International Journal of Pure and Applied Mathematics
Volume 71 No. 4 2011, 665-669

INTERPOLATION PROBLEMS WITH RESPECT TO REDUCIBLE SUBCURVES OF PROJECTIVE SPACES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X_1, \dots, X_s \subset \mathbb{P}^n$ be integral curves (repetitions are allowed). Fix $P \in \mathbb{P}^n$. Here we give several results on the possibilities of finding a small integer $k \le s$ and $P_i \in X_i$, $1 \le i \le k$, such that $P \in \langle \{P_1, \dots, P_k\} \rangle$.

AMS Subject Classification: 14N05, 14H50
Key Words: rank, linear span, reducible curve, X-rank

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field. For any finite set $S \subset \mathbb{P}^n$ let $\langle S \rangle \subseteq \mathbb{P}^n$ denote its linear span. For any $P \in \mathbb{P}^n$ and any integral and non-degenerate variety $X \subset \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$ (see [6], [4]). Several different definitions of X-rank may be defined if X is reducible (see [2]). Here we focus on one them. Our aim is to show that the corresponding interpolation problem is quite different if we allow X to be reducible and that the difference depends more from the mutual position of the irreducible components than from the geometry of the single irreducible components. Look at the following two statements.

Theorem 1. *Fix an odd integer $n \ge 3$, a 3-dimensional linear space $U_0 \subset \mathbb{P}^n$ and 2-dimensional linear subspaces V_i, $1 \le i \le (n-3)/2$. For all $i \in \{1, \dots, (n-3)/2\}$, define recursively the linear space U_i by the formula $U_i := \langle U_{i-1} \cup V_i \rangle$. Assume $\dim(U_i) = 3+2i$ for all i and call Q_i, $1 \le i \le (n-3)/2$,*

Received: July 18, 2011

the unique point of $V_i \cap U_{i-1}$. *Fix curves* $C_1, C_2 \subset U_0$ *such that for every* $Q \in U_0$ *there is* $(P_1, P_2) \in C_1 \times C_2$ *such that* $Q \in \langle \{P_1, P_2\} \rangle$. *Fix integral curves* $D_i \subset V_i$; *if* $Q_i \in D_i$, *then assume that every line of* V_i *though* Q_i *intersects* $D_i \setminus \{Q_i\}$. *Then for all* $P \in \mathbb{P}^n$ *there are* $A_j \in C_j$, $j = 1, 2$, *and* $B_i \in D_i$, $1 \le i \le (n-3)/2$, *such that* $P \in \langle \{A_1, A_2, B_1, \dots, B_{(n-3)/2}\} \rangle$.

In the statement of Theorem 1 we allow the case $C_1 = C_2$. We omit the proof of the following result, since it is equal to the proof of Theorem 1 that we will give below.

Theorem 2. *Fix an even integer* $n \ge 4$, *and* 2*-dimensional linear subspaces* $U_0 \subset \mathbb{P}^n$ *and* $V_i \subset \mathbb{P}^n$, $1 \le i \le (n-2)/2$. *For all* $i \in \{1, \dots, (n-2)/2\}$, *define recursively the linear space* U_i *by the formula* $U_i := \langle U_{i-1} \cup V_i \rangle$. *Assume* $\dim(U_i) = 2 + 2i$ *for all* i *and call* Q_i, $1 \le i \le (n-2)/2$, *the unique point of* $V_i \cap U_{i-1}$. *Fix an integral curve* $C \subset U_0$ *such that* $\deg(C) \ge 2$. *Fix integral curves* $D_i \subset V_{i-1}$; *if* $Q_i \in D_i$, *then assume that every line of* V_i *though* Q_i *intersects* $D_i \setminus \{Q_i\}$. *Assume that* C *is not a strange curve. Then for all* $P \in \mathbb{P}^n$ *there are* $A_1, A_2 \in C$, *and* $B_i \in D_i$, $1 \le i \le (n-2)/2$, *such that* $P \in \langle \{A_1, A_2, B_1, \dots, B_{(n-2)/2}\} \rangle$.

