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A CONSERVATIVE FINITE DIFFERENCE SCHEME FOR THE GENERALIZED ROSENAU EQUATION

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Abstract: In this paper, a finite difference scheme of the generalized Rosenau equation is proposed. Existence and uniqueness of numerical solution are proved. The convergent in the order of is showed. Numerical simulations show the method is efficient.

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1. Introduction

In the study of the dynamic of dense of discrete systems, the case of wave-wave and wave-wall interactions cannot be described by using the well-known KdV equation. To overcome this shortcoming of the KdV equation, Rosenau (see [1], [2]) proposed the so-called Rosenau equation:

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{1.1}$$

A lot of work has been done on the numerical method for the Rosenau equation (see [3]-[8]). This equation is generalized to become a general equation, that is

$$u_t + u_{xxxxt} + u_x + u^p u_x = 0, \tag{1.2}$$

with the boundary conditions

$$u(0,t) = u(L,t) = 0, u_{xx}(0,t) = u_{xx}(L,t) = 0, \tag{1.3}$$

and an initial condition

$$u(x,0) = u_0(x). \tag{1.4}$$

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The following conservation law is well known:

$$\forall t \in [0,T], E(t) = \|u(\cdot,t)\|^2 + \|u_{xx}(\cdot,t)\|^2 = \cdots = \|u_0\|^2 + \|u_{0xx}\|^2 = E(0).$$

In this paper, we derive a linear conservative scheme for (1.2).

The rest of this paper is organized as follows: in Section 2, we will describe a finite difference scheme for the equation and discuss the energy conservation of the difference scheme. In Section 3, we will show the scheme is uniquely solvable. In Section 4, we will prove the convergence and stability of the difference scheme. At last, some numerical tests are given in Section to verify our theoretical analysis.

2. Finite Difference Scheme and Energy Conservation Law

Let h and τ be the uniform step in the spatial and temporal direction, respectively. Denote $x_j = jk$ $(0 \le j \le J)$, $t_n = n\tau \quad (0 \le n\tau \le N)$, $u_j^n \approx u(hj, n\tau)$ and $Z_h^0 = \{u = u(u_j) | u_0 = u_J = 0, j = 0,1,2,\cdots,J\}$. Throughout this paper, we will denote C a generic constant independent of step sizes h and τ. We define the difference operators as follow:

$$(u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\overline{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$

$$(u_j^n)_{\widehat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \quad (u_j^n)_t = \frac{u_j^{n+1} - u_j^n}{\tau},$$

$$(u_j^n)_{x\overline{x}} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}, \quad (\overline{u}_j^n) = \frac{u_j^{n+1} + u_j^n}{2},$$

$$(u_j^n)_{\widehat{t}} = \frac{u_j^{n+1} - u_j^{n-1}}{2\tau}, \quad (u^n, v^n) = h\sum_{j=0}^{J-1} u_j^n v_j^n,$$

$$\|u^n\|^2 = (u^n, u^n), \quad \|u^n\|_\infty = \max_{0 \le j \le J-1} |u_j^n|.$$

Our analysis is based on techniques of zhang (see [9]), where conservative finite difference scheme for the generalized long-wave equation was shown. We propose a conservative difference scheme for the solution of problem (1.2), (1.3) and (1.4).

$$(u_j^n)_{\widehat{t}} + u_j^n)_{xx\overline{x}\overline{x}\widehat{t}} + (\overline{u}_j^n)_{\widehat{x}} + \frac{1}{p+2}\{(u_j^n)^p(\overline{u}_j^n)_{\widehat{x}} + [(u_j^n)^p\overline{u}_j^n]_{\widehat{x}}\} = 0, \tag{2.1}$$

$$u_k^0 = u_0(x_j), \tag{2.2}$$

$$u_0^n = u_J^n = 0(u_0^n)_{x\overline{x}} = (u_J^n)_{\overline{x}x} = 0, \tag{2.3}$$

For a simple notation, the function Q is introduced

$$Q = \frac{1}{p+2}\{(u_j^n)^p(\overline{u}_j^n)_{\widehat{x}} + [(u_j^n)^p\overline{u}_j^n]_{\widehat{x}}\}.$$

Lemma 1. *For $u \in Z_h$, we have:*

(1) $(u_x^n, v^n) = -(u^n, v_{\overline{x}}^n)$;

(2) $(u_{\widehat{x}}^n, u^n) = 0$;

(3) $\|u_x^n\| = \|u_{\overline{x}}^n\|$;

(4) $\|u_{x\overline{x}}^n\| = \|u_{xx}^n\|$.

