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THE VARIATIONAL PROBLEM IN LAGRANGE SPACES ENDOWED WITH (γ, β) METRICS

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Abstract: In the present paper we studied the variational problem of Lagrange spaces with (γ, β)-metrics. The results follow the classical ones and some results of R. Miron concerning Lagrange spaces.

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1. Introduction

In the paper B. Nicolaescu [3, 4] has studied the variational problem of Lagrange spaces with (α, β)-metrics. In the present paper we studied the variational problem of Lagrange spaces with (γ, β)-metrics. Result obtain in the paper has been put in the form of seven propositions.

2. Preliminaries

Let (TM, τ, M) be the tangent bundle of a C^∞-differentiable real, n-dimensional manifold M. If (U, ϕ) is a local chart on M, then the coordinates of a point $u = (x, y) \in \tau^{-1}(U) \subset TM$ will be denoted by (x, y). R. Miron [1] given following defintions:

Definition 1. a) A differentiable Lagrangian on TM is a mapping $L : (x, y) \in TM \longrightarrow L(x, y) \in R, \forall u = (x, y) \in TM$, which is of class C^∞ on

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$\widehat{TM} = TM \setminus (0)$ and is continuous on the null section of the projection $\tau : TM \longrightarrow M$, such that

$$g_{ij} = \frac{1}{2}\frac{\partial^2 L(x,y)}{\partial y^i \partial y^j} \tag{1}$$

is a (0, 2)-type symmetric d-tensor field on TM.

b) A differential Lagrangian L on TM is said to be regular if

$$rank\|g_{ij}(x,y)\| = n, \qquad \forall (x,y) \in \widehat{TM}.$$

We will further use its contrvariant d-tensor $g^{ij}(x,y)$ given by $g^{ik}g_{jk} = \delta^i_j$.

c) A Lagrange space is a pair $L^n = (M, L)$ formed by a smooth real n-dimensional manifold M and a regular differentiable Lagrangian L on M, for which the d-tensor field g_{ij} from (1) has constant signature on $\widehat{TM}$.

Let $L : TM \longrightarrow R$ be a differentiable Lagrangian on the manifold M, which is not necessarily regular. A curve $c : t \in [0,1] \longrightarrow (x^i(t)) \in U \subset M$ having the image in a domain of a chart U of M, has the extension to $\widehat{TM}$ given by $c^* : t \in [0,1] \longrightarrow (x^i(t), \frac{dx^i(t)}{dt}) \in \tau^{-1}(U)$.

The integral of action of the Lagrangian L on the curve c is given by the functional

$$I(c) = \int_0^1 L\left(x(t), \frac{dx}{dt}\right) dt.$$

Consider the curve $c_\epsilon : t \in [0,1] \longrightarrow (x^i(t) + \epsilon v^i(t)) \in M$, which have the same endpoints $x^i(0), x^i(1)$ as the curve c, $v^i(0) = v^i(1) = 0$ and ϵ is a real number, sufficiently small in absolute value, such that $Imc_\epsilon \in U$. The extension of the curve c_ϵ to TM is

$$c^*_\epsilon : t \in [0,1] \longrightarrow \left(x^i(t) + \epsilon v^i(t), \frac{dx^i}{dt} + \epsilon \frac{dv^i}{dt}\right) \in \tau^{-1}(U).$$

The integral of action of the Lagrangian L on the curve c_ϵ is,

$$I(c_\epsilon) = \int_0^1 L(x + \epsilon v, \frac{dx}{dt} + \epsilon \frac{dv}{dt})dt$$

A necessary condition for I(c) to be an extremal value $I(c_\epsilon)$ is

$$\frac{dI(c_\epsilon)}{d\epsilon}|_{\epsilon=0} = 0$$

In order that the functional $I(c)$ be an extremal value of $I(c_\epsilon)$ it is necessary that c be the solution of the Euler-Lagrange equations,

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt}(\frac{\partial L}{\partial y^i}) = 0, \quad y^i = \frac{dx^i}{dt}.$$

3. The Fundamental Tensor of a Lagrange Space with (γ, β)-Metric

In 2011, Pandey and Chaubey [2], introduce the concept of (γ, β)-metric, where $\gamma^3 = a_{ijk}(x)y^iy^jy^k$ is a cubic metric and $\beta = b_i(x)y^i$ is a one form metric on TM.