The assumption in Theorem 2 that C is not strange (always satisfied if $\mathrm{char}(\mathbb{K}) = 0$) is only made to be sure that $r_C(Q) \le 2$ for every $Q \in U_0$ (see [3]).

A simple dimensional count with joins (or the corresponding notion of X-border recant and the dimension of the secant varieties of X), show that Theorem 1 and 2 are sharp. They were also quite unexpected: for $n \ge 5$ we do not know any irreducible curve $X \subset \mathbb{P}^n$ such that $r_X(P) \le \lfloor (n+2)/2 \rfloor$ for all $P \in \mathbb{P}^n$.

Conjecture 1. *There is no integral non-degenerate curve* $X \subset \mathbb{P}^{2k+1}$, $k \ge 3$, *such that* $r_X(P) \le k+1$ *for all* $P \in \mathbb{P}^{2k+1}$.

For any integral subvarieties $A_i \subset \mathbb{P}^n$, $1 \le i \le k$, $k \ge 2$, let $J(A_1, \dots, A_k) \subseteq \mathbb{P}^n$ denote the join of $A_1, \dots, A_k$.

Theorem 3. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix an integer* $n \ge 3$ *and integral curves* X_i, $1 \le i \le n-1$, *such that* $X_1 \cap X_2 = \emptyset$. *If* $n > 3$ *fix general* $P_i \in X_i$, $i > 2$, *and assume* $J(X_1, X_2) \cap \langle \{P_3, \dots, P_{n-1}\} \rangle = \emptyset$ *and* $\dim(\langle \{P_3, \dots, P_{n-1}\} \rangle) = n-4$. *Then there are* $P_1 \in X_1$ *and* $P_2 \in X_2$ *such that* $P \in \langle \{P_1, \dots, P_{n-1}\} \rangle$.

Theorem 4. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Fix positive integer* n, a, b *such that* $n > a+b$ *and integral curves* X_i, $1 \le i \le n+1-a-b$, *such that* $X_1 \cap X_2 = \emptyset$ *and each* X_i *is non-degenerate. If* $n \ge a+b+2$ *fix a general* $(P_{a+b+1}, \dots, P_{n-3}) \in X_{a+b+1} \times \cdots \times X_{n-1}$. *If* $a > 1$, *fix a general* $P_i \in X_1$, $3 \le i \le a+2$. *If* $b > 1$

then fix general $P_i \in X_2$, $a+3 \le i \le a+b$. *Then for every* $P \in \mathbb{P}^n$ *there is* $P_1 \in X_1$ *and* $P_2 \in X_2$ *such that* $P \in \langle\{A_1, A_2, B_1, \dots, B_{(n-2)/2}\}\rangle$.

2. Proofs and Related Results

Lemma 1. *Let* $D_1, D_2 \subset \mathbb{P}^3$ *be reduced curves* $\mathbb{P}^3$ *such that* $D_1 \cap D_2 = \emptyset$. *Then for every* $P \in \mathbb{P}^3$ *there is* $P_1 \in D_1$ *and* $P_2 \in D_2$ *such that* $P_1 \neq P_2$ *and* $P \in \langle\{P_1, P_2\}\rangle$.

Proof. If $P \in D_1 \cup D_2$, say $P \in D_1$, take $P_1 := P$ and as P_2 a general point of D_2. If $P \notin (D_1 \cup D_2)$, then use that any two plane curves meet, while $D_1 \cap D_2 = \emptyset$. □