Proof. (1) We have

$$(u_x^n, v^n) = h\sum_{j=0}^{J-1}(u_j^n)_x v_j^n = h\sum_{j=0}^{J-1}\frac{(u_{j+1}^n - u_j^n)}{h}v_j^n = \sum_{j=0}^{J-1}(u_{j+1}^n v_j^n - u_j^n v_j^n) - (u^n, v_{\overline{x}}^n)$$

$$= -h\sum_{j=0}^{J-1}(u_j^n)(v_j^n)_{\overline{x}} = -h\sum_{j=0}^{J-1}(u_j^n)\frac{(v_j^n - v_{j-1}^n}{h}) = \sum_{j=0}^{J-1}(u_j^n v_{j-1}^n - u_j^n v_j^n).$$

According to (2.2), (1) is proved.

(2) We have

$$(u_{\widehat{x}}^n, u^n) = h\sum_{j=0}^{J-1}(u_j^n)_{\widehat{x}}u_j^n = h\sum_{j=0}^{J-1}\frac{(u_{j+1}^n - u_{j-1}^n)}{2h}u_j^n$$

$$= \frac{1}{2}\sum_{j=0}^{J-1}(u_{j+1}^n u_j^n - u_j^n u_{j-1}^n) = 0.$$

(3) We have

$$\|u_x^n\|^2 = (u_x^n, u_x^n) = h\sum_{j=0}^{J-1}(u_j^n)_x(u_j^n)_x = \frac{1}{h}\sum_{j=0}^{J-1}(u_{j+1}^n u_{j+1}^n - 2u_{j+1}^n u_j^n + u_{j-1}^n u_j^n),$$

$$\|u_{\overline{x}}^n\| = h\sum_{j=0}^{J-1}(u_j^n)_{\overline{x}}(u_j^n)_{\overline{x}} = \frac{1}{h}\sum_{j=0}^{J-1}(u_j^n u_j^n - 2u_{j-1}^n u_j^n + u_{j-1}^n u_{j-1}^n)$$

Therefore $\|u_x^n\| = \|u_{\overline{x}}^n\|$.

So from (3) we can obtain (4). □

Theorem 2. *The difference scheme (2.1), (2.2) and (2.3) is conservative for discrete energy:*

$$E^n = \|u^n\|^2 + \|u^n_{xx})\|^2 = \cdots = \|u^0\|^2 + \|u^0_{xx}\|^2 = E^0.$$

Proof. Taking an inter product of (2.1) with $2\overline{u}^n_j$, we obtain

$$\begin{aligned}(u_{\widehat{t}}, 2\overline{u}^n) &= h\sum_{j=0}^{J-1}(u^n_j)\widehat{t}(u^{n+1}_j + u^{n-1}_j)\\ &= \frac{h}{2\tau}\sum_{j=0}^{J-1}(u^{n+1}_j - u^{n-1}_j)(u^{n+1}_j + u^{n-1}_{j-1}) = \frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2),\end{aligned}$$

$$\begin{aligned}(u^n_j)_{xx\overline{xx}\widehat{t}}, 2\overline{u}^n) &= h\sum_{j=0}^{J-1}(u^n_j)_{xx\overline{xx}\widehat{t}}(u^{n+1}_j + u^{n-1}_j)\\ &= \frac{h}{2\tau}\sum_{j=0}^{J-1}(u^{n+1}_j - u^{n-1}_j)_{xx\overline{xx}}(u^{n+1}_j + u^{n-1}_j)\\ &= \frac{h}{2\tau}\sum_{j=0}^{J-1}(u^{n+1}_j - u^{n-1}_j)_{xx}(u^{n+1}_j + u^{n-1}_j)_{xx} = \frac{1}{2\tau}(\|u^{n+1}_{xx}\|^2 - \|u^{n-1}_{xx}\|^2),\end{aligned}$$

$$\begin{aligned}(Q, 2\overline{u}^n) &= \frac{h}{p+2}\sum_{j=0}^{J-1}[(u^n_j)^p(\overline{u}^n_j)_{\widehat{x}} + [(u^n_j)^p\overline{u}^n_j]_{\widehat{x}}](u^{n+1}_j + u^{n-1}_j)\\ &= \frac{1}{4(p+2)}\sum_{j=0}^{J-1}(u^n_j)^p(u^{n+1}_{j+1} + u^{n-1}_{j+1} - u^{n+1}_{j-1} - u^{n-1}_{j-1})(u^{n+1}_j + u^{n-1}_j)\\ &\quad + [(u^n_{j+1})^p(u^{n+1}_{j+1} + u^{n-1}_{j+1}) + (u^n_{j-1})^p(u^{n+1}_{j-1} + u^{n-1}_{j-1})](u^{n+1}_j + u^{n-1}_j) = 0.\end{aligned}$$

