**POSITIVE PERIODIC SOLUTIONS OF NEUTRAL LOTKA-VOLterra COMPETITION SYSTEMS ON TIME SCALES**

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**Abstract:** By using a fixed point theorem of strict-set-contraction, some sufficient conditions are obtained for the existence of positive periodic solutions for a periodic neutral Lotka-Volterra competition system on time scales of the form

\[
x_i^\Delta(t) = x_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) - \sum_{j=1}^{n} c_{ij}(t)x_j^\Delta(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))),
\]

where \(r_i, a_{ij}, b_{ij}, c_{ij} \in C(\mathbb{T}, \mathbb{R}^+)\) \((i, j = 1, 2, \ldots, n)\) are \(\omega\)-periodic functions, and \(\tau_{ij}, \sigma_{ij} \in C(\mathbb{T} \times \mathbb{R}^n, \mathbb{T})\) \((i = 1, 2, \ldots, n)\) are \(\omega\)-periodic functions with respect to their first arguments, respectively.

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1. Introduction

In the past decades, the application of theories of functional differential equations in mathematical ecology has been developed rapidly. Various mathematical models with delays have been proposed in the study of population dynamics. One of the famous models for dynamics of population is the Lotka-Volterra competition system. Owing to its theoretical and practical significance, the Lotka-Volterra systems have been studied extensively [1-9].

Recently, by using an existence theorem for neutral functional differential equations developed in [10,11], Fang [12], Liu and Chen [13] studied the existence of positive periodic solutions of the neutral Lotka-Volterra competition system

\[
\frac{dN_i(t)}{dt} = N_i(t) \left[ a_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)N_j(t) - \sum_{j=1}^{n} b_{ij}(t)N_j(t - \tau_{ij}(t)) \right. \\
\left. - \sum_{j=1}^{n} c_{ij}(t)N'_{ij}(t - \gamma_{ij}(t)) \right], \quad i = 1, 2, \ldots, n, \quad (1.1)
\]

respectively, where \(a_i, \beta_{ij}, b_{ij}, c_{ij}, \tau_{ij}, \gamma_{ij}\) are nonnegative continuous \(\omega\)-periodic functions. Under the transformation \(N_i(t) = e^{x_i(t)}, i = 1, 2, \ldots, n\), they first rewrote the above system in the following form

\[
\left[ x_i(t) + \sum_{j=1}^{n} \frac{c_{ij}(t)}{1 - \tau'_{ij}(t)} e^{x_j(t - \tau_{ij}(t))} \right]' \\
= a_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)e^{x_j(t)} - \sum_{j=1}^{n} \left( b_{ij}(t) - \left( \frac{c_{ij}(t)}{1 - \tau'_{ij}(t)} \right)' \right) e^{x_j(t - \tau_{ij}(t))}, \quad (1.2)
\]

where \(i = 1, 2, \ldots, n\), and made use of the existence theorem to obtain the existence of at least one periodic solution \(x^*(t) = (x_1^*(t), x_2^*(t), \ldots, x_n^*(t))^T\) of (1.2). Then they claimed that \(N^*(t) = (e^{x_1^*(t)}, e^{x_2^*(t)}, \ldots, e^{x_n^*(t)})^T\) is a positive periodic solution of (1.1). Unfortunately, according to their proofs, they only proved that (1.2) has at least one continuous periodic solution \(x^*(t)\) satisfying that

\[
x_i^*(t) + \sum_{j=1}^{n} \frac{c_{ij}(t)}{1 - \tau'_{ij}(t)} e^{x_j^*(t - \tau_{ij}(t))}, \quad i = 1, 2, \ldots, n,
\]

are differentiable. However, in general, \(x^*(t)\) is not differentiable. So, \(N^*(t)\) is not necessarily a solution of (1.1).
Since the study on periodic solutions of a population model is of great interest in mathematical biology [2,14] and many authors [15,16] have argued that the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations, also, discrete time models can provide efficient computational models of continuous models for numerical simulations. It is reasonable to study discrete time Lotka-Volterra model governed by difference equations. Also, the study of differential equations on time scales, which has been created in order to unify the study of differential and difference equations, is an area of mathematics that has recently gained a lot of attention, moreover, many results on this issue have been well documented in the monographs [17-20]. In this paper, we are concerned with the following neutral Lotka-Volterra competition system with state dependent delays on time scales

$$x_i^\Delta(t) = x_i(t) \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) - \sum_{j=1}^{n} c_{ij}(t)x_j^\Delta(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right], \quad i = 1, 2, \ldots, n, \quad (1.3)$$

where $r_i, a_{ij}, b_{ij}, c_{ij} \in C(T, \mathbb{R}^+)(j = 1, 2, \ldots, n)$ are $\omega$-periodic functions, $\mathbb{R}^+ = [0, \infty)$, and $\tau_{ij}, \sigma_{ij} \in C(T \times \mathbb{R}^n, T)(i = 1, 2, \ldots, n)$ are $\omega$-periodic functions with respect to their first arguments, respectively, $T$ is a periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. Obviously, the following differential equation

$$\frac{dx_i(t)}{dt} = x_i(t) \left[ r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t) - \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) - \sum_{j=1}^{n} c_{ij}(t)x_j'(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right], \quad i = 1, 2, \ldots, n \quad (1.4)$$

and its following discrete analogue

$$\Delta x_i(k) = x_i(k) \left[ r_i(k) - \sum_{j=1}^{n} a_{ij}(k)x_j(k) - \sum_{j=1}^{n} b_{ij}(k)x_j(k - \tau_{ij}(k, x_1(k), \ldots, x_n(k))) - \sum_{j=1}^{n} c_{ij}(k)\Delta x_j(k - \sigma_{ij}(k, x_1(k), \ldots, x_n(k))) \right], \quad k \in \mathbb{Z}, \quad i = 1, 2, \ldots, n$$
are two special cases of (1.3) when the time scale $\mathbb{T}$ is chosen as $\mathbb{R}$ or $\mathbb{Z}$, respectively.