Proof of Theorem 3. If $n = 3$, then use Lemma 1. Now assume $n > 3$. If $P \in \langle\{P_3, \dots, P_{n-1}\}\rangle$, then we may take arbitrary $P_1 \in X_1$ and $P_2 \in X_2$. Hence we may assume $P \notin \langle\{P_3, \dots, P_{n-1}\}\rangle$. Let $\ell : \mathbb{P}^{n-1} \setminus \langle\{P_3, \dots, P_{n-1}\}\rangle \to \mathbb{P}^3$ denote the linear projection from $\langle\{P_3, \dots, P_{n-1}\}\rangle$. Our assumptions imply $\ell(X_1) \cap \ell(X_2) = \emptyset$. Apply Lemma 1 to the point $\ell(P)$ and the curves $\ell(X_1)$ and $\ell(X_2)$. □

Proof of Theorem 1. We use induction $n-2 \mapsto n$. Fix $P \in \mathbb{P}^n$. If $P \in U_{(n-5)/2}$ then we may find A_j, B_i with as $B_{(n-3)/2}$ an arbitrary point of $D_{(n-3)/2}$. Now assume $P \notin U_{(n-5)/2}$. First assume $Q_{(n-3)/2} \notin D_{(n-3)/2}$. Since $\dim(J(V_{(n-5)/2}, D_{(n-3)/2})) = \dim(V_{(n-5)/2}) + 2 = n$ (see [1], Corollary 1.5), we have $P \in J(V_{(n-5)/2}, D_{(n-3)/2})$ (notice that this is true even if $D_{(n-3)/2}$ is a line, because in this case our assumptions implies $D_{(n-3)/2} \cap V_{(n-5)/2} = \emptyset$). By the definition of join the are a smooth and connected quasi-projective curve W, $o \in W$, a flat family $\{D_t\}_{t \in W}$ of lines of $\mathbb{P}^n$ and $P_{1,t} \in V_{(n-5)/2}$, $P_{2,t} \in D_{(n-3)/2}$, $t \in W \setminus \{o\}$, such that $P \in D_o$, $P_{1,t} \neq P_{2,t}$ for all $t \in W \setminus \{o\}$ and $D_t = \langle\{P_{1,t}, P_{2,t}\}\rangle$ for all $t \in W \setminus \{o\}$. Since the Hilbert scheme $\mathrm{Hilb}^2(V_{(n-5)/2} \cup D_{(n-3)/2})$ is proper, there is a degree 2 zero-dimensional scheme $Z \subset V_{(n-5)/2} \cup D_{(n-3)/2}$ which is a flat limit of the flat family $\{P_{1,t}, P_{2,t}\}_{t \in W \setminus \{o\}}$. Since $V_{(n-5)/2}$ is projective, the flat family $\{P_{1,t}\}_{t \in W \setminus \{o\}}$ has at least one flat limit $P_1 \in V_{(n-5)/2}$. Similarly, $\{P_{2,t}\}_{t \in W \setminus \{o\}}$ has a flat limit $P_2 \in D_{(n-5)/2}$. Since $D_{(n-3)/2} \cap V_{(n-5)/2} = \emptyset$, we have $P_1 \neq P_2$. Hence $Z = \{P_1, P_2\}$ and $P \in \langle\{P_1, P_2\}\rangle$. By the inductive assumption there is $(A_1, A_2, \dots, B_{(n-5)/2}) \in C_1 \times C_2 \times \cdots D_{(n-5)/2}$ such that $P_1 \in \langle\{A_1, A_2, \dots, B_{(n-5)/2}$. Take $B_{(n-3)/2} := P_2$.

Now assume $Q_{(n-3)/2} \notin D_{(n-3)/2}$. Since $P \notin U_{(n-5)/2}$ the linear space $D := \langle U_{(n-5)/2} \cup \{P\}\rangle \cap V_{(n-3)/2}$ is a line through D. By our assumption on $V_{(n-3)/2}$ there is $B_{(n-3)/2} \in D \setminus \{Q_{(n-3)/2}\}$. By the inductive assumption there is $(A_1, A_2, \dots, B_{(n-5)/2}) \in C_1 \times C_2 \times \cdots D_{(n-5)/2}$ such that $Q_{(n-3)/2} \in \langle\{A_1, A_2, \dots, B_{(n-5)/2}\}\rangle$. Since $P \in \langle\{Q_{(n-3)/2}, B_{(n-3)/2}\}\rangle$, we have $P \in \langle\{A_1, A_2, B_1, \dots, B_{(n-3)/2}\}\rangle$. □

