International Journal of Pure and Applied Mathematics
Volume 71 No. 4 2011, 551-558

FIXED POINT AND RN-STABILITY OF FUNCTIONAL EQUATIONS

Hassan Azadi Kenary[1], Afshin Erami[2], Habibi Hoseini[3] §

[1]Department of Mathematics
College of Sciences
Yasouj University, Yasouj 75914-353, IRAN

[2] Department of Mathematics
Marvdasht Branch
Islamic Azad University
Marvdasht, IRAN

[3]Department of Mathematics
Firoozabad Branch
Islamic Azad University
Firoozabad, IRAN

Abstract: In this paper, using direct method, we prove the generalized Hyers-Ulam stability of the following quadratic functional equation:

$$f(3x \pm y) = f(x \pm y) + 16f(x)$$

in random normed spaces.

AMS Subject Classification: 39B82, 39B52
Key Words: generalized Hyers-Ulam stability, random normed spaces, fixed point method

1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?". If the problem accepts a solution, we say that the equation is *stable*. The first stability

Received: March 19, 2011

§Correspondence author

problem concerning group homomorphisms was raised by Ulam [11] in 1940. In the next year, Hyers [6] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [8] proved a generalization of Hyers's theorem for additive mappings. The result of Rassias has provided a lot of influence during the last three decades in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as generalized Hyers-Ulam stability or Hyers-Ulam-Rassias stability of functional equations. Furthermore, in 1994, a generalization of Rassias's theorem was obtained by Găvruta [5] by replacing the bound $\epsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$.

The functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. In 1983, a generalized Hyers-Ulam stability problem for the quadratic functional equation was proved by Skof [10] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. In 1984, Cholewa [3] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group and, in 2002, Czerwik [4] proved the generalized Hyers-Ulam stability of the quadratic functional equation.

2. Preliminaries

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [9].

Throughout this paper (in random stability section), let Γ^+ denote the set of all probability distribution functions $F : \mathbb{R} \cup [-\infty, +\infty] \to [0, 1]$ such that F is left-continuous and nondecreasing on $\mathbb{R}$ and $F(0) = 0, F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Gamma^+ : l^- F(-\infty) = 1\}$, where $l^- f(x) = \lim\limits_{t \to x^-} f(t)$, is a subset of Γ^+. The set Γ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. For any $a \geq 0$, the element $H_a(t)$ of D^+ is defined by

$$H_a(t) = \begin{cases} 0 & \text{if} \quad t \leq a, \\ \\ 1 & \text{if} \quad t > a. \end{cases}$$

We can easily show that the maximal element in Γ^+ is the distribution function $H_0(t)$.

Definition 2.1. A function $T : [0,1]^2 \to [0,1]$ is a *continuous triangular norm* (briefly, a t-norm) if T satisfies the following conditions: (a) T is commutative and associative; (b) T is continuous; (c) $T(x,1) = x$ for all $x \in [0,1]$; (d) $T(x,y) \le T(z,w)$ whenever $x \le z$ and $y \le w$ for all $x,y,z,w \in [0,1]$.

Three typical examples of continuous t-norms are as follows: $T(x,y) = xy$, $T(x,y) = \max\{a+b-1, 0\}$, $T(x,y) = \min(a,b)$.

Recall that, if T is a t-norm and $\{x_n\}$ is a sequence in $[0,1]$, then $T_{i=1}^n x_i$ is defined recursively by $T_{i=1}^1 x_1 = x_1$ and $T_{i=1}^n x_i = T(T_{i=1}^{n-1} x_i, x_n)$ for all $n \ge 2$. $T_{i=n}^\infty x_i$ is defined by $T_{i=1}^\infty x_{n+i}$.

Definition 2.2. A *random normed space* (briefly, RN-space) is a triple (X,μ,T), where X is a vector space, T is a continuous t-norm and $\mu : X \to D^+$ is a mapping such that the following conditions hold:

(a) $\mu_x(t) = H_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;

(b) $\mu_{\alpha x}(t) = \mu_x(\frac{t}{|\alpha|})$ for all $\alpha \in \mathbb{R}$ with $\alpha \ne 0$, $x \in X$ and $t \ge 0$;

(c) $\mu_{x+y}(t+s) \ge T(\mu_x(t), \mu_y(s))$ for all $x,y \in X$ and $t,s \ge 0$.

Definition 2.3. Let (X,μ,T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be *convergent* to a point $x \in X$ (write $x_n \to x$ as $n \to \infty$) if $\lim_{n\to\infty} \mu_{x_n - x}(t) = 1$ for all $t > 0$.

(2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* in X if $\lim_{n\to\infty} \mu_{x_n - x_m}(t) = 1$ for all $t > 0$.

(3) The RN-space (X,μ,T) is said to be *complete* if every Cauchy sequence in X is convergent.