Definition 2. A Lagrange space $L^n = L(M, L(x, y))$ is called with (γ, β)-metric if the fundamental function L(x, y) is a function $\overline{L}$, which depends only on $\gamma(x, y)$ and $\beta(x, y)$,

$$L = \overline{L}(\gamma(x, y), \beta(x, y)).$$

Here, we shall use the following notations throughout the whole paper,

$$\dot{\partial}_i\gamma = \frac{\partial\gamma}{\partial y^i},\ \dot{\partial}_i\beta = \frac{\partial\beta}{\partial y^i},\ \dot{\partial}_i\dot{\partial}_j\gamma = \frac{\partial^2\gamma}{\partial y^i\partial y^j},\ \overline{L}_\gamma = \frac{\partial\overline{L}}{\partial\gamma},\ \overline{L}_\beta = \frac{\partial\overline{L}}{\partial\beta},$$

$$\overline{L}_{\gamma\gamma} = \frac{\partial^2\overline{L}}{\partial\gamma^2},\ \overline{L}_{\gamma\beta} = \frac{\partial^2\overline{L}}{\partial\gamma\partial\beta},\ \overline{L}_{\beta\beta} = \frac{\partial^2\overline{L}}{\partial\beta^2}.$$

Proposition 1. *We have the relations*

$$\dot{\partial}_i\gamma = \gamma^{-1}y_i,\ \dot{\partial}_i\dot{\partial}_j\gamma = 2\gamma^{-1}a_{ij}(x, y) - \gamma^{-3}y_iy_j,\ \dot{\partial}_i\beta = b_i(x),\ \dot{\partial}_i\dot{\partial}_j\beta = 0,$$

where,

$$y_i = a_{ij}(x, y)y^j,\ a_{ijk}y^jy^k = a_i(x, y),\ 2a_{ijk}y^k = a_{ij}.$$

We introduce the moments of the Lagrangian $L(x, y) = \overline{L}(\gamma(x, y), \beta(x, y))$,

$$p_i = \frac{1}{2}\frac{\partial L}{\partial y^i} = \frac{1}{2}(\overline{L}_\gamma\dot{\partial}_i\gamma + \overline{L}_\beta\dot{\partial}_i\beta)$$

and we get the following propositions.

Proposition 2. *The moments of the Lagrangian L(x, y) are given by*

$$p_i = \rho y_i + \rho_1 b_i \tag{2}$$

where, $\rho = \frac{1}{2}\gamma^{-1}\overline{L}_\gamma$ *and* $\rho_1 = \frac{1}{2}\overline{L}_\beta$.

The two scalar functions defined in (3) are called the principal invarints of the Lagrange space L^n.

Proposition 3. *The derivatives of principal invariants of the Lagrange space* L^n *are given by*

$$\dot{\partial}_i \rho = \rho_{-2} y_i + \rho_{-1} b_i, \quad \dot{\partial}_i \rho_1 = \rho_{-1} y_i + \rho_0 b_i,$$

where, $\rho_{-2} = \frac{1}{2}\gamma^{-2}(\overline{L}_{\gamma\gamma} - \gamma^{-1}\overline{L}_\gamma)$, $\rho_{-1} = \frac{1}{2}\gamma^{-1}\overline{L}_{\gamma\beta}$, $\rho_0 = \frac{1}{2}\overline{L}_{\beta\beta}$.

Proposition 4. *The Energy*

$$E_L = y^i \frac{\partial L}{\partial y^i} - L$$

of a Lagrangian with (γ, β)*-metric is given by,*

$$E_L = \gamma^{-1}\overline{L}_\gamma + \beta \overline{L}_\beta - \overline{L}.$$

We can determine the fundametal tensor g_{ij} *of the Lagrange space with* (γ, β)*-metric, as follows.*

Proposition 5. *The fundamental tensor* g_{ij} *of the Lagrange space with* (γ, β)*-metric is*

$$g_{ij} = 2\rho a_{ij} + c_i c_j \tag{3}$$

where, $c_i = q_{-1} y_i + q_0 b_i$ *and* q_{-1}, q_0 *satisfy the equations* $\rho_0 = (q_0)^2, \quad \rho_{-1} = q_0 q_{-1}, \quad \rho_{-2} = (q_{-1})^2$.

Proposition 6. *The reciprocal tensor* g^{ij} *of the fundamental tensor* g_{ij} *is given by*

$$g^{ij} = \frac{1}{2\rho} a^{ij} - \frac{1}{(1 + c^2)} c^i c^j \tag{4}$$

where, $c^i = \frac{1}{2}\rho^{-1} a^{ij} c_j$ *and* $c^i c_i = c^2$.