The main purpose of this paper is by using a fixed point theorem of strict-set contraction [Theorem 3, 21] to establish criteria to guarantee the existence of positive periodic solutions of (1.3). To the best of our knowledge, this is the first paper to study the existence of periodic solutions of neutral Lotka-Volterra competition systems with state dependent delays on time scales.

2. Preliminaries

In this section, we shall recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$  

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$.

If $\mathbb{T}$ has a left-scattered maximum $m$, then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

Let $\omega \in \mathbb{R}, \omega > 0$, $\mathbb{T}$ is an $\omega$-periodic time scale if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ such that $t + \omega \in \mathbb{T}$ and $\mu(t) = \mu(t + \omega)$ whenever $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If $f$ is continuous at each right-dense point and each left-dense point, then $f$ is said to be continuous function on $\mathbb{T}$. We define $C[\mathbb{J}, \mathbb{R}] = \{u(t) \text{ is continuous on } \mathbb{J}\}$, and $C^1[\mathbb{J}, \mathbb{R}] = \{u^\Delta(t) \text{ is continuous on } \mathbb{J}\}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$||y(\sigma(t)) - y(s)| - y^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If $y$ is continuous, then $y$ is right-dense continuous, and if $y$ is delta differentiable at $t$, then $y$ is continuous at $t$. 
Let $y$ be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

**Definition 2.1.** (see [22]) We say that a time scale $\mathbb{T}$ is periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $p$ is called the period of the time scale.

**Definition 2.2.** (see [22]) Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $p$. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period $\omega$ if there exists a natural number $n$ such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$ and $\omega$ is the smallest number such that $f(t + \omega) = f(t)$. If $\mathbb{T} = \mathbb{R}$, we say that $f$ is periodic with period $\omega > 0$ if $\omega$ is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$.

A function $r : \mathbb{T} \to \mathbb{R}$ is called regressive if

$$1 + \mu(t)r(t) \neq 0$$

for all $t \in \mathbb{T}^k$.

If $r$ is regressive function, then the generalized exponential function $e_r$ is defined by

$$e_r(t, s) = \exp \left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta \tau \right\}, \text{ for } s, t \in \mathbb{T},$$

with the cylinder transformation

$$\xi_h(z) = \frac{\log(1 + hz)}{h} \quad \text{if } h \neq 0,$$

$$z \quad \text{if } h = 0.$$  

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, we define

$$p \oplus q := p + q + \mu pq, \quad \ominus p := -\frac{p}{1 + \mu p}, \quad p \ominus q := p \oplus (\ominus q).$$

Then the generalized exponential function has the following properties.

**Lemma 2.1.** Assume that $p, q : \mathbb{T} \to \mathbb{R}$ are two regressive functions, then

(i) $e_p(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;

(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;

(iii) $e_p(t, \sigma(s)) = \frac{e_p(t, s)}{1 + \mu(s)p(s)}$. 
(iv) \( \frac{1}{e_p(t,s)} = e \ominus p(t,s) \);
(v) \( e_p(t,s) = \frac{1}{e_{p(t,s)}} = e \ominus p(s,t) \);
(vi) \( e_p(t,s) e_p(s,r) = e_p(t,r) \);
(vii) \( e_p(t,s) e_q(t,s) = e_p \oplus q(t,s) \);
(viii) \( \frac{e_p(t,s)}{e_q(t,s)} = e_p \ominus q(t,s) \).

Lemma 2.2. (see [23]) Let \( r : \mathbb{T} \rightarrow \mathbb{R} \) be right-dense continuous and regressive, \( a \in \mathbb{T} \), and \( y_a \in \mathbb{R} \). Then the unique solution of the initial value problem
\[
y^\Delta(t) = r(t) y(t) + h(t), \quad y(a) = y_a
\]
is given by
\[
y(t) = e_r(t,a) y_a + \int_a^t e_r(t,\sigma(s)) h(s) \Delta s.
\]

By using Lemma 2.2, we get the following lemma which is fundamental in our discussion.

Lemma 2.3. \( x(t) \) is an \( \omega \)-periodic solution of Eq.(1.3) is equivalent to \( x(t) \) is an \( \omega \)-periodic solution of the following:

\[
x_i(t) =
\int_t^{t+\omega} G_i(t,s) x_i(s) \left[ \sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) + \sum_{j=1}^n c_{ij}(s) x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s, \quad (2.1)
\]

where
\[
G_i(t,s) = \frac{e_{r_i}(t,\sigma(s))}{1 - e_{r_i}(0,\omega)}, \quad t \in \mathbb{T}, \ s \in [t,t+\omega]_\mathbb{T}, \ i = 1, 2, \ldots, n. \quad (2.2)
\]

Proof. If \( x = (x_1, x_2, \ldots, x_n)^T \) is an \( \omega \)-periodic solution of (1.3), then \( x_i(t+\omega) = x_i(t) \). From Lemma 2.2, we know that

\[
x_i(t) = e_{r_i}(t,t+\omega) x_i(t) + \int_t^{t+\omega} e_{r_i}(t,\sigma(s)) x_i(s) \left[ \sum_{j=1}^n a_{ij}(s) x_j(s) + \sum_{j=1}^n b_{ij}(s) x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) + \sum_{j=1}^n c_{ij}(s) x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s.
\]
in view of $e_{ri}(t, t + \omega) = e_{ri}(0, \omega)$, we see that $x$ satisfies (2.1).

