

BIOECONOMIC MODEL WITH GOMPERTZ POPULATION
GROWTH AND SPECIES CONSERVATION

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Abstract: This paper proposes a bioeconomic model destined to show how the human activity may alter the natural evolution of biological species. The model is formulated in term of optimal control theory, and then is transformed into “augmented” dynamic system by meas of the optimal choice of control variables resulting form the application of Pontryagin’s maximum principle. In the case of species abundance, we prove that saddle-point stability of the original bioeconomic system is transferred to the “augmented” system under optimal choice of control variables.

AMS Subject Classification: 92D25, 49K15

Key Words: Gompertz population growth, bioeconomic model, Pontryagin maximum principle, saddle-point stability

1. Introduction

The goal of this paper is to build and validate the model that shows how the human activity may alter the natural evolution of ecosystems. To formalize our study, we introduce a simple *bioeconomic* model in continuous time that includes two state variables – the population of a specie (biological component) and the capital (economical component). This model also contains two control variables – the consumption related to aggregated production with its negative effects (such as pollution, reduction of species habitat, etc.) and the defensive expenditures aimed at the conservation of the species (positive effects). We do not consider the harvesting (i.e., direct consumption of the species) and

Received: June 16, 2011

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suppose that the population dynamics of biological species is altered only by the negative side-effects of aggregated production.

The representative agents choose the consumption and defensive expenditures so as to maximize the present discounted value of their utility function subject to physical capital accumulation dynamics and population growth affected by side-effects, that is, the natural population dynamics modified by human intervention.

Therefore, the problem can be formulated in terms of optimal control. Applying the maximum principle and introducing two additional variables (co-states or shadow-price variables) we arrive to a four-dimensional dynamic system and revise its stability properties. Using the necessary and sufficient conditions for stability of such augmented system [1], we show that saddle-point property of the original bioeconomic system is transferrable to the augmented system under the optimal choice of control variables when the population of species is sufficiently high in the fixed point. This suggests that the decision makers may implement a policy that guarantees species abundance in the future and also obtain maximum utility.

Additionally, the proposed model is validated by means of numerical simulations. This validation is carried out by consequent performance of variations (*ceteris paribus*) of ecological, economical, and biological parameters of the model.

This research is inspired by the articles [2] and [3] that deal with linear population dynamics. In our approach, we focus on nonlinear dynamics with carrying capacity (Gompertz population growth).

2. The Model

In this section we will construct the bio-economic model in continuous time and formulate the core problem in terms of optimal control.

For the sake of simplicity, consider a single-species unstructured population model given by *Gompertz* equation

$$\frac{dx}{dt} = \alpha x \ln \left(\frac{K}{x} \right), \quad x(t_0) = x_0 \quad (1)$$

where $x(t)$ describes the population size at time t (i.e., number of individuals, population abundance or biomass), $x_0 > 0$ is the initial size of population, constant $\alpha > 0$ defines the growth rate of population and constant $K > 0$ is a carrying capacity parameter. The carrying capacity of a biological species in

an environment is the maximum population size to be achieved in infinite time given the food, habitat, water and other necessities available in the environment. It is easily verified that (1) has two fixed points: $x_1^* = 0$ (unstable node) and $x_2^* = K$ (stable node).

Let us suppose that species population obeys the evolutionary biological dynamics given by Gompertz equation (1) under the course of nature without human intervention. Eventually, the biological dynamics can be affected by side-effects of human activity such as aggregated production, pollution, urbanization, etc. Almost all scientists acknowledge that the rate of species loss is greater now than at any time in human history. According to P.R.Ehrlich [4], the primary cause of the decay of organic diversity is not direct human exploitation or malevolence, but the habitat destruction that inevitably results from expansion of human population and human activities. To counterbalance this tendency, policy makers should implement defensive actions aiming at prevention of single species from extinction.

