$ and s such that $(n; a_1, \dots, a_n, s)$ is not one of the cases (a), (b), (c) or (d) in the list. Then $\sigma_s(X(a_1, \dots, a_s))$ has the expected dimension.*

We may take $\gamma = 1 + (625!) \cdot (306)^{625}$ (see Remark 3).

For any smooth n-dimensional connected variety A, any $P \in A$ a *double of* A (or just a double point $\{2P, A\}$ is the closed subscheme of A with $(\mathcal{I}_{P,A})^2$ as its ideal sheaf. As a consequence, $\{2P, A\}_{red} = \{P\}$ and $\deg(\{2P, A\}) = n+1$. For any finite subset $S \subset A$ set $\{2S, A\} := \cup_{P \in S}\{2P, A\}$. The characteristic free part of Terracini's lemma and the fact that we took a linearly embedding to define $X(a_1, \dots, a_n)$ show that Theorem 1 (is a consequence of the following result (in both theorems the unknown integer γ is the same, in the sense that any γ good for Theorem 2 is good for Theorem 1 (see Remark 1).

Theorem 2. *There is a positive integer γ with the following property. Fix any prime $p \ge \gamma$ and any algebraically closed field $\mathbb{K}$ such that* $\mathrm{char}(\mathbb{K}) = p$. *Take X over $\mathbb{K}$ and fix $0 < a_1 \le \cdots \le a_n$ and s such that $(n; a_1, \dots, a_n, s)$ is not one of the cases (a), (b), (c) or (d) in the list. Let $S \subset X$ be a general subset with cardinality s. Then either $h^0(X, \mathcal{I}_{2S} \otimes \mathcal{O}_X(a_1, \dots, a_n)) = 0$ (case $s(n+1) \ge \prod_{i=1}^{n}(a_i + 1)$) or $h^1(X, \mathcal{I}_{2S} \otimes \mathcal{O}_X(a_1, \dots, a_n)) = 0$ (case $s(n+1) \le \prod_{i=1}^{n}(a_i + 1)$).*

Theorem 2 and the characteristic free part of the computations in [14] give the following result (in which we may take as γ the same positive integer as in Theorem 2).

Proposition 1. *Fix an integer $k > 0$ and set $E := \mathbb{P}^k \times X \times \mathbb{P}^k$. Fix positive integers $a_1, \dots, a_n$ such that $a_1 \le \cdots \le a_n$ and set $a_0 := 1$. Assume that $(n+1; a_0, \dots, a_n; s)$ is not as in case (b) or case (d) of the list, i.e., assume $(n; a_1, \dots, a_n; s) \notin \{((2; 1, 2a; a+1), (3; 1, 1, 1; 3)\}$. Then X is not $(1, s-1)$-defective in the sense of [13]. Moreover, X is not $(1, s-1)$-defective for any prime p for which Theorem 2 holds.*

1. The Proofs

Remark 1. Theorem 2 for a fixed prime p implies Theorem 1 for the same prime p, by the characteristic free part of Terracini's lemma (see [4]).

We only need to follow the proofs in the literature.

Remark 2. It seems that to apply [7], 2.3 and 8.1, for double points one needs to assume $p \ne 2$ (or $p > m$ for fat points with multiplicity $\le m$). However, the case $r = 1$ of [17], theorem 1.2, is characteristic free.

Proof of Theorem 2 (Modulo the Quoted References. (a) In [15] it is essential to specialize X to a reducible n-dimensional variety $X_1 \cup X_2$. This is done essentially taking the product of $(\mathbb{P}^1)^{n-1}$ with a degeneration of $\mathbb{P}^1$ to a reducible conic. It is easy to do that in arbitrary characteristic (with a control of the degeneration of the Picard groups). We only need to deform some zero-dimensional subschemes of $X_1 \cup X_2$ to some zero-dimensional schemes and all line bundles on $X_1 \cup X_2$ to line bundles on X (not the converse, which would be very delicate). Also the double degenerations claimed in [15] are obviously characteristic free if $n \ge 3$, while the case $n = 2$ is elementary from several points of view. Their proof requires the main theorem of [12], i.e. the case

$n_i = 1$ for all i (see step (c) below). Then they make a very clever (and characteristic free) induction. They preferred to start a part of the induction living to the computer the check of some cases. We discuss this problem in step (b).

(b) In [15], Proposition 5.1, a finite number of cases cases were done using a computer and a freely available program (see the web-page quoted in [15]). Laface and Postinghel worked over $\mathbb{F}_{307}$. Each of these cases may be translated to the fact that a certain rectangular matrix, say $a \times b$ with $a \leq b$, with as entries integers between 0 and 306 has at least one $a \times a$ minor whose determinant, Δ, is an integer not divisible by 307. In particular $\Delta \in \mathbb{Z} \setminus \{0\}$. Hence Δ is divisible only by finitely many primes. We may take as γ any positive integer bigger than the primes arising in this way for all the finitely many cases listed needed for [15], Proposition 5.1. Since the program quoted in [15] works over $\mathbb{F}_{307}$ and steps (a) and (c) are characteristic free by Remark 2, a statement like Theorem 1, 2 or Proposition 1 is true over any algebraically closed field with characteristic 307.

(c) Assume $a_i = 1$ for all i. In [9], [10] and [12] the authors developed a clever method which give a complete description of the case $a_i = 1$ for all i. They proved that the problem related to Theorem 2 for $(n; 1, \dots, 1; s)$ is equivalent to a certain postulation problem in $\mathbb{P}^n$. This equivalence is true with the same proof over an arbitrary field. They proved their main theorem using in a clever way the Differential Horace Method (characteristic free by Remark 2). At one point (line 15 of page 376 of [10]) they used (in the published version) a computer check (the case $(5; 1, 1, 1, 1, 1; 5)$). However, this case is one of the cases listed in [15], Proposition 5.1.

Proof of Proposition 1. With minor modifications the Terracini's lemma for Grassmann's defectivity proved in [14], Proposition 1.3, works in positive characteristic in the following sense. There are two maps from pairs of generically smooth varieties and their differential at general points of their domain are given by two suitable matrices M_1, M_2. The coranks of M_1 and M_2 are the same. Therefore if the rank of M_i is maximal, then the rank of M_{2-i} is maximal. Hence Theorem 2 implies Proposition 1. Joint work in progress (and characteristic free) will imply that a statement like Theorem 1 for a single prime p implies Proposition 1 for the same prime p. $\square$

Remark 3. We may take as γ the integer $1 + (625!) \cdot (306)^{625}$ for the following reason. In the cases listed in [15], Proposition 5.1, discussed in step (b) all the matrices have at most 625 rows or colums, the largest matrix arising in the case $n = 4$ and $a_1 = a_2 = a_3 = a_4 = 4$ (here $h^0(X, \mathcal{O}_X(4,4,4,4)) = 5^4$). Hence no prime $p \geq 1 + (625!) \cdot (306)^{625}$ divides one of the integers Δ arising in step (b).

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