From the definition of inner product and Lemma 1, we have

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^{n-1}\|^2) + \frac{1}{2\tau}(\|u^{n+1}_{xx}\|^2 - \|u^{n-1}_{xx}\|^2) = 0.$$

Therefore, the theorem is completed. □

3. Solvability

Below, we are going to prove the solvability of the finite difference scheme (2.1).

Theorem 3. *The difference scheme (2.1) is uniquely solvable.*

Proof. It is obvious that u^0 and u^1 are uniquely determined by (2.2) and (2.3). Now suppose $u^0, u^1, \cdots, u^n$ $(0 \leq n \leq N-1)$ be soled uniquely. Consider the equation of (2.1) for u^{n+1}.

$$\frac{1}{2\tau}u_j^{n+1} + \frac{1}{2\tau}(u_j^{n+1})_{xx\overline{x}\overline{x}} + \frac{1}{2}(u_j^{n+1})_{\widehat{x}} + \frac{1}{2(p+2)}\{(u_j^n)^p(u_j^{n+1})_{\widehat{x}} + [(u_j^n)^p u_j^{n+1}]_{\widehat{x}}\} = 0. \quad (3.1)$$

Computing the inner product of (3.1) with $2u^{n+1}$, and using Theorem 1, we obtain

$$(\frac{1}{2\tau}u^{n+1}, 2u^{n+1}) = \frac{h}{\tau}\sum_{j=0}^{J-1} u_j^{n+1},$$

$$u_j^{n+1} = \frac{1}{\tau}\|u^{n+1}\|^2,$$

$$(\frac{1}{2\tau}u_{xx\overline{x}\overline{x}}^{n+1}, 2u^{n+1}) = \frac{1}{\tau}\|u_{xx}^{n+1}\|^2,$$

$$\begin{aligned}
&(\frac{1}{2(p+2)}\{(u_j^n)^p(u_j^{n+1})_{\widehat{x}} + [(u_j^n)^p u_j^{n+1}]_{\widehat{x}}\}, 2u^{n+1}) \\
&= \frac{h}{p+2}\sum_{j=0}^{J-1}[(u_j^n)^p(u_{j+1}^{n+1} - u_{j-1}^{n+1}) + (u_{j+1}^n)^p u_{j+1}^{n+1} - (u_{j-1}^n)^p u_{j-1}^{n+1}]u_j^{n+1} \\
&= \frac{h}{p+2}\sum_{j=0}^{J-1}[(u_j^n)^p u_{j+1}^{n+1}u_j^{n+1} - (u_j^n)^p u_{j-1}^{n+1}u_j^{n+1} + (u_{j+1}^n)^p u_{j+1}^{n+1}u_j^{n+1} \\
&\qquad - (u_{j-1}^n)^p u_{j-1}^{n+1}u_j^{n+1}] = 0.
\end{aligned}$$

So, we get

$$\frac{1}{\tau}\|u^{n+1}\|^2 + \frac{1}{\tau}\|u_{xx}^{n+1}\|^2 = 0.$$

Which is held if and just only if $u^{n+1} = 0$. $\square$

4. Convergence and Stability

The following lemmas will be used in this section.

Lemma 4. (Discrete Sobolev's Inequality (see [10])) *There exist two constants C_1 and C_2 such that*

$$\|u^n\|_\infty \le C_1\|u^n\| + C_2\|u_x^n\|.$$

Lemma 5. *Suppose u_0 is smooth enough. Then, the solution of the difference scheme (2.1), (2.2) and (2.3) is estimated as follow:*

$$\|u^n\| \le C, \quad \|u_x^n\| \le C, \quad \|u^n\|_\infty \le C.$$

Proof. Directly from (2.4) and (2.5), We have

$$\|u^n\| \le C, \quad \|u_{xx}^n\| \le C.$$

From Lemma 1 and Cauchy-Schwarz Inequality, we derive

$$\|u_x^n\| \le \|u^n\|\|u_{x\overline{x}}^n\| = \|u^n\|\|u_{xx}^n\| \le \frac{\|u^n\|^2 + \|u_{xx}^n\|^2}{2} \le C. \tag{4.1}$$