If $x = (x_1, x_2, \ldots, x_n)^T$ is an $\omega$-periodic solution of (2.1), then from (2.1) we have

$$x_i^\Delta(t) = G_i(\sigma(t), t + \omega)x_i(t + \omega) \left[ \sum_{j=1}^{n} a_{ij}(t + \omega)x_j(t + \omega) ight.$$ 

$$+ \sum_{j=1}^{n} b_{ij}(t + \omega)x_i(t + \omega - \tau_{ij}(t + \omega, x_1(t + \omega), \ldots, x_n(t + \omega)))$$ 

$$+ \sum_{j=1}^{n} c_{ij}(t + \omega)x^\Delta_j(t + \omega - \sigma_{ij}(t, x_1(t + \omega), \ldots, x_n(t + \omega)))) \right]$$ 

$$- G_i(\sigma(t), t)x_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)x_j(t) ight.$$ 

$$+ \sum_{j=1}^{n} b_{ij}(t)x_i(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t)))$$ 

$$+ \sum_{j=1}^{n} c_{ij}(t)x^\Delta_i(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t)))) \right] + r_i(t)x_i(t)$$ 

$$= r_i(t)x_i(t) - x_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)x_j(t) ight.$$ 

$$+ \sum_{j=1}^{n} b_{ij}(t)x_i(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t)))$$ 

$$+ \sum_{j=1}^{n} c_{ij}(t)x^\Delta_i(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t)))) \right], \quad i = 1, 2, \ldots, n.$$

So we know that, $x$ is also an $\omega$-periodic solution of (1.3). The proof is complete. \square
For convenience, we introduce the notation

\[ M_{ij} = \int_0^\omega \left[ e_{r_j}(0, \omega) a_{ij}(s) + e_{r_j}(0, \omega) b_{ij}(s) - c_{ij}(s) \right] \Delta s, \quad i, j = 1, 2, \ldots, n. \]

\[ \hat{M}_{ij} = \int_0^\omega \left[ a_{ij}(s) + b_{ij}(s) + c_{ij}(s) \right] \Delta s, \quad i, j = 1, 2, \ldots, n. \]

\[ f^M = \max_{t \in [0, \omega]} \{ f(t) \}, \quad f^m = \min_{t \in [0, \omega]} \{ f(t) \}, \]

where \( f \) is a continuous \( \omega \)-periodic function.

Throughout this paper, we assume that

\((H1)\) \( r_i(t) \) is not identically vanishing.

\((H2)\) \( e_{r_j}(0, \omega) a_{ij}(t) + e_{r_j}(0, \omega) b_{ij}(t) - c_{ij}(t) \geq 0, \quad i, j = 1, 2, \ldots, n. \)

\((H3)\) \( (1 + r_i^m) \frac{e_{r_i}(0, \omega)}{1 - e_{r_i}(0, \omega)} M_{ij} \geq \max_{t \in [0, \omega]} \left\{ a_{ij}(t) + b_{ij}(t) + c_{ij}(t) \right\}, \quad i, j = 1, 2, \ldots, n. \)

\((H4)\) \( \frac{\hat{M}_{ij}(r_i^M - 1)}{e_{r_i}(0, \omega)(1 - e_{r_i}(0, \omega))} \leq \min_{t \in [0, \omega]} \left\{ e_{r_j}(0, \omega) a_{ij}(t) + e_{r_j}(0, \omega) b_{ij}(t) - c_{ij}(t) \right\}, \quad i, j = 1, 2, \ldots, n. \)

\((H5)\) \( \max_{1 \leq i \leq n} \left\{ \frac{1 - e_{r_i}(0, \omega)}{e_{r_i}(0, \omega)} \min_{1 \leq j \leq n} \left\{ M_{ij} \right\} \right\} \left[ \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n c_{ij}^M \right\} \right] < 1. \)

In order to obtain the existence of a periodic solution of system (1.3), we make the following preparations:

Let \( E \) be a Banach space and \( K \) be a cone in \( E \). The semi-order induced by the cone \( K \) is denoted by "\( \leq \)". That is, \( x \leq y \) if and only if \( y - x \in K \). In addition, for a bounded subset \( A \subset E \), let \( \alpha_E(A) \) denote the (Kuratowski) measure of non-compactness defined by

\[ \alpha_E(A) = \inf \left\{ \delta > 0 : \text{there is a finite number of subsets } A_i \subset A \text{ such that } A = \bigcup_i A_i \text{ and diam}(A_i) \leq \delta \right\}, \]

where \( \text{diam}(A_i) \) denotes the diameter of the set \( A_i \).
Let $E, F$ be two Banach spaces and $D \subset E$, a continuous and bounded map $\Phi : \Omega \rightarrow F$ is called $k$-set contractive if for any bounded set $S \subset D$ we have

$$\alpha_E(\Phi(S)) \leq k\alpha_E(S).$$

$\Phi$ is called strict-set-contractive if it is $k$-set-contractive for some $0 \leq k < 1$.

The following lemma cited from [21,24] which is useful for the proof of our main results of this paper.

**Lemma 2.4.** (see [21, 24]) Let $K$ be a cone of the real Banach space $X$ and $K_{r,R} = \{x \in K | r \leq ||x|| \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \rightarrow K$ is strict-set-contractive such that one of the following two conditions is satisfied:

(i) $\Phi x \not\subseteq x$, $\forall x \in K$, $||x|| = r$ and $\Phi x \not\subseteq x$, $\forall x \in K$, $||x|| = R$.

(ii) $\Phi x \not\subseteq x$, $\forall x \in K$, $||x|| = r$ and $\Phi x \not\subseteq x$, $\forall x \in K$, $||x|| = R$.

Then $\Phi$ has at least one fixed point in $K_{r,R}$.

In order to apply Lemma 2.4 to system (1.3), we set

$$C^0_\omega = \{x = (x_1, x_2, \ldots, x_n)^T : x \in C^0(\mathbb{T}, \mathbb{R}^n), x(t + \omega) = x(t)\}$$

with the norm defined by $||x|| = \sum_{i=1}^{n} |x_i|_0$ where $|x_i|_0 = \max_{t \in [0, \omega]_{\mathbb{T}}} \{|x_i(t)|\}$, $i = 1, 2, \ldots, n$, and

$$C^1_\omega = \{x = (x_1, x_2, \ldots, x_n)^T : x \in C^1(\mathbb{T}, \mathbb{R}^n), x(t + \omega) = x(t)\}$$

with the norm defined by $||x||_1 = \sum_{i=1}^{n} |x_i|_1$, where $|x_i|_1 = \max\{|x_i|_0, |x_i^\Delta|_0\}$, $i = 1, 2, \ldots, n$. Then $C^0_\omega$ and $C^1_\omega$ are all Banach spaces.