The objective of our study is to find out how the economic growth and human intervention could alter the natural evolution of biological species. Let us introduce a simple growth model that links the biological dynamics (1) to a capital accumulation dynamics. For the sake of simplicity, we suppose that there is a single good which is produced by capital alone $k(t)$ according to Solow classical model [5]:

$$\frac{dk}{dt} = k^\gamma - c - d \quad (2)$$

Here k^γ is per capita production function of Cobb-Douglas type with constant parameter γ such that $0 < \gamma < 1$. Thus, the capital accumulation, described by Eq. (2), is used for aggregate production k^γ , consumption c and defensive measures for environmental protection d .

Economic activity modeled by Eq. (2) can produce changes in natural biological dynamics of a specie given by Eq. (1). Namely, the pollution, habitat reduction and other consequences attributed to aggregated production k^γ , will have *negative* effect for population $x(t)$ while the defensive measures d will constitute *positive* effect for $x(t)$. Thus, the biological dynamics (modified by negative and positive effects of human activity) can be described by the differential equation

$$\frac{dx}{dt} = \alpha x \ln \left(\frac{K}{x} \right) - \epsilon k^\gamma + \sigma d^\theta \quad (3)$$

where $\epsilon > 0$ measures the negative impact of aggregated production on the population dynamics of x while $\sigma > 0$ measures the positive impact of defensive expenditures destined for conservation of the species population x . The

parameter $\theta \in (0, 1)$ is introduced in order to emphasize that the positive effect of defensive investment on specie evolution is not directly proportional to population growth; in other words, extra-spending on specie conservation (increase on d) may decrease the positivity of such impact on evolution of x due to carrying capacity limitations of the environment. Without loss of generality, we may assume for further analysis that $\theta = 1/2$ and propose the following *bioeconomic* dynamic system:

$$\begin{cases} \dot{x} = \alpha x \ln\left(\frac{K}{x}\right) - \epsilon k^\gamma + \sigma\sqrt{d}, & x(t_0) = x_0, \\ \dot{k} = k^\gamma - c - d, & k(t_0) = k_0, \end{cases} \quad (4)$$

where x_0 and k_0 are initial values for species population and capital, respectively. It should be noted that $x = x(t)$ and $k = k(t)$ are referred to as *state* variables, and their respective trajectories are uniquely defined from the system (4) for specified c and d . The latter two parameters $c > 0$ and $d \geq 0$ are referred to as *control* or *decision* variables and their values are determined by a decision maker. Strictly positive parameters $\alpha, K, \epsilon, \sigma$ and $\gamma \in (0, 1)$ are constants estimated by adjustment of real observation data to the model.

The dynamics of this bioeconomic system will not be as stable as biological dynamics given by Gompertz equation (1) alone, and $x_1^* = 0, x_2^* = K$ will not be its fixed points. In effect, the fixed points of (4) will naturally depend on the parameters c, d and can be viewed as (x^*, k^*) where

$$k^* = (c + d)^{1/\gamma}$$

while x^* solves the following nonlinear equation

$$\alpha x^* \ln\left(\frac{K}{x^*}\right) - \epsilon(c + d) + \sigma\sqrt{d} = 0. \quad (5)$$

The Jacobian matrix of the system (4) is given by

$$\mathcal{J}(x, k, c, d) = \begin{bmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{bmatrix} = \begin{bmatrix} \alpha \ln\left(\frac{K}{x}\right) - \alpha & -\epsilon \gamma k^{\gamma-1} \\ 0 & \gamma k^{\gamma-1} \end{bmatrix}$$

whose eigenvalues are of the form

$$\nu_1 = \mathcal{J}_{11} = \alpha \ln\left(\frac{K}{x}\right) - \alpha, \quad \nu_2 = \mathcal{J}_{22} = \gamma k^{\gamma-1}. \quad (6)$$

It is clear that $\nu_2 > 0$ independently of the choice of c, d while ν_1 may have either positive or negative value. From the economics point of view, it is interesting

to study the case when $\nu_1 < 0$ that is, when Eq. (5) has solution(s) x^* such that $x^* > K/e$.