We obtain the result by the use of Lemma 4. □

Let $v(x,t)$ be the solution of problem (1.2)-(1.4) and

$$v_j^n = (hj, n\tau).$$

Then the truncation of the difference scheme (2.1)-(2.3) is

$$R_j^n = (v_j^n)_{\hat{t}} + v_j^n)_{xx\overline{x}\overline{x}\hat{t}} + (\overline{v}_j^n)_{\hat{x}} + \frac{1}{p+2}\{(v_j^n)^p(\overline{v}_j^n)_{\hat{x}} + [(v_j^n)^p\overline{v}_j^n]_{\hat{x}}\}. \tag{4.2}$$

Making use of Taylor expansion, we can easily get the follow theorem.

Theorem 6. *Let is smooth enough, and then the truncation error of the difference scheme (2.1) is $O(\frac{h^2}{\tau} + \frac{\tau^2}{h})$.*

Lemma 7. (Discrete Gronwall Inequality (see [10])) *Suppose $\omega(k), \rho(k)$ are nonnegative function and $\rho(k)$ is nondecreasing. If $C > 0$, and*

$$\omega(k) \le \rho(k) + C\tau\sum_{l=0}^{k-1}\omega(l), \quad \forall k,$$

then

$$\omega(k) \le \rho(k)e^{C\tau k}, \quad \forall k.$$

Theorem 8. *Suppose* $u_{0\in Z_0^2[0,L]}$, *then the solution of the difference scheme (2.1)-(2.3) converges to the solution of problem (1.2)-(1.4) in norm* $\|\cdot\|_\infty$ *and the rate of convergence is* $O(\frac{\tau^2}{h}+\frac{h^2}{\tau})$.

Proof. Subtracting (2.1) from (4.2) and letting $e_j^n=v_j^n-u_j^n$, we have

$$\begin{aligned}R_j^n &= (e_j^n)_{\hat{t}}+e_j^n)_{xx\overline{xx}\hat{t}}+(\overline{e}_j^n)_{\hat{x}}\\&+\frac{1}{p+2}\{(v_j^n)^p(\overline{v}_j^n)_{\hat{x}}+[(v_j^n)^p\overline{v}_j^n]_{\hat{x}}\}-\frac{1}{p+2}\{(u_j^n)^p(\overline{u}_j^n)_{\hat{x}}+[(u_j^n)^p\overline{u}_j^n]_{\hat{x}}\}.\end{aligned}\tag{4.3}$$

Computing the inner product with $2\overline{e}^n$, we obtain

$$(R^n,2\overline{e}^n)=\frac{1}{2\tau}(\|e^{n+1}\|^2-\|e^{n-1}\|^2)+\frac{1}{2\tau}(\|e_{xx}^{n+1}\|^2-\|e_{xx}^{n-1}\|^2)+(A+B,2\overline{e}^n).$$

That is

$$\|e^{n+1}\|^2-\|e^{n-1}\|^2+\|e_{xx}^{n+1}\|^2-\|e_{xx}^{n-1}\|^2=2\tau(R^n,2\overline{e}^n)-2\tau(A+B,2\overline{e}^n),\tag{4.4}$$

where

$$A=\frac{1}{p+2}[(v_j^n)^p(\overline{v}_j^n)_{\hat{x}}-(u_j^n)^p(\overline{u}_j^n)_{\hat{x}}],$$

$$B=\frac{1}{p+2}\{[(v_j^n)^p\overline{v}_j^n]_{\hat{x}}-[(u_j^n)^p\overline{u}_j^n]_{\hat{x}}\},$$

$$\begin{aligned}(A,2\overline{e}^n) &= \frac{h}{p+2}\sum_{j=0}^{J-1}[(v_j^n)^p(\overline{v}_j^n)_{\hat{x}}-(u_j^n)^p(\overline{u}_j^n)_{\hat{x}}](e_j^{n+1}+e_j^{n-1})\\&=\frac{h}{2(p+2)}\sum_{j=0}^{J-1}\{(v_j^n)^p(e_j^{n+1}+e_j^{n-1}+u_j^{n+1}+u_j^{n-1})_{\hat{x}}\\&\quad-(u_j^n)^p(u_j^{n+1}+u_j^{n-1})_{\hat{x}}\}(e_j^{n+1}+e_j^{n-1})\\&=\frac{h}{2(p+2)}\sum_{j=0}^{J-1}\{(v_j^n)^p(e_j^{n+1}+e_j^{n-1})_{\hat{x}}\\&\quad+[(v_j^n)^p-(u_j^n)^p](u_j^{n+1}+u_j^{n-1})_{\hat{x}}\}(e_j^{n+1}+e_j^{n-1})\\&\le\frac{Ch}{p+2}\sum_{j=0}^{J-1}(|e_j^{n+1}+e_j^{n-1}|_{\hat{x}}+|e_j^n|)|e_j^{n+1}+e_j^{n-1}|\\&\le C(\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2+\|e^n\|^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2),\end{aligned}\tag{4.5}$$