Define the cone $K$ in $C^1_\omega$ by

$$K = \{x : x = (x_1, x_2, \ldots, x_n)^T \in C^1_\omega, x_i(t) \geq e_{r_i}(0, \omega)|x_i|_1, t \in [0, \omega]_{\mathbb{T}},$$

$$i = 1, 2, \ldots, n\}. \quad (2.3)$$

Let the map $\Phi$ be defined by

$$(\Phi x)(t) = ((\Phi_1 x)(t), (\Phi_2 x)(t), \ldots, (\Phi_n x)(t))^T, \quad (2.4)$$

where $x \in K, t \in \mathbb{T}$

$$\Phi_i(t) = \int_t^{t+\omega} G_i(t, s)x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_j(s) \right]$$
\begin{align*}
\sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \\
+ \sum_{j=1}^{n} c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \big| \Delta s, \quad i = 1, 2, \ldots, n
\end{align*}

and \( G_i(t, s) \) is given by (2.2). It is easy to see that \( G_i(t + \omega, s + \omega) = G_i(t, s) \).

By (H1), we have \( 0 < e_{r_i}(0, \omega) < 1 \), then
\[
\frac{e_{r_i}(0, \omega)}{1 - e_{r_i}(0, \omega)} \leq G_i(t, s) \leq \frac{1}{1 - e_{r_i}(0, \omega)}, \quad s \in [t, t + \omega] \cap T, \quad i = 1, 2, \ldots, n.
\]

In the following, we will give some lemmas concerning \( K \) and \( \Phi \) defined by (2.3) and (2.4), respectively.

**Lemma 2.5.** Assume that \((H_1)-(H_3)\) hold.

(i) If \( \max\{r_i^M, i = 1, 2, \ldots, n\} \leq 1 \), then \( \Phi : K \to K \) is well defined.

(ii) If \((H_4)\) holds and \( \min\{r_i^M, i = 1, 2, \ldots, n\} > 1 \), then \( \Phi : K \to K \) is well defined.

**Proof.** For any \( x \in K \), it is clear that \( \Phi x \in C^1(T, \mathbb{R}^n) \). In view of (2.4), for \( t \in T \), we obtain
\[
(\Phi_i x)(t + \omega) = \int_{t + \omega}^{t + 2\omega} G_i(t + \omega, s)x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_j(s) \\
+ \sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \\
+ \sum_{j=1}^{n} c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s
\]
\[
= \int_{t}^{t + \omega} G_i(t + \omega, u + \omega)x_i(u + \omega) \left[ \sum_{j=1}^{n} a_{ij}(u + \omega)x_j(u + \omega) \\
+ \sum_{j=1}^{n} b_{ij}(u + \omega)x_j(u + \omega - \tau_{ij}(u + \omega, x_1(u + \omega), \ldots, x_n(u + \omega))) \\
+ \sum_{j=1}^{n} c_{ij}(u + \omega)x_j^\Delta(u + \omega - \sigma_{ij}(u + \omega, x_1(u + \omega), \ldots, x_n(u + \omega))) \right]
\]
\[
\begin{align*}
  x_n(u + \omega)) \quad \Delta u \\
  = \int_t^{t+\omega} G_i(t, u)x_i(u) \\
  \times \left[ \sum_{j=1}^{n} a_{ij}(u)x_j(u) + \sum_{j=1}^{n} b_{ij}(u)x_j(u - \tau_{ij}(u, x_1(u), \ldots, x_n(u))) \\
  + \sum_{j=1}^{n} c_{ij}(u)x^\Delta_j(u - \sigma_{ij}(u, x_1(u), \ldots, x_n(u))) \right] \Delta u \\
  = (\Phi_i x)(t), \quad i = 1, 2, \ldots, n.
\end{align*}
\]

That is, \((\Phi x)(t + \omega) = (\Phi x)(t), t \in \mathbb{T}\). So \(\Phi x \in C^1_\omega\). In view of \((H_2)\), for \(x \in K, t \in [0, \omega)_T\), we have

\[
\begin{align*}
  \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) \\
  + \sum_{j=1}^{n} c_{ij}(t)x^\Delta_j(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \\
  \geq \sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) \\
  - \sum_{j=1}^{n} c_{ij}(t)|x^\Delta_j(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t)))| \\
  \geq \sum_{j=1}^{n} e_{r_i}(0, \omega)a_{ij}(t)|x_j|_1 + \sum_{j=1}^{n} e_{r_i}(0, \omega)b_{ij}(t)|x_j|_1 - \sum_{j=1}^{n} c_{ij}(t)|x_j|_1 \\
  = \sum_{j=1}^{n} \left[ e_{r_j}(0, \omega)a_{ij}(t) + e_{r_j}(0, \omega)b_{ij}(t) - c_{ij}(t) \right]|x_j|_1 \geq 0, \quad (2.5)
\end{align*}
\]

where \(i = 1, 2, \ldots, n\). Therefore, for \(x \in K, t \in [0, \omega)_T\), we find

\[
|\Phi_i x|_0 \leq \frac{1}{1 - e_{r_i}(0, \omega)} \int_0^\omega x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_j(s) \\
  + \sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \right]
\]
\[ + \sum_{j=1}^{n} c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \] \Delta s \]

and

\[
(\Phi_i x)(t) \geq \frac{e_{r_i}(0, \omega)}{1 - e_{r_i}(0, \omega)} \int_{t}^{t+\omega} x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_j(s) \\
+ \sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \\
+ \sum_{j=1}^{n} c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s
\]

\[
\geq e_{r_i}(0, \omega) |\Phi_i x|_0, \quad (2.6)
\]