In this case, (x^*, k^*) will be a *saddle* point and there will exist a stable manifold from which the initial values of x, k should be chosen in order to guarantee the stability of the system (4). Thus, instead of stable equilibrium of Gompertz equation, we now have (in the best case) only a stable manifold of the saddle point (x^*, k^*) with $x^* > K/e$ where both x^* and k^* depend on the choice of c and d .

Let us go for an *optimal choice* of the control variables c and d by proposing an optimal control problem.

3. Optimal Control Problem

Suppose that a representative agent that acts in economy has a task to choose the parameters c and d in order to maximize the present discounted value of its *utility function* $U(x, c)$ over infinite horizon. The agent's welfare depends, in each instant t , on the consumption c and abundance of the specie x , while d represents investments for saving x from extinction. By supposing that $c = c(t) > 0$ and $d = d(t) \geq 0$ are real-valued piecewise continuous functions, we can now formulate the following optimal control problem:

$$\int_{t_0}^{\infty} U(x, c) e^{-\rho t} dt \rightarrow \max \quad (7)$$

subject to dynamic system (4). Here the constant parameter $\rho > 0$ indicates the intertemporal discount rate.

The definition of utility function $U(x, c)$ must clearly reflect the priorities of decision-making. By setting

$$U(x, c) = U_1(x, c) = p x + q \ln c \quad (8)$$

where $p > 0$ and $q > 0$ represent weight parameters, it is indicated that consumption $c(t)$ is more important than species conservation. In other words, total extinction of the species ($x = 0$) can be paid off by aggregated consumption level. On the other hand, by defining

$$U(x, c) = U_2(x, c) = p \ln x + q \ln c \quad (9)$$

we can indicate that the abundance of $x(t)$ is essential for decision-maker and that the extinction of $x(t)$ in the future would cause tremendous loss of utility.

Under the function $U_2(x, c)$, if $x(t) \rightarrow 0^+$ when $t \rightarrow \infty$, the representative agent will suffer an infinite utility loss.

Remark 1. Note that here $x(t)$ expresses the abundance of species and can be scaled to hundreds, thousands, millions, etc. of individuals; therefore, if $0 < x < 1$ then $\ln x < 0$ and the total utility $U_2(x, c)$ will decrease.

The current-value Hamiltonian of the optimal control problem with infinite horizon (7) subject to bioeconomic dynamic system (4) is defined as

$$\begin{aligned} H(x, k, \lambda, \mu, c, d) &= U(x, c) \\ &+ \lambda \left[\alpha x \ln \left(\frac{K}{x} \right) - \epsilon k^\gamma + \sigma \sqrt{d} \right] + \mu [k^\gamma - c - d] \end{aligned} \quad (10)$$

for utility function $U(x, c)$ given either by (8) or by (9). Here λ and μ are adjoint variables (or *co-states* also called *shadow prices* in economics) associated to x and k , respectively, and satisfy the following equations:

$$\begin{cases} \dot{\lambda} = \rho \lambda - H_x(x, k, \lambda, \mu) = \lambda \left[\rho + \alpha - \alpha \ln \left(\frac{K}{x} \right) \right] - U_x(x, c) \\ \dot{\mu} = \rho \mu - H_k(x, k, \lambda, \mu) = (\rho - \gamma k^{\gamma-1}) \mu + \epsilon \gamma k^{\gamma-1} \lambda \end{cases} \quad (11)$$

The transversality conditions for system (11) are given by

$$\lim_{T \rightarrow \infty} e^{-\rho T} x(T) \lambda(T) = 0, \quad \lim_{T \rightarrow \infty} e^{-\rho T} k(T) \mu(T) = 0. \quad (12)$$