Theorem 2.1. (see [9]) *If (X,μ,T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$, then $\lim_{n\to\infty} \mu_{x_n}(t) = \mu_x(t)$.*

Definition 2.4. Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following conditions:

(a) $d(x,y) = 0$ if and only if $x = y$ for all $x,y \in X$;

(b) $d(x,y) = d(y,x)$ for all $x,y \in X$;

(c) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in X$.

Theorem 2.2. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either*

$$d(J^n x, J^{n+1} x) = \infty \tag{2.1}$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that:*

(a) $d(J^n x, J^{n+1}x) < \infty$ *for all* $n_0 \geq n_0$*;*

(b) *the sequence* $\{J^n x\}$ *converges to a fixed point* y^* *of* J*;*

(c) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X : d(J^{n_0}x, y) < \infty\}$*;*

(d) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$*.*

In this paper, we prove the generalized Hyers-Ulam stability of the following functional equation:

$$f(3x \pm y) = f(x \pm y) + 16f(x) \tag{2.2}$$

in random normed spaces.

3. Generalized Hyers-Ulam Stability of Functional Equation (2.2)

In this section, using the fixed point alternative approach, we prove the generalized Hyers-Ulam stability of functional equation (2.2) in random normed spaces.

Theorem 3.1. *Let* X *be a linear space,* (Y, μ, T_M) *be a complete RN-space and* Φ *be a mapping from* X^2 *to* D^+ *(*$\Phi(x, y)$ *is denoted by* $\Phi_{x,y}$*) such that there exists* $0 < \alpha < \frac{1}{9}$ *such that*

$$\Phi_{3x,0}(t) \leq \Phi_{x,0}(\alpha t) \tag{3.1}$$

for all $x, y \in X$ *and* $t > 0$*. Let* $f : X \to Y$ *be an odd mapping satisfying*

$$\mu_{f(3x \pm y) - f(x \pm y) - 16f(x)}(t) \geq \Phi_{x,y}(t) \tag{3.2}$$

for all $x, y \in X$ *and* $t > 0$*. Then, for all* $x \in X$

$$A(x) := \lim_{n\to\infty} 9^n f\Big(\frac{x}{3^n}\Big)$$

exists and $A : X \to Y$ *is a unique additive mapping such that*

$$\mu_{f(x) - A(x)}(t) \geq \Phi_{x,0}\Big(\frac{(2 - 18\alpha)t}{\alpha}\Big) \tag{3.3}$$

for all $x \in X$ *and* $t > 0$*.*

Proof. Putting $y = 0$ in (3.2), we have

$$\mu_{9f(\frac{x}{3})-f(x)}(t) \geq \Phi_{\frac{x}{3},0}(2t) \tag{3.4}$$

for all $x \in X$ and $t > 0$. Consider the set

$$S := \{g : X \to Y\}$$

and the generalized metric d in S defined by

$$d(f,g) = \inf\{u \in \mathbb{R}^+ : \mu_{g(x)-h(x)}(ut) \geq \Phi_{x,0}(t),\ \forall x \in X,\ t > 0\}, \tag{3.5}$$

where $\inf \emptyset = +\infty$. It is easy to show that (S,d) is complete (see [7], Lemma 2.1). Now, we consider a linear mapping $J : S \to S$ such that $Jh(x) := 9h(\frac{x}{3})$ for all $x \in X$. First, we prove that J is a strictly contractive mapping with the Lipschitz constant 9α.

In fact, let $g, h \in S$ be such that $d(g,h) < \epsilon$. Then we have $\mu_{g(x)-h(x)}(\epsilon t) \geq \Phi_{x,0}(t)$ for all $x \in X$ and $t > 0$ and so

$$\begin{aligned}
\mu_{Jg(x)-Jh(x)}(9\alpha\epsilon t) &= \mu_{9g(\frac{x}{3})-9h(\frac{x}{3})}(9\alpha\epsilon t)\\
&= \mu_{g(\frac{x}{3})-h(\frac{x}{3})}(\alpha\epsilon t)\\
&\geq \Phi_{\frac{x}{3},0}(\alpha t)\\
&\geq \Phi_{x,0}(t)
\end{aligned}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) < \epsilon$ implies that $d(Jg, Jh) < 2\alpha\epsilon$. This means that $d(Jg, Jh) \leq 2\alpha d(g,h)$ for all $g, h \in S$. It follows from (3.4) that $d(f, Jf) \leq \frac{\alpha}{2}$. By Theorem 2.2, there exists a mapping $A : X \to Y$ satisfying the following:

(1) A is a fixed point of J, that is,

$$A\Big(\frac{x}{2}\Big) = \frac{1}{2}A(x) \tag{3.6}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that A is a unique mapping satisfying (3.6) such that there exists $u \in (0,\infty)$ satisfying $\mu_{f(x)-A(x)}(ut) \geq \Phi_{x,0}(t)$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, A) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 9^n f\Big(\frac{x}{3^n}\Big) = A(x)$$

for all $x \in X$.