where \( i = 1, 2, \ldots, n \). Now, we show that \( (\Phi_i x)(t) \geq e_{r_i}(0, \omega)(|\Phi_i x|^\Delta|_0, t \in [0, \omega]_\mathbb{T}. \) From (2.4), we have

\[
(\Phi_i x)^\Delta(t) = G_i(\sigma(t), t + \omega)x_i(t + \omega) \left[ \sum_{j=1}^{n} a_{ij}(t + \omega)x_j(t + \omega) \\
+ \sum_{j=1}^{n} b_{ij}(t + \omega)x_j(t + \omega - \tau_{ij}(t + \omega, x_1(t + \omega), \ldots, x_n(t + \omega))) \\
+ \sum_{j=1}^{n} c_{ij}(t + \omega)x_j^\Delta(t + \omega - \sigma_{ij}(t, x_1(t + \omega), \ldots, x_n(t + \omega))) \right] \\
- G_i(\sigma(t), t)x_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)x_j(t) \\
+ \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) \right]
\]
It follows from (2.5) and (2.7) that if $(\Phi_i x)^\Delta(t) \geq 0$, then

$$(\Phi_i x)^\Delta(t) \leq r_i(t) (\Phi_i x)(t) \leq \nu_i^M (\Phi_i x)(t) \leq (\Phi_i x)(t), \quad i = 1, 2, \ldots, n. \tag{2.8}$$

On the other hand, from (2.6), (2.7) and (H3), if $(\Phi_i x)^\Delta(t) < 0$, then

$$-(\Phi_i x)^\Delta(t) = x_i(t) \left[ \sum_{j=1}^n a_{ij}(t)x_j(t) + \sum_{j=1}^n b_{ij}(t)x_j(t) - r_i(t)(\Phi_i x)(t) \right.\nonumber$$

$$\left. + \sum_{j=1}^n c_{ij}(t)x^\Delta_j(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right] - r_i(t)(\Phi_i x)(t) \nonumber$$

$$\leq |x_i|_1 \sum_{j=1}^n [a_{ij}(t) + b_{ij}(t) + c_{ij}(t)]|x_j|_1 - r_i^m(\Phi_i x)(t) \nonumber$$

$$\leq (1 + r_i^m) \frac{e_{r_i}^2(0, \omega)}{1 - e_{r_i}(0, \omega)} |x_i|_1 \nonumber$$

$$\times \sum_{j=1}^n \left\{ \int_0^\omega \left[ e_{r_j}(0, \omega)a_{ij}(s) + e_{r_j}(0, \omega)b_{ij}(s) \right.\nonumber$$

$$\left. - c_{ij}(s) \right] |x_j|_1 \Delta s \right\} - r_i^m(\Phi_i x)(t) \nonumber$$

$$= (1 + r_i^m) \int_0^\omega \frac{e_{r_i}(0, \omega)}{1 - e_{r_i}(0, \omega)} e_{r_i}(0, \omega)|x_i|_1 \sum_{j=1}^n \left[ e_{r_j}(0, \omega)a_{ij}(s) \right.\nonumber$$

$$+ e_{r_j}(0, \omega)b_{ij}(s) - c_{ij}(s) \right] |x_j|_1 \Delta s - r_i^m(\Phi_i x)(t) \nonumber$$

$$\leq (1 + r_i^m) \int_t^{t+\omega} G_i(t, s)x_i(s) \left[ \sum_{j=1}^n a_{ij}(s)x_j(s) \right.\nonumber$$

$$\left. + \sum_{j=1}^n b_{ij}(s)x_j(s) + \sum_{j=1}^n c_{ij}(s)x^\Delta_j(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s - r_i^m(\Phi_i x)(t). \nonumber$$
+ \sum_{j=1}^{n} b_{ij}(s)x_{j}(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s)))
- \sum_{j=1}^{n} c_{ij}(s)|x_{j}^\Delta| (s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s - r_i^m(\Phi_i x)(t)
\leq (1 + r_i^m) \int_{t}^{t+\omega} G_i(t, s)x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_{j}(s)
+ \sum_{j=1}^{n} b_{ij}(s)x_{j}(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s)))
+ \sum_{j=1}^{n} c_{ij}(s)x_{j}^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s - r_i^m(\Phi_i x)(t)
= (1 + r_i^m)(\Phi_i x)(t) - r_i^m(\Phi_i x)(t)
= (\Phi_i x)(t), \ i = 1, 2, \ldots, n. \quad (2.9)

It follows from (2.8) and (2.9) that \(|(\Phi_i x)^\Delta|_0 \leq |\Phi_i x|_0, i = 1, 2, \ldots, n.\) So \(|\Phi_i x|_1 = |\Phi_i x|_0, i = 1, 2, \ldots, n.\) By (2.6) we have \((\Phi_i x)(t) \geq e_{r_i}(0, \omega)|\Phi_i x|_1.\) Hence, \(\Phi x \in K.\) The proof of (i) is complete.

(ii) In view of the proof of (i), we only need to prove that \((\Phi_i x)^\Delta(t) \geq 0, i = 1, 2, \ldots, n\) imply \((\Phi_i x)^\Delta(t) \leq (\Phi_i x)(t), \ i = 1, 2, \ldots, n.\)