Applying the *maximum principle* we obtain the first-order necessary conditions for optimality:

$$\frac{\partial H}{\partial c} = \frac{\partial U}{\partial c} - \mu = 0, \quad \frac{\partial H}{\partial d} = \frac{\sigma}{2\sqrt{d}} \lambda - \mu = 0 \quad (13)$$

and observe that

$$D^2 H = \begin{bmatrix} H_{cc} & H_{cd} \\ H_{dc} & H_{dd} \end{bmatrix} = \begin{bmatrix} -\frac{q}{c^2} & 0 \\ 0 & -\frac{\sigma \lambda}{4\sqrt{d^3}} \end{bmatrix}$$

with $\det D^2 H > 0$ and $H_{cc} < 0$ for both $U = U_1(x, c)$ and $U = U_2(x, c)$ as long as $\lambda > 0$. Therefore, the Hamiltonian (10) attains its maximum when

$$c^* = \frac{q}{\mu}, \quad d^* = \frac{\sigma^2}{4} \left(\frac{\lambda}{\mu} \right)^2. \quad (14)$$

Note that c^* and d^* are evaluated according to (14) in the same way for $U = U_1(x, c)$ and for $U = U_2(x, c)$. Additionally, it should be fulfilled that $\mu > 0$ in order to guarantee that $c^* > 0$.

The substitution the optimal choices c^* and d^* from (14) into the dynamic systems (4) and (11) will result in (4×4) *augmented* dynamic system with respect to four variables (x, k, λ, μ) :

$$\begin{aligned} \dot{x} &= \alpha x \ln \left(\frac{K}{x} \right) - \epsilon k^\gamma + \sigma \left(\frac{\sigma \lambda}{2\mu} \right) \\ \dot{k} &= k^\gamma - \frac{q}{\mu} - \left(\frac{\sigma \lambda}{2\mu} \right)^2 \\ \dot{\beta} &= \left(\rho + \alpha \ln \left(\frac{K}{x} \right) - \alpha \right) \lambda - \frac{\partial U_i}{\partial x} \\ \dot{\lambda} &= \epsilon \gamma k^{\gamma-1} \lambda + (\rho - \gamma k^{\gamma-1}) \mu \end{aligned} \quad (15)$$

with initial conditions $x(t_0) = x_0, k(t_0) = k_0$ and the transversality conditions (12). Solution of this ODE system is quite a challenge, since there is no known technique for solving it analytically and for numerical solution the initial conditions $\lambda(t_0)$ and $\mu(t_0)$ are not provided.

However, it is clear that transversality conditions (12) will be always satisfied by any state trajectory approaching some fixed point $(x^*, k^*, \lambda^*, \mu^*)$ of the system (15). Therefore, by choosing $\lambda(t_0)$ and $\mu(t_0)$ closely enough to such fixed-point values we can manage to satisfy the transversality conditions (12) and obtain fair initial guesses $\lambda(t_0) = \lambda^*$ and $\mu(t_0) = \mu^*$ for numerical integration of the system (15).

By analyzing local stability of this augmented system, we shall see if the stability properties of bioeconomic system (4) can be preserved under the optimal choice (14). Namely, for any initial state (x_0, k_0) sufficiently close the fixed-point values of (x, k) , would it be possible to determine the initial values of multipliers (λ_0, μ_0) such that the optimal economic growth trajectory could eventually converge to the fixed point of the augmented system (15). The latter will be possible only if the Jacobian matrix of the augmented system, evaluated in the fixed point, admits at least two eigenvalues with strictly negative real part.