(3) $d(f, A) \leq \frac{d(f,Jf)}{1-9\alpha}$ with $f \in \Omega$, which implies the inequality $d(f, A) \leq \frac{\alpha}{2-18\alpha}$ and so

$$\mu_{f(x)-A(x)}\Big(\frac{\alpha t}{2-18\alpha}\Big) \geq \Phi_{x,0}(t)$$

for all $x \in X$ and $t > 0$. This implies that the inequality (3.3) holds. Now, we have

$$\mu_{9^n[f(\frac{3x\pm y}{3^n})-f(\frac{x\pm y}{3^n})-16f(\frac{x}{3^n})]}(t) \geq \Phi_{\frac{x}{3^n},\frac{y}{3^n}}\Big(\frac{t}{9^n}\Big)$$

for all $x, y \in X$, $t > 0$ and $n \geq 1$ and so, from (3.1), it follows that

$$\Phi_{\frac{x}{3^n},\frac{y}{3^n}}\Big(\frac{t}{9^n}\Big) \geq \Phi_{x,y}\Big(\frac{t}{(9\alpha)^n}\Big)$$

Since $\lim_{n\to\infty} \Phi_{x,y}(\frac{t}{(9\alpha)^n}) = 1$ for all $x, y \in X$ and $t > 0$, we have

$$\mu_{A(3x\pm y)-A(x\pm y)-16A(x)}(t) = 1,$$

for all $x, y \in X$ and $t > 0$. Thus the mapping $A : X \to Y$ is quadratic. This completes the proof. □

Corollary 3.1. *Let X be a real normed space, $\theta \geq 0$ and p be a real number with $p \in (1, +\infty)$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{f(3x\pm y)-f(x\pm y)-16f(x)}(t) \geq \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) = \lim_{n\to\infty} 9^n f\Big(\frac{x}{3^n}\Big)$$

exists and $A : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{2(1-9^{1-p})t}{2(1-9^{1-p})t + 9^{-p}\theta\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.1 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 9^{-p}$, then we get the desired result. □

Similarly, we can obtain the following and so we omit the proof.

Theorem 3.2. *Let X be a linear space, (Y, μ, T_M) be a complete RN-space and Φ be a mapping from X^2 to D^+ ($\Phi(x, y)$ is denoted by $\Phi_{x,y}$) such that for some* $0 < \alpha < 9$

$$\Phi_{x,0}(t) \leq \Phi_{3x,0}(\alpha t)$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a mapping satisfying

$$\mu_{f(3x\pm y)-f(x\pm y)-16f(x)}(t) \geq \Phi_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) := \lim_{n\to\infty} \frac{1}{9^n} f(3^n x)$$

exists and $A : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \Phi_{x,0}((18 - 2\alpha)t)$$

for all $x \in X$ and $t > 0$.

Corollary 3.2. *Let X be a real normed space, $\theta \geq 0$ and p be a real number with $p \in (0, 1)$. Let $f : X \to Y$ be a mapping satisfying*

$$\mu_{f(3x\pm y)-f(x\pm y)-16f(x)}(t) \geq \frac{t}{t + \theta\big(\|x\|^p + \|y\|^p\big)}$$

for all $x, y \in X$ and $t > 0$. Then, for all $x \in X$,

$$A(x) = \lim_{n\to\infty} \frac{1}{9^n} f(3^n x)$$

exists and $A : X \to Y$ is a unique quadratic mapping such that

$$\mu_{f(x)-A(x)}(t) \geq \frac{2(9 - 9^p)t}{2(9 - 9^p)t + \theta\|x\|^p}$$

for all $x \in X$ and $t > 0$.

Proof. The proof follows from Theorem 3.2 if we take

$$\Phi_{x,y}(t) = \frac{t}{t + \theta(\|x\|^p + \|y\|^p)}$$

for all $x, y \in X$ and $t > 0$. In fact, if we choose $\alpha = 9^p$, then we get the desired result. □

References

[1] H. Azadi Kenary, Hyres-Rassias stability of the Pexiderial functional equation, *Ital. J. Pure Appl. Math.*, To Appear.

[2] H. Azadi Kenary, The probabilistic stability of a Pexiderial functional equation in random normed spaces, *Rend. Del Circolo Math. di Palermo*, To Appear.

[3] P.W. Cholewa, Remarks on the stability of functional equations, *Aequat. Math.*, **27** (1984), 76-86.

[4] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific, River Edge, NJ, 2002.

[5] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.*, **184** (1994), 431-436.

[6] D.H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.*, USA, **27** (1941), 222-224.

[7] D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces, *J. Math. Anal. Appl.*, **343** (2008), 567-572.

[8] Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297-300.

[9] B. Schewizer, A. Sklar, *Probabilistic Metric Spaces*, North-Holland Series in Probability and Applied Mathematics, North-Holland and New York, USA (1983).

[10] F. Skof, Local properties and approximation of operators, *Rend. Sem. Mat. Fis. Milano*, **53** (1983), 113-129.

[11] S.M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons (1964).