From (2.5), (2.7), (H2) and (H4), we obtain

\[(\Phi_i x)^\Delta(t) \leq r_i(t)(\Phi_i x)(t) - e_{r_i}(0, \omega)|x_i|_1 \left[ \sum_{j=1}^{n} a_{ij}(t)x_{j}(t)
+ \sum_{j=1}^{n} b_{ij}(t)x_{j}(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t)))
- \sum_{j=1}^{n} c_{ij}(t)|x_{j}^\Delta| (t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right]
\leq r_i(t)(\Phi_i x)(t) - e_{r_i}(0, \omega)|x_i|_1 \times \left[ e_{r_j}(0, \omega)a_{ij}(t) + e_{r_j}(0, \omega)b_{ij}(t) - c_{ij}(t) \right]|x_j|_1
\leq r_i^M(\Phi_i x)(t) - e_{r_i}(0, \omega)|x_i|_1 \frac{r_i^M - 1}{e_{r_i}(0, \omega)(1 - e_{r_i}(0, \omega))} \]
× \int_0^\omega \sum_{j=1}^n \left[ a_{ij}(s) + b_{ij}(s) + c_{ij}(s) \right] |x_j|_1 \Delta s \\
\leq r_i^M(\Phi_i x)(t) - (r_i^M - 1) \int_t^{t+\omega} \frac{1}{1 - \epsilon_r(0, \omega)} |x_i|_1 \\
\times \sum_{j=1}^n \left[ a_{ij}(s)|x_j|_1 + b_{ij}(s)|x_j|_1 + c_{ij}(s)|x_j|_1 \right] \Delta s \\
\leq r_i^M(\Phi_i x)(t) - (r_i^M - 1) \int_t^{t+\omega} G_i(t, s)x_i(s) \left[ \sum_{j=1}^n a_{ij}(s)x_j(s) \\
+ \sum_{j=1}^n b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \\
+ \sum_{j=1}^n c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s))) \right] \Delta s \\
= r_i^M(\Phi_i x)(t) - (r_i^M - 1)(\Phi_i x)(t) \\
= (\Phi_i x)(t), \ i = 1, 2, \ldots, n.

The proof of (ii) is complete. \qed

Lemma 2.6. Assume that (H_1)-(H_3) hold and \( R \max_{1 \leq i \leq n} \{ \sum_{j=1}^n c_{ij}^M \} < 1, \)

(i) If \( \max\{r_i^M, i = 1, 2, \ldots, n\} \leq 1, \) then \( \Phi : K \cap \bar{\Omega}_R \to K \) is strict-set-contractive,

(ii) If (H_4) holds and \( \min\{r_i^M, i = 1, 2, \ldots, n\} > 1, \) then \( \Phi : K \cap \bar{\Omega}_R \to K \) is strict-set-contractive,

where \( \Omega_R = \{ x \in C_1^\omega : \|x\|_1 < R \}. \)
Proof. We only need to prove (i), since the proof of (ii) is similar. It is easy to see that $\Phi$ is continuous and bounded. Now we prove that $\alpha_{C^0_w}(\Phi(S)) \leq \left(R \max_{1 \leq i \leq n} \{\sum_{j=1}^{n} M_{ij}\}\right) \alpha_{C^0_w}(S)$ for any bounded set $S \subset \overline{\Omega}_R$. Let $\eta = \alpha_{C^0_w}(S)$. Then, for any positive number $\varepsilon < \left(R \max_{1 \leq i \leq n} \{\sum_{j=1}^{n} M_{ij}\}\right) \eta$, there is a finite family of subsets $\{S_i\}$ satisfying $S = \bigcup_i S_i$ with $\text{diam}(S_i) \leq \eta + \varepsilon$. Therefore

$$\|x - y\|_1 \leq \eta + \varepsilon \quad \text{for any } x, y \in S_i. \quad (2.10)$$

As $S$ and $S_i$ are precompact in $C^0_w$, it follows that there is a finite family of subsets $\{S_{ij}\}$ of $S_i$ such that $S_i = \bigcup_j S_{ij}$ and

$$\|x - y\| \leq \varepsilon \quad \text{for any } x, y \in S_{ij}. \quad (2.11)$$

In addition, for any $x \in S$ and $t \in [0, \omega)_T$, we have

$$|((\Phi_i x)(t)| = \int_t^t + \omega G_i(t, s)x_i(s) \left[\sum_{j=1}^{n} a_{ij}(s)x_j(s) + \sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) + \sum_{j=1}^{n} c_{ij}(s)x_j^\tau(s - \sigma_{ij}(s, x_1(s), \ldots, x_n(s)))\right] \Delta s \leq \frac{R^2}{1 - \varepsilon_{r_i}(0, \omega)} \int_0^\omega \sum_{j=1}^{n} \left[a_{ij}(s) + b_{ij}(s) + c_{ij}(s)\right] \Delta s := H_i,$$

where $i = 1, 2, \ldots, n$, and

$$|((\Phi_i x)^\triangledown(t)| = |r_i(t)(\Phi_i x)(t) - x_i(t) \left[\sum_{j=1}^{n} a_{ij}(t)x_j(t) + \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) + \sum_{j=1}^{n} c_{ij}(t)x_j^\tau(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t)))\right] | \leq r_i^M H_i + R^2 \sum_{j=1}^{n} (a_{ij}^M + b_{ij}^M + c_{ij}^M), i = 1, 2, \ldots, n.$
Hence,
\[ \| \Phi x \| \leq \sum_{i=1}^{n} H_i \]
and
\[ \| \Phi x \Delta \| \leq \sum_{i=1}^{n} (r_i M_i H_i) + R^2 \sum_{i=1}^{n} \sum_{j=1}^{n} (a_{ij} M_j + b_{ij} M_j + C_{ij} M_j). \]

It follows from Lemma 2.4 in [25] that \( \Phi(S) \) is precompact in \( C_0^\omega \). Then, there is a finite family of subsets \( \{ S_{ijk} \} \) of \( S_{ij} \) such that \( S_{ij} = \bigcup_k S_{ijk} \) and
\[ \| \Phi x - \Phi y \| \leq \varepsilon \text{ for any } x, y \in S_{ijk}. \]  