4. Conditions for Saddle-Point Stability

Fixed point(s) $P^* = (x^*, k^*, \lambda^*, \mu^*)$ of the system (15) are solutions of the following non-linear algebraic system:

$$\alpha x \ln\left(\frac{K}{x}\right) - \epsilon k^\gamma + \frac{\sigma^2}{2} \cdot \frac{\lambda}{\mu} = 0 \quad (16a)$$

$$k^\gamma - \frac{q}{\mu} - \frac{\sigma^2}{4} \left(\frac{\lambda}{\mu}\right)^2 = 0 \quad (16b)$$

$$\lambda \left[\rho + \alpha - \alpha \ln\left(\frac{K}{x}\right) \right] - \frac{\partial U_i}{\partial x} = 0 \quad (16c)$$

$$(\rho - \gamma k^{\gamma-1}) \mu + \epsilon \gamma k^{\gamma-1} \lambda = 0 \quad (16d)$$

The Jacobian matrix of this system is given by

$$J(x, k, \lambda, \mu) = \begin{bmatrix} H_{\lambda x} & H_{\lambda k} & H_{\lambda \lambda} & H_{\lambda \mu} \\ 0 & H_{\mu k} & H_{\mu \lambda} & H_{\mu \mu} \\ -H_{xx} & 0 & \rho - H_{x\lambda} & 0 \\ 0 & -H_{kk} & -H_{k\lambda} & \rho - H_{k\mu} \end{bmatrix} \quad (17)$$

According to [6], the characteristic polynomial of (17) can be written as

$$\det |J - zI| = z^4 - \text{Tr}|J_1|z^3 + Az^2 - Bz + \det |J| \quad (18)$$

where $\text{Tr}|J| = 2\rho$, A is the sum of all (2×2) principal minors of J and B is the sum of all (3×3) principal minors of J :

$$A = \rho^2 + \rho [H_{x\lambda} + H_{k\mu}] - H_{x\lambda}^2 - H_{k\mu}^2 + H_{xx}H_{\lambda\lambda} + H_{kk}H_{\mu\mu},$$

$$B = \rho^2 [H_{k\mu} + H_{x\lambda}] + \rho [H_{xx}H_{\lambda\lambda} + H_{kk}H_{\mu\mu} - H_{x\lambda}^2 - H_{k\mu}^2] = \rho(A - \rho^2).$$

According to [1], polynomial (18) will have two eigenvalues with strictly negative real part *if and only if* the following conditions

$$W = A - \rho^2 < 0, \quad 0 < \det J < \left(\frac{W}{2}\right)^2 \quad (19)$$

are fulfilled in the fixed point P^* . Additionally, the same paper provides an explicit formula to calculate all eigenvalues of (17) evaluated in P^* :

$$z_j = \frac{\rho}{2} \pm \sqrt{\left(\frac{\rho}{2}\right)^2 - \frac{W}{2} \pm \frac{1}{2} \sqrt{W^2 - 4 \det J}}, \quad j = 1, 2, 3, 4.$$

The latter implies that z_j are symmetric around $\frac{\rho}{2}$ and that there will be no eigenvalues with real part equal to zero (since $\rho > 0$); therefore, limit cycles are ruled out.

5. Case Studies

It was mentioned in Section 3, that different attitudes of the policy-makers can be expressed by defining an appropriate utility function. To validate our model, let us consider two situations:

(a) representative agents do not care for species conservation (utility function $U_1(x, c)$ defined by (8));

(b) representative agents care for species conservation (utility function $U_2(x, c)$ defined by (9)).

5.1. Policy that does not favor species conservation

In this case, $\frac{\partial U_i(x, c)}{\partial x} = p$ in (15) and (16). Therefore, the Jacobian matrix J_1 associated with $U_1(x, c)$ is expressed as:

$$J_1 = \begin{bmatrix} \alpha \ln\left(\frac{K}{x}\right) - \alpha & -\epsilon\gamma k^{\gamma-1} & \frac{\sigma^2}{2\mu} & -\frac{\sigma^2\lambda}{2\mu^2} \\ 0 & \gamma k^{\gamma-1} & -\frac{\sigma^2\lambda}{2\mu^2} & \frac{q}{\mu^2} + \frac{\sigma^2\lambda^2}{2\mu^3} \\ \frac{\alpha\lambda}{x} & 0 & \rho + \alpha - \alpha \ln\left(\frac{K}{x}\right) & 0 \\ 0 & \gamma(\gamma-1)k^{\gamma-2}(\epsilon\lambda - \mu) & \epsilon\gamma k^{\gamma-1} & \rho - \gamma k^{\gamma-1} \end{bmatrix}.$$

Following the idea of [2], it is observed that if the fixed-point value x^* is high enough, then $\frac{\alpha\lambda}{x}$ in the third row of J_1 is close to zero and is therefore negligible.