$$
\begin{aligned}
(B, 2\overline{e}^n) = \frac{h}{p+2}\sum_{j=0}^{J-1}\{[(v_j^n)^p\overline{v}_j^n]_{\widehat{x}} - [(u_j^n)^p\overline{u}_j^n]_{\widehat{x}}\}(e_j^{n+1}+e_j^{n-1})\\
= \frac{h}{2(p+2)}\sum_{j=0}^{J-1}\{[(v_j^n)^p(v_j^{n+1}+v_j^{n-1})]_{\widehat{x}}\\
- [(u_j^n)^p(u_j^{n+1}+u_j^{n-1})]_{\widehat{x}}\}(e_j^{n+1}+e_j^{n-1})\\
= \frac{h}{2(p+2)}\sum_{j=0}^{J-1}\{[(v_j^n)^p(e_j^{n+1}+e_j^{n-1}+u_j^{n+1}+u_j^{n-1})]_{\widehat{x}}\\
- [(u_j^n)^p(u_j^{n+1}+u_j^{n-1})]_{\widehat{x}}\}(e_j^{n+1}+e_j^{n-1})\\
= \frac{h}{2(p+2)}\sum_{j=0}^{J-1}\{[(v_j^n)^p(e_j^{n+1}+e_j^{n-1})]_{\widehat{x}}\\
+ [((v_j^n)^p-(u_j^n)^p)(u_j^{n+1}+u_j^{n-1})]_{\widehat{x}}(e_j^{n+1}+e_j^{n-1})\\
\le \frac{Ch}{p+2}\sum_{j=0}^{J-1}(|e_j^{n+1}+e_j^{n-1}|_{\widehat{x}}+|e_j^n|)|e_j^{n+1}+e_j^{n-1}|\\
\le C(\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2+\|e^n\|^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2). \quad (4.5)
\end{aligned}
$$

We note that

$$
(R^n, 2\overline{e}^n) \le \|R^n\|^2 + 2(\|e^{n+1}\|^2+\|e^{n-1}\|^2). \tag{4.7}
$$

From (4.4)-(4.7), we have

$$
\begin{aligned}
\|e^{n+1}\|^2 - \|e^{n-1}\|^2 + \|e_{xx}^{n+1}\|^2 - \|e_{xx}^{n-1}\|^2\\
\le 2\tau(|R^n\|^2 + 2(\|e^{n+1}\|^2+\|e^{n-1}\|^2) - 2\tau(A+B, 2\overline{e}^n)\\
\le 2\tau(|R^n\|^2 + 4\tau(\|e^{n+1}\|^2+\|e^{n-1}\|^2)\\
+ 2C\tau(\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2+\|e^n\|^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2)\\
\|e^{n+1}\|^2+\|e_{xx}^{n+1}\|^2+\|e^n\|^2+\|e_{xx}^n\|^2-\|e^n\|^2-\|e_{xx}^n\|^2-\|e^{n-1}\|^2-\|e_{xx}^{n-1}\|^2\\
\le 2\tau(|R^n\|^2 + 4\tau(\|e^{n+1}\|^2+\|e^{n-1}\|^2)\\
+ 2C\tau(\|e_x^{n+1}\|^2+\|e_x^{n-1}\|^2+\|e^n\|^2+\|e^{n+1}\|^2+\|e^{n-1}\|^2).
\end{aligned}
$$

