THE VARIATIONAL PROBLEM IN LAGRANGE SPACES
ENDOWED WITH \((\gamma, \beta)\) METRICS

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Abstract: In the present paper we studied the variational problem of Lagrange spaces with \((\gamma, \beta)\)-metrics. The results follow the classical ones and some results of R. Miron concerning Lagrange spaces.

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1. Introduction

In the paper B. Nicolaescu [3, 4] has studied the variational problem of Lagrange spaces with \((\alpha, \beta)\)-metrics. In the present paper we studied the variational problem of Lagrange spaces with \((\gamma, \beta)\)-metrics. Result obtain in the paper has been put in the form of seven propositions.

2. Preliminaries

Let \((TM, \tau, M)\) be the tangent bundle of a \(C^\infty\)-differentiable real, n-dimensional manifold M. If \((U, \phi)\) is a local chart on M, then the coordinates of a point \(u = (x, y) \in \tau^{-1}(U) \subset TM\) will be denoted by \((x, y)\). R. Miron [1] given following defintions:

**Definition 1.** a) A differentiable Lagrangian on TM is a mapping \(L : (x, y) \in TM \rightarrow L(x, y) \in R, \forall u = (x, y) \in TM\), which is of class \(C^\infty\) on
\( \hat{T}M = TM \setminus \{0\} \) and is continuous on the null section of the projection \( \tau : TM \rightarrow M \), such that

\[
g_{ij} = \frac{1}{2} \frac{\partial^2 L(x, y)}{\partial y^i \partial y^j}
\]  

is a \((0, 2)\)-type symmetric d-tensor field on \( TM \).

b) A differential Lagrangian \( L \) on \( TM \) is said to be regular if

\[
\text{rank}\|g_{ij}(x, y)\| = n, \quad \forall (x, y) \in \hat{T}M.
\]

We will further use its contrvariant d-tensor \( g^{ij}(x, y) \) given by \( g^{ik}g_{kj} = \delta^i_j \).

c) A Lagrange space is a pair \( \mathcal{L} = (M, L) \) formed by a smooth real \( n \)-dimensional manifold \( M \) and a regular differentiable Lagrangian \( L \) on \( M \), for which the d-tensor field \( g_{ij} \) from (1) has constant signature on \( \hat{T}M \).

Let \( L : TM \rightarrow R \) be a differentiable Lagrangian on the manifold \( M \), which is not necessarily regular. A curve \( c : t \in [0, 1] \rightarrow (x^i(t)) \in U \subset M \) having the image in a domain of a chart \( U \) of \( M \), has the extension to \( \hat{T}M \) given by \( c^\star : t \in [0, 1] \rightarrow (x^i(t), \frac{dx^i(t)}{dt}) \in \tau^{-1}(U) \).

The integral of action of the Lagrangian \( L \) on the curve \( c \) is given by the functional

\[
I(c) = \int_0^1 L(x(t), \frac{dx}{dt}) \, dt.
\]

Consider the curve \( c_\epsilon : t \in [0, 1] \rightarrow (x^i(t) + \epsilon v^i(t)) \in M \), which have the same endpoints \( x^i(0), x^i(1) \) as the curve \( c \), \( v^i(0) = v^i(1) = 0 \) and \( \epsilon \) is a real number, sufficiently small in absolute value, such that \( Imc_\epsilon \in U \). The extension of the curve \( c_\epsilon \) to \( TM \) is

\[
c_\epsilon^\star : t \in [0, 1] \rightarrow (x^i(t) + \epsilon v^i(t), \frac{dx^i}{dt} + \epsilon \frac{dv^i}{dt}) \in \tau^{-1}(U).
\]

The integral of action of the Lagrangian \( L \) on the curve \( c_\epsilon \) is,

\[
I(c_\epsilon) = \int_0^1 L(x + \epsilon v, \frac{dx}{dt} + \epsilon \frac{dv}{dt}) \, dt
\]

A necessary condition for \( I(c) \) to be an extremal value \( I(c_\epsilon) \) is

\[
\left. \frac{dI(c_\epsilon)}{d\epsilon} \right|_{\epsilon=0} = 0
\]
In order that the functional $I(c)$ be an extremal value of $I(c_\epsilon)$ it is necessary that $c$ be the solution of the Euler-Lagrange equations,

$$E_i(L) = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}.$$  

3. The Fundamental Tensor of a Lagrange Space with $(\gamma, \beta)$-Metric

In 2011, Pandey and Chaubey [2], introduce the concept of $(\gamma, \beta)$-metric, where $\gamma^3 = a_{ijk}(x)y^i y^j y^k$ is a cubic metric and $\beta = b_i(x)y^i$ is a one form metric on TM.