From (2.5), (2.7) and (2.10)-(2.12) and \((H_2)\), for any \( x, y \in S_{ijk} \), we obtain
\[
\| \Phi x \Delta - \Phi y \Delta \| = \max_{t \in [0, \omega]_\mathbb{T}} \left\{ r_i(t)(\Phi_i x)(t) - r_i(t)(\Phi_i y)(t) 
- x_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)x_j(t) 
+ \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) 
+ \sum_{j=1}^{n} c_{ij}(t)x_j(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right] \right.
+ y_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j(t - \tau_{ij}(t, y_1(t), \ldots, y_n(t))) 
+ \sum_{j=1}^{n} c_{ij}(t)y_j(t - \sigma_{ij}(t, y_1(t), \ldots, y_n(t)))) \right] \} 
\leq \max_{t \in [0, \omega]_\mathbb{T}} \{ |r_i(t)(\Phi_i x)(t) - (\Phi_i y)(t)| \} + \max_{t \in [0, \omega]_\mathbb{T}} \left\{ x_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)x_j(t) 
+ \sum_{j=1}^{n} b_{ij}(t)x_j(t - \tau_{ij}(t, x_1(t), \ldots, x_n(t))) 
+ \sum_{j=1}^{n} c_{ij}(t)x_j(t - \sigma_{ij}(t, x_1(t), \ldots, x_n(t))) \right] \right. 
+ y_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t)y_j(t) + \sum_{j=1}^{n} b_{ij}(t)y_j(t - \tau_{ij}(t, y_1(t), \ldots, y_n(t))) 
+ \sum_{j=1}^{n} c_{ij}(t)y_j(t - \sigma_{ij}(t, y_1(t), \ldots, y_n(t)))) \right] \} 
\]
\[- y_i(t) \left[ \sum_{j=1}^{n} a_{ij}(t) y_j(t) + \sum_{j=1}^{n} b_{ij}(t) y_i(t - \tau_{ij}(t, y_1(t) \ldots, y_n(t))) \right]
+ \sum_{j=1}^{n} c_{ij}(t) y_j^\Delta(t - \sigma_{ij}(t, y_1(t) \ldots, y_n(t))) \right]\right]\right) \leq \tau_i^M |(\Phi_i x) - (\Phi_i y)|_0 + \max_{t \in [0, \omega]_T} \left\{ x_i(t) \left[ \left( \sum_{j=1}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} b_{ij}(t) x_j(t - \tau_{ij}(t, x_1(t) \ldots, x_n(t))) \right) + \sum_{j=1}^{n} c_{ij}(t) x_j^\Delta(t - \sigma_{ij}(t, x_1(t) \ldots, x_n(t))) \right] \right\}
+ \max_{t \in [0, \omega]_T} \left\{ \left( \sum_{j=1}^{n} a_{ij}(t) y_j(t) + \sum_{j=1}^{n} b_{ij}(t) y_j(t - \tau_{ij}(t, y_1(t) \ldots, y_n(t))) \right) + \sum_{j=1}^{n} c_{ij}(t) y_j^\Delta(t - \sigma_{ij}(t, y_1(t) \ldots, y_n(t))) \right] \right\}
+ \max_{t \in [0, \omega]_T} \left\{ x_i(t) - y_i(t) \right\} \left( \sum_{j=1}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} b_{ij}(t) x_j(t - \tau_{ij}(t, x_1(t) \ldots, x_n(t))) \right) + \sum_{j=1}^{n} c_{ij}(t) x_j^\Delta(t - \sigma_{ij}(t, x_1(t) \ldots, x_n(t))) - y_j^\Delta(t - \sigma_{ij}(t, y_1(t) \ldots, y_n(t))) \right) \right\}
+ \max_{t \in [0, \omega]_T} \left\{ x_i(t) - y_i(t) \right\} \left( \sum_{j=1}^{n} a_{ij}(t) x_j(t) + \sum_{j=1}^{n} b_{ij}(t) x_j(t - \tau_{ij}(t, x_1(t) \ldots, x_n(t))) \right) + \sum_{j=1}^{n} c_{ij}(t) x_j^\Delta(t - \sigma_{ij}(t, x_1(t) \ldots, x_n(t))) - y_j^\Delta(t - \sigma_{ij}(t, y_1(t) \ldots, y_n(t))) \right) \right\}
\[\sum_{j=1}^{n} c_{ij}(t)|y_{ij}(t - \sigma(t, y(t)))[x_i(t) - y_i(t)]\right\}
\leq r_i^M \varepsilon + R\varepsilon \left(\sum_{j=1}^{n} |a_{ij}^M + b_{ij}^M| + |x_i|_0(\eta + \varepsilon) \left(\sum_{j=1}^{n} c_{ij}^M\right)\right)
\quad + R\varepsilon \left(\sum_{j=1}^{n} |a_{ij}^M + b_{ij}^M + c_{ij}^M|\right)
\quad = |x_i|_0\eta \sum_{j=1}^{n} c_{ij}^M + \hat{H}_i \varepsilon, \quad (2.13)
\]

where \(\hat{H} = r_i^M + 2R \sum_{j=1}^{n} a_{ij}^M + 2R \sum_{j=1}^{n} b_{ij}^M + 2R \sum_{j=1}^{n} c_{ij}^M, i = 1, 2, \ldots, n\).

From (2.12) and (2.13) we have
\[
||\Phi x - \Phi y||_1 \leq \left(\sum_{j=1}^{n} |x_i|_0 \sum_{j=1}^{n} c_{ij}^M\right)\eta + \varepsilon \sum_{j=1}^{n} \hat{H}_i
\quad = R \max_{1 \leq i \leq n} \left\{\sum_{j=1}^{n} c_{ij}^M\right\}\eta + \varepsilon \sum_{j=1}^{n} \hat{H}_i \quad \text{for any } x, y \in S_{ijk}.
\]

As \(\varepsilon\) is arbitrary small, it follows that
\[
\alpha_{C_1^\omega}(\Phi(S)) \leq \left(R \max_{1 \leq i \leq n} \left\{\sum_{j=1}^{n} c_{ij}^M\right\}\right)\alpha_{C_1^\omega}(S).
\]

Therefore, \(\Phi\) is strict-set-contractive. The proof of Lemma 2.6 is complete. \(\square\)

### 3. Main Results

Our main result of this paper is as follows:

**Theorem 3.1.** Assume that \((H_1) - (H_3), (H_5)\) hold.

(i) If \(\max\{r_i^M, i = 1, 2, \ldots, n\} \leq 1\), then system (1.3) has at least one positive \(\omega\)-periodic solution.