Similarly to (4.1) we can derive

$$
\|e_x^{n+1}\|^2 \le \frac{1}{2}(\|e^{n+1}\|^2 + \|e_{xx}^{n+1}\|^2), \tag{4.8}
$$

$$\|e_x^{n-1}\|^2 \le \frac{1}{2}(\|e^{n-1}\|^2 + \|e_{xx}^{n-1}\|^2), \tag{4.9}$$

$$\begin{aligned}\|e^{n+1}\|^2 + \|e_{xx}^{n+1}\|^2 + \|e^n\|^2 + \|e_{xx}^n\|^2 - \|e^n\|^2 - \|e_{xx}^n\|^2 - \|e^{n-1}\|^2 - \|e_{xx}^{n-1}\|^2 \\ \le C\tau(|R^n\|^2 + C\tau(\|e^{n+1}\|^2 + \|e_{xx}^{n+1}\|^2 + \|e^n\|^2 + \|e_{xx}^n\|^2 + \|e^n\|^2 \\ + \|e_{xx}^n\|^2 + \|e_{xx}^{n-1}\|^2 + \|e^{n-1}\|^2).\end{aligned}$$

Let

$$B^n = \|e^{n+1}\|^2 + \|e_{xx}^{n+1}\|^2 + \|e^n\|^2 + \|e_{xx}^n\|^2,$$

$$B^n - B^{n-1} \le C\tau(|R^n\|^2 + C\tau(B^n + B^{n-1}), \tag{4.10}$$

Summing (4.10) from 0 to $n-1$, we obtain

$$B^n \le B^0 + C\tau \sum_{l=0}^{n-1} \|R^l\|^2 + C\tau \sum_{l=0}^{n-1} B^l,$$

$$\tau \sum_{l=0}^{n-1} \|R^l\|^2 \le n\tau \|R^l\|^2 \le TO(\frac{\tau^2}{h} + \frac{h^2}{\tau})^2,$$

$$B^n \le O(\frac{\tau^2}{h} + \frac{h^2}{\tau})^2 + C\tau \sum_{l=0}^{n-1} B^l.$$

From Lemma 7, we obtain

$$B^n \le O(\frac{\tau^2}{h} + \frac{h^2}{\tau}),$$

$$\|e^n\| \le O(\frac{\tau^2}{h} + \frac{h^2}{\tau}),$$

$$\|e_{xx}^n\| \le O(\frac{\tau^2}{h} + \frac{h^2}{\tau}).$$

Using Lemma 4, we easily complete the proof of theorem. □

Theorem 9. *Under the conditions of theorem 4.2, the solution of (2.1)-(2.3) is stable for initial data in norm* $\|\cdot\|_\infty$.

	$\|e^n\|_\infty$		
t	h=0.1	h=0.05	h=0.0125
0.1	1.2210e-007	5.8792e-008	8.5935e-009
0.2	2.4429e-007	1.1762e-007	1.7193e-008
0.3	3.6655e-007	1.7649e-007	2.5797e-008
0.4	4.8890e-007	2.3539e-007	3.4407e-008
0.5	6.1133e-007	2.9433e-007	4.3022e-008
0.6	7.3385e-007	3.5331e-007	5.1642e-008
0.7	8.5644e-007	4.1232e-007	6.0268e-008
0.8	9.7912e-007	4.7137e-007	6.8899e-008
0.9	1.1019e-006	5.3045e-007	7.7536e-008
1.0	1.2247e-006	5.8958e-007	8.6178e-008

Table 1

5. Numerical Simulations

We consider the general Rosenau-Buegers, when $p = 2$

$$u_t + u_{xxxxt} + u_x + u^2 u_x = 0, \tag{5.1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad u_{xx}(0,t) = u_{xx}(1,t) = 0, \tag{5.2}$$

$$u(x,0) = u_0(x) = x^4(1-x)^4, \quad x \in [0,1]. \tag{5.3}$$

We construct a difference scheme to (5.1)-(5.3) similar to (2.1)-(2.3). We get a linear system of difference equation, when $p = 2$:

$$u_j^1 + (u_j^1)_{xx\overline{x}\overline{x}} = u_0(x_j) + \frac{\partial^4 u_0}{\partial x^4}(x_j) - 2\tau(u_0^2(x_j)\frac{\partial u_0(x_j)}{\partial x} + \frac{\partial u_0(x_j)}{\partial x}). \tag{5.4}$$

Since we do not have the exact solution to (5.1)-(5.4), we consider the solution on mesh $\frac{1}{160}$ as the reference solution, and obtain the error estimates by comparing the numerical solutions on mesh 0.1, 0.05 and 0.0125 with those on mesh $\frac{1}{160}$, respectively. Table 1 shows the error estimates in the maximum norm which testify the scheme in our paper is efficient.

6. Conclusions

We propose a linear three level finite difference scheme for a nonlinear equation in this article. Taking the advantage of its convenience to solve the problem without iteration, we can apply it into another class of energy conserving problem.

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