**Definition 2.** A Lagrange space $L^n = L(M, L(x, y))$ is called with $(\gamma, \beta)$-metric if the fundamental function $L(x, y)$ is a function $L$, which depends only on $\gamma(x, y)$ and $\beta(x, y)$,

$$L = T(\gamma(x, y), \beta(x, y)).$$

Here, we shall use the following notations throughout the whole paper,

$$\dot{\gamma} = \frac{\partial \gamma}{\partial y^i}, \quad \dot{\beta} = \frac{\partial \beta}{\partial y^i}, \quad \dot{\gamma} \dot{\gamma} = \frac{\partial^2 \gamma}{\partial y^i \partial y^j}, \quad T_\gamma = \frac{\partial T}{\partial \gamma}, \quad T_\beta = \frac{\partial T}{\partial \beta},$$

$$T_{\gamma \gamma} = \frac{\partial^2 T}{\partial \gamma^2}, \quad T_{\gamma \beta} = \frac{\partial^2 T}{\partial \gamma \partial \beta}, \quad T_{\beta \beta} = \frac{\partial^2 T}{\partial \beta^2}.$$  

**Proposition 1.** We have the relations

$$\dot{\gamma} = \gamma^{-1} y_i, \quad \dot{\beta} = \gamma^{-1} a_{ii}(x, y) - \gamma^{-3} y_1 y_j, \quad \dot{\gamma} \dot{\beta} = b_i(x), \quad \dot{\beta} = 0,$$

where,

$$y_i = a_{ij}(x, y)y^j, \quad a_{ijk}y^j y^k = a_i(x, y), \quad 2a_{ijk}y^k = a_{ij}.$$  

We introduce the moments of the Lagrangian $L(x, y) = T(\gamma(x, y), \beta(x, y))$,

$$p_i = \frac{1}{2} \frac{\partial L}{\partial y^i} = \frac{1}{2} \left( T_\gamma \dot{\gamma} + T_\beta \dot{\beta} \right)$$

and we get the following propositions.

**Proposition 2.** The moments of the Lagrangian $L(x, y)$ are given by

$$p_i = \rho y_i + \rho_1 b_i$$  

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where, \( \rho = \frac{1}{2} \gamma^{-1} \mathcal{L}_\gamma \) and \( \rho_1 = \frac{1}{2} \mathcal{L}_\beta \).

The two scalar functions defined in (3) are called the principal invariants of the Lagrange space \( L^n \).

**Proposition 3.** The derivatives of principal invariants of the Lagrange space \( L^n \) are given by

\[
\dot{\rho}_i = \rho - 2 y_i + \rho_{-1} b_i, \quad \dot{\rho}_1 = \rho_{-1} y_i + \rho_0 b_i,
\]

where, \( \rho_{-2} = \frac{1}{2} \gamma^{-2}(\mathcal{L}_{\gamma\gamma} - \gamma^{-1} \mathcal{L}_\gamma) \), \( \rho_{-1} = \frac{1}{2} \gamma^{-1} \mathcal{L}_{\gamma\beta} \), \( \rho_0 = \frac{1}{2} \mathcal{L}_{\beta\beta} \).

**Proposition 4.** The Energy

\[
E_L = y^i \frac{\partial L}{\partial y^i} - L
\]

of a Lagrangian with \((\gamma, \beta)\)-metric is given by,

\[
E_L = \gamma^{-1} \mathcal{L}_\gamma + \beta \mathcal{L}_{\beta\beta} - \mathcal{L}.
\]

We can determine the fundamental tensor \( g_{ij} \) of the Lagrange space with \((\gamma, \beta)\)-metric, as follows.

**Proposition 5.** The fundamental tensor \( g_{ij} \) of the Lagrange space with \((\gamma, \beta)\)-metric is

\[
g_{ij} = 2 \rho a_{ij} + c_i c_j
\]

where, \( c_i = q_{-1} y_i + q_0 b_i \) and \( q_{-1}, q_0 \) satisfy the equations \( \rho_0 = (q_0)^2 \), \( \rho_{-1} = q_0 q_{-1} \), \( \rho_{-2} = (q_{-1})^2 \).

**Proposition 6.** The reciprocal tensor \( g^{ij} \) of the fundamental tensor \( g_{ij} \) is given by

\[
g^{ij} = \frac{1}{2\rho} a^{ij} - \frac{1}{(1 + c^2)} c^i c^j
\]

where, \( c^i = \frac{1}{2} \rho^{-1} a^{ij} c_j \) and \( c^i c_i = c^2 \).