Lemma 2. *Assume* $\mathrm{char}(\mathbb{K}) = 0$. *Let* $D_i \subset \mathbb{P}^n$, $i = 1, 2, 3$, $n \geq 3$, *be integral and non-degenerate projective curves such that* $D_i \neq D_j$ *for all* $i \neq j$. *Then for a general* $Q \in D_1$ *the images (call it* C_1, C_2 *and* C_3*) of* D_1, D_2 *and* D_3 *by the linear projection from* Q *are different, respectively birational to* X_1, X_2 *and* X_3, $\deg(C_1) = \deg(X_1) - 1$, $\deg(C_2) = \deg(X_2)$ *and* $\deg(C_3) = \deg(X_3)$.

Proof. Since $D_1 \neq X_2$ and Q is general in X_1 we have $Q \notin D_2$. Since X_1 is infinite we get that C_2 is birational to D_2 and $\deg(C_2) = \deg(D_2)$ (see [5]). Since Q is general in X_1 the curve D_1 is birational to X_1 and $\deg(C_1) = \deg(D_1) - 1$. Hence $\deg(D_2) = d - 1$. A dimensional count shows that the following conditions are equivalent (in particular note that (c) implies (d)):

(a) $C_1 = C_2$;

(b) a general secant line of D_1 meets D_2;

(c) a general line intersecting both $D_1 \setminus D_1 \cap D_2$ and $D_2 \setminus D_1 \cap D_2$ is secant to D_1,

Assume $C_1 = C_2$. Fix a general $(O_1, O_2) \in D_1 \times D_2$. By (c) there is $Q_1 \in D_1$ such that $Q_1 \neq O_1$ and $Q_1 \in \langle\{O_1, O_2\}\rangle$. Fix O_2. Varying O_1 in D_1 we get that the linear projection of D_1 from O_2 is not birational onto its image. Since D_2 is infinite, [5] gives a contradiction. Using X_3 instead of X_2 we get $D_1 \neq D_3$. Hence to conclude the proof it is sufficient to prove $D_2 \neq D_3$. Assume $D_2 = D_3$, i.e. assume that a general line intersecting both $D_1 \setminus D_1 \cap D_2$ and $D_2 \setminus D_1 \cap D_2$ intersects D_3. By symmetry we get that a general line intersecting both $D_1 \setminus D_1 \cap D_3$ and $D_3 \setminus D_1 \cap D_3$ intersects D_2. By (a) we get $C_1 = C_3$, a contradiction. □

Proof of Theorem 4. Use Lemma 2 to modify the proof of Theorem 3. □

Acknowledgements

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

References

[1] B. Ådlandsvik, Joins and higher secant varieties, *Math. Scand.*, **61** (1987), 213-222.

[2] E. Ballico, Ranks of reducible subvarieties of projective spaces, *Int. J. Pure Appl. Math.*, **62**, No. 4 (2010), 413-418.

[3] E. Ballico, An upper bound for the X-ranks of points of $\mathbb{P}^n$ in positive characteristic, *Albanian J. Math.*, **5**, No. 1 (2011), 3-10.

[4] A. Bernardi, A. Gimigliano, M. Idà, Computing symmetric rank for symmetric tensors, *J. Symbolic. Comput.*, **46** (2011), 34-55.

[5] A. Calabri, C. Ciliberto, On special projections of varieties: epitome to a theorem of Beniamino Segre, *Adv. Geom.*, **1** (2001), 97-106.

[6] J.M. Landsberg and Z. Teiler, On the ranks and border ranks of symmetric tensors, *Found. Comput. Math.*, **10**, No. 3 (2010), 339-366.