(ii) If \((H_4)\) holds and \(\min\{r_i^M, i = 1, 2, \ldots, n\} > 1\), then system (1.3) has at least one positive \(\omega\)-periodic solution.
Proof. We only need to prove (i), since the proof of (ii) is similar. Let
\[ R = \max_{1 \leq i \leq n} \left\{ \frac{1 - e_{ri}(0, w)}{e_{ri}^2(0, w) \min_{1 \leq i \leq n} \{M_{ij}\}} \right\} \]
and
\[ 0 < r < \frac{e_{ri}(0, w)(1 - e_{ri}(0, w))}{\max_{1 \leq j \leq n} \{\hat{M}_{ij}\}}. \]
Then we have \( 0 < r < R \). From Lemmas 2.5 and 2.6, we know that \( \Phi \) is strict-set-contractive on \( K_r, R \). In view of Lemma 2.3 and 2.4, we see that if there exists \( x^* \in K \) such that \( \Phi x^* = x^* \), then \( x^* \) is one positive \( \omega \)-periodic solution of system (1.3). Now, we shall prove that condition (ii) of Lemma 2.4 holds.

First, we prove that \( \Phi x \not\geq x \), \( \forall x \in K, |x|_1 = r \). Otherwise, there exists \( x \in K, |x|_1 = r \) such that \( \Phi x \geq x \). So \( |x| > 0 \) and \( \Phi x - x \in K \), which implies that
\[
(\Phi_i x)(t) - x_i(t) \geq e_{ri}(0, w)|\Phi_i x - x_i|_1 \geq 0,
\]
for any \( t \in [0, \omega]_\mathbb{T}, i = 1, 2, \ldots, n. \) (3.1)

Moreover, for \( t \in [0, \omega]_\mathbb{T} \), we have
\[
(\Phi_i x)(t) = \int_t^{t+\omega} \sum_{j=1}^n a_{ij}(s)x_j(s) + \sum_{j=1}^n b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) + \sum_{j=1}^n c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(t, x(s))) \Delta s
\]
\[
\leq \frac{1}{1 - e_{ri}(0, w)} |x|_1 \max_{1 \leq j \leq n} \{\hat{M}_{ij}\} |x|_1 \sum_{j=1}^n (a_{ij}(s) + b_{ij}(s) + c_{ij}(s)) |x_j|_1 \Delta s
\]
\[
\leq \frac{1}{1 - e_{ri}(0, w)} \max_{1 \leq j \leq n} \{\hat{M}_{ij}\} |x|_1 \sum_{j=1}^n |x_j|_1
\]
\[
= \frac{\max_{1 \leq j \leq n} \{\hat{M}_{ij}\}}{1 - e_{ri}(0, w)} |x|_0 r
\]
\[
< e_{ri}(0, w)|x|_0, \quad i = 1, 2, \ldots, n.
\] (3.2)

In view of (3.1) and (3.2), we have
\[
|x| \leq |\Phi x|_0 < \max_{1 \leq i \leq n} \{e_{ri}(0, \omega)\} |x| < ||x||,
\]
which is a contradiction. Finally, we prove that $\Phi x \not\leq x$, $\forall x \in K$, $|x|_1 = R$ also holds. For this case, we only need to prove that

$$\Phi x \not< x \quad x \in K, \|x\|_1 = R.$$ 

Suppose, for the sake of contradiction, that there exists $x \in K$ and $\|x\|_1 = R$ such that $\Phi x < x$. Thus $x - \Phi x \in K \setminus \{0\}$. Furthermore, for any $t \in [0, \omega]_T$, we have

$$x_i(t) - (\Phi_i x)(t) \geq e_{ri}(0, w)|x_i - \Phi_i x|_1 > 0, \ i = 1, 2, \ldots, n. \quad (3.3)$$

In addition, for any $t \in [0, \omega]_T$, we find

$$(\Phi_i x)(t) = \int_t^{t+\omega} G_i(t, s)x_i(s) \left[ \sum_{j=1}^{n} a_{ij}(s)x_j(s) ight. \\
+ \sum_{j=1}^{n} b_{ij}(s)x_j(s - \tau_{ij}(s, x_1(s), \ldots, x_n(s))) \\
+ \sum_{j=1}^{n} c_{ij}(s)x_j^\Delta(s - \sigma_{ij}(t, x(s))) \bigg] \Delta s \\
\geq \frac{e_{ri}^2(0, w)}{1 - e_{ri}(0, w)}|x_i|_1 \sum_{j=1}^{n} |x_j|_1 \\
\times \int_0^{\omega} \left[ e_{ri}(0, w)a_{ij}(s) + e_{rij}(0, w)b_{ij}(s) - c_{ij}(s) \right] \Delta s \\
= \frac{e_{ri}^2(0, w)}{1 - e_{ri}(0, w)}|x_i|_1 \sum_{j=1}^{n} M_{ij}|x_j|_1, \ i = 1, 2, \ldots, n.$$

Thus,

$$\|\Phi x\| = \sum_{j=1}^{n} |(\Phi_i x)|_0 \\
\geq \sum_{j=1}^{n} e_{ri}^2(0, w)\frac{n}{1 - e_{ri}(0, w)}|x_i|_1 \sum_{j=1}^{n} M_{ij}|x_j|_1 \\
\geq \sum_{j=1}^{n} e_{ri}^2(0, w)\min_{1 \leq j \leq n}\{M_{ij}\}|x_i|_1 \sum_{j=1}^{n} |x_j|_1.$$
From (3.3) and (3.4), we obtain

$$\|x\| > \|\Phi x\| \geq R,$$

which is a contradiction. Therefore, conditions (i) and (ii) hold. By Lemma 2.4, we see that \( \Phi \) has at least one nonzero fixed point in \( K \). Therefore, system (1.3) has at least one positive \( \omega \)-periodic solution. The proof of Theorem 3.1 is complete.

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References