### 4. Euler-Lagrange Equations in Lagrange Spaces with \((\gamma, \beta)\)-Metric

The Euler-Lagrange equations of the Lagrange spaces with \((\gamma, \beta)\)-metric are,

\[
E_i(\mathcal{L}) = \frac{\partial \mathcal{L}}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial y^i} \right) = 0, \quad y^i = \frac{dx^i}{dt}
\]
considering the relations,
\[ \frac{\partial \mathcal{L}}{\partial x^i} = \mathcal{L}_\gamma \frac{\partial \gamma}{\partial x^i}, \quad \frac{\partial \mathcal{L}}{\partial y^i} = \mathcal{L}_\gamma \frac{\partial \gamma}{\partial y^i} + \mathcal{L}_\beta \frac{\partial \beta}{\partial y^i}, \]
\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial y^i} \right) = \mathcal{L}_\gamma \frac{d\gamma}{dt} \frac{\partial \gamma}{\partial y^i} + \mathcal{L}_\beta \frac{d\beta}{dt} \frac{\partial \beta}{\partial y^i} + \mathcal{L}_\gamma \frac{d}{dt} \left( \frac{\partial \gamma}{\partial y^i} \right) + \mathcal{L}_\beta \frac{d}{dt} \left( \frac{\partial \beta}{\partial y^i} \right). \]

By direct calculation, we have
\[ E_i(\mathcal{L}) = \mathcal{L}_\gamma E_i(\gamma) + \mathcal{L}_\beta E_i(\beta) - \frac{\partial \gamma}{\partial y^i} \frac{d\mathcal{L}_\gamma}{dt} - \frac{\partial \beta}{\partial y^i} \frac{d\mathcal{L}_\beta}{dt}, \]
where
\[ \frac{d\mathcal{L}_\gamma}{dt} = \mathcal{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\gamma\beta} \frac{d\beta}{dt} \]
and
\[ \frac{d\mathcal{L}_\beta}{dt} = \mathcal{L}_{\beta\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\beta\beta} \frac{d\beta}{dt}. \]

Then, we get
\[ E_i(\mathcal{L}) = \mathcal{L}_\gamma E_i(\gamma) + \mathcal{L}_\beta E_i(\beta) - \frac{\partial \gamma}{\partial y^i} (\mathcal{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\gamma\beta} \frac{d\beta}{dt}) - \frac{\partial \beta}{\partial y^i} (\mathcal{L}_{\beta\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\beta\beta} \frac{d\beta}{dt}). \]

As well we have
\[ E_i(\gamma) = \frac{1}{3\gamma^2} E_i(\gamma^3) + \frac{1}{\gamma^2} \frac{d\gamma}{dt}, \quad E_i(\beta) = F_{ir} \frac{dx^r}{dt}, \]
where
\[ F_{ir} = \frac{\partial A_r}{\partial x^i} - \frac{\partial A_i}{\partial x^r} \]
is the electromagnetic tensor field. Finally we have the following relation
\[ E_i(\mathcal{L}) = \frac{2}{3\gamma} \rho E_i(\gamma^3) + \frac{2}{3\gamma} \rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma}{dt}^2 + 2 \rho_1 F_{ir} \frac{dx^r}{dt} - \frac{\partial \gamma}{\partial y^i} (\mathcal{L}_{\gamma\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\gamma\beta} \frac{d\beta}{dt}) - \frac{\partial \beta}{\partial y^i} (\mathcal{L}_{\beta\gamma} \frac{d\gamma}{dt} + \mathcal{L}_{\beta\beta} \frac{d\beta}{dt}). \]

**Proposition 7.** The Euler-Lagrange equation in the Lagrange space \( L^n \) endowed with \((\gamma, \beta)\)-metric are,
\[ E_i(\mathcal{L}) = 0, \quad y^i = \frac{dx^i}{dt}. \]

If we have the natural parametrization of the curve \( \in [0, 1] \rightarrow (x^i(t) \in M) \) relative to the cubic metric \( a_{ijk}(x) \), then \( \gamma(x, \frac{dx}{dt}) = 1 \). Then we get
Proposition 8. In the canonical parametrization the Euler-Lagrange equations in $L^n$ spaces with $(\gamma, \beta)$-metric are,

$$E_i(\mathcal{L}) = \frac{2}{3\gamma}\rho E_i(\gamma^3) + \frac{2}{3\gamma}\rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma}{dt} + 2\rho_1 F_{ir} \frac{dx^r}{dt} - \frac{\partial \beta}{\partial y^i}(T_{\beta\gamma} \frac{d\gamma}{dt} + T_{\beta\beta} \frac{d\beta}{dt}) = 0.$$ (5)

Proposition 9. If the 1-form $\beta$ is constant on the integral curve $c$ of the Euler-Lagrange equations, then (6) rewrite as the Lorentz equations of the space $L^n$,

$$E_i(\mathcal{L}) = \frac{2}{3\gamma}\rho E_i(\gamma^3) + \frac{2}{3\gamma}\rho \frac{\partial \gamma}{\partial y^i} \frac{d\gamma}{dt} + 2\rho_1 F_{ir} \frac{dx^r}{dt} = 0$$

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