4. Eular-Lagrange Equations in Lagrange Spaces with (γ, β)-Metric

The Eular-Lagrange equations of the Lagrange spaces with (γ, β)-metric are,

$$E_i(\overline{L}) = \frac{\partial \overline{L}}{\partial x^i} - \frac{d}{dt}\left(\frac{\partial \overline{L}}{\partial y^i}\right) = 0, \quad y^i = \frac{dx^i}{dt}$$

considering the relations,

$$\frac{\partial\overline{L}}{\partial x^i}=\overline{L}_\gamma\frac{\partial\gamma}{\partial x^i},\quad \frac{\partial\overline{L}}{\partial y^i}=\overline{L}_\gamma\frac{\partial\gamma}{\partial y^i}+\overline{L}_\beta\frac{\partial\beta}{\partial y^i},$$

$$\frac{d}{dt}(\frac{\partial\overline{L}}{\partial y^i})=\frac{d\overline{L}_\gamma}{dt}\frac{\partial\gamma}{\partial y^i}+\frac{d\overline{L}_\beta}{dt}\frac{\partial\beta}{\partial y^i}+\overline{L}_\gamma\frac{d}{dt}(\frac{\partial\gamma}{\partial y^i})+\overline{L}_\beta\frac{d}{dt}(\frac{\partial\beta}{\partial y^i}).$$

By direct calculation, we have

$$E_i(\overline{L})=\overline{L}_\gamma E_i(\gamma)+\overline{L}_\beta E_i(\beta)-\frac{\partial\gamma}{\partial y^i}\frac{d\overline{L}_\gamma}{dt}-\frac{\partial\beta}{\partial y^i}\frac{d\overline{L}_\beta}{dt},\quad y^i=\frac{dx^i}{dt},$$

where

$$\frac{d\overline{L}_\gamma}{dt}=\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt}+\overline{L}_{\gamma\beta}\frac{d\beta}{dt}\text{ and }\frac{d\overline{L}_\beta}{dt}=\overline{L}_{\beta\gamma}\frac{d\gamma}{dt}+\overline{L}_{\beta\beta}\frac{d\beta}{dt}.$$

Then, we get

$$E_i(\overline{L})=\overline{L}_\gamma E_i(\gamma)+\overline{L}_\beta E_i(\beta)-\frac{\partial\gamma}{\partial y^i}(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt}+\overline{L}_{\gamma\beta}\frac{d\beta}{dt})-\frac{\partial\beta}{\partial y^i}(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt}+\overline{L}_{\beta\beta}\frac{d\beta}{dt}).$$

As well we have

$$E_i(\gamma)=\frac{1}{3\gamma^2}E_i(\gamma^3)+\frac{1}{\gamma^2}\frac{\partial\gamma}{\partial y^i}\frac{d\gamma^2}{dt},\quad E_i(\beta)=F_{ir}\frac{dx^r}{dt},$$

where

$$F_{ir}=\frac{\partial A_r}{\partial x^i}-\frac{\partial A_i}{\partial x^r}$$

is the electromagnetic tensor field. Finally we have the following relation

$$\begin{aligned}E_i(\overline{L})=&\frac{2}{3\gamma}\rho E_i(\gamma^3)+\frac{2}{3\gamma}\rho\frac{\partial\gamma}{\partial y^i}\frac{d\gamma^2}{dt}+2\rho_1F_{ir}\frac{dx^r}{dt}-\frac{\partial\gamma}{\partial y^i}(\overline{L}_{\gamma\gamma}\frac{d\gamma}{dt}+\overline{L}_{\gamma\beta}\frac{d\beta}{dt})\\&-\frac{\partial\beta}{\partial y^i}(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt}+\overline{L}_{\beta\beta}\frac{d\beta}{dt}).\end{aligned}$$

Proposition 7. *The Euler-Lagrange equation in the Lagrange space L^n endowed with (γ,β)-metric are,*

$$E_i(\overline{L})=0,\quad y^i=\frac{dx^i}{dt}.$$

If we have the natural parametrization of the curve $\in[0,1]\longrightarrow(x^i(t)\in M)$ relative to the cubic metric $a_{ijk}(x)$, then $\gamma(x,\frac{dx}{dt})=1$. Then we get

Proposition 8. *In the canonical parametrization the Euler-Lagrange equations in L^n spaces with (γ,β)-metric are,*

$$\begin{aligned} E_i(\overline{L}) =& \frac{2}{3\gamma}\rho E_i(\gamma^3) + \frac{2}{3\gamma}\rho\frac{\partial\gamma}{\partial y^i}\frac{d\gamma^2}{dt} + 2\rho_1 F_{ir}\frac{dx^r}{dt} \\ & - \frac{\partial\beta}{\partial y^i}(\overline{L}_{\beta\gamma}\frac{d\gamma}{dt} + \overline{L}_{\beta\beta}\frac{d\beta}{dt}) \\ =& 0. \end{aligned} \tag{5}$$

Proposition 9. *If the 1-form β is constant on the integral curve c of the Euler-Lagrange equations, then (6) rewrite as the Lorentz equations of the space L^n,*

$$E_i(\overline{L}) = \tfrac{2}{3\gamma}\rho E_i(\gamma^3) + \tfrac{2}{3\gamma}\rho\tfrac{\partial\gamma}{\partial y^i}\tfrac{d\gamma^2}{dt} + 2\rho_1 F_{ir}\tfrac{dx^r}{dt} = 0$$

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