Remark 2. This assumption is reasonable since we are interested to find strictly positive solution $P^* = (x^*, k^*, \lambda^*, \mu^*)$ of the fixed-point system (16) which will be consistent with economical sense. Moreover, this solution must also agree with saddle-point property of the original bioeconomic system (4), that is, satisfy the condition $x^* > \frac{K}{e}$. Additionally, solvability of (16d) is guaranteed under the condition $0 < \lambda^* < \frac{\mu^*}{\epsilon}$.

Therefore, the eigenvalues of $J_1(P^*)$ will have the same sign as the eigenvalues of $\hat{J}(P^*)$ given by

$$\hat{J}(x, k, \lambda, \mu) = \begin{bmatrix} H_{\lambda x} & H_{\lambda k} & H_{\lambda\lambda} & H_{\lambda\mu} \\ 0 & H_{\mu k} & H_{\mu\lambda} & H_{\mu\mu} \\ 0 & 0 & \rho - H_{x\lambda} & 0 \\ 0 & -H_{kk} & -H_{k\lambda} & \rho - H_{k\mu} \end{bmatrix} \quad (20)$$

$$= \begin{bmatrix} \alpha \ln \left(\frac{K}{x} \right) - \alpha & -\epsilon \gamma k^{\gamma-1} & \frac{\sigma^2}{2\mu} & -\frac{\sigma^2 \lambda}{2\mu^2} \\ 0 & \gamma k^{\gamma-1} & -\frac{\sigma^2 \lambda}{2\mu^2} & \frac{q}{\mu^2} + \frac{\sigma^2 \lambda^2}{2\mu^3} \\ 0 & 0 & \rho + \alpha - \alpha \ln \left(\frac{K}{x} \right) & 0 \\ 0 & \gamma(\gamma-1)k^{\gamma-2}(\epsilon\lambda - \mu) & \epsilon \gamma k^{\gamma-1} & \rho - \gamma k^{\gamma-1} \end{bmatrix}.$$

Effectively, the characteristic polynomial of \hat{J} admits the following factorization:

$$\det \left| \hat{J} - zI \right| = Q_1(z) \cdot Q_2(z)$$

with

$$\begin{aligned} Q_1(z) &= (H_{x\lambda} - z) \cdot (\rho - H_{x\lambda} - z), \\ Q_2(z) &= z^2 - \rho z + H_{k\mu}(\rho - H_{k\mu}) - H_{kk}H_{\mu\mu}, \end{aligned}$$

where

$$\begin{aligned} H_{x\lambda} &= \alpha \ln \left(\frac{K}{x} \right) - \alpha, & H_{k\mu} &= \gamma k^{\gamma-1}, \\ H_{kk} &= \gamma(\gamma-1)k^{\gamma-2}(\epsilon\lambda - \mu), & H_{\mu\mu} &= \frac{q}{\mu^2} + \frac{\sigma^2 \lambda^2}{2\mu^3}. \end{aligned}$$

The roots of $Q_1(z)$ in the fixed point $P^* = (x^*, k^*, \lambda^*, \mu^*)$ are:

$$z_1 = \alpha \ln \left(\frac{K}{x^*} \right) - \alpha, \quad z_2 = \rho - \alpha \ln \left(\frac{K}{x^*} \right) + \alpha$$

where x^* is such that $x^* > \frac{K}{e}$ (see Section 2, Eq. (6)). The latter implies that $\ln \left(\frac{K}{x^*} \right) < 1$ and, therefore, $z_1 < 0$ while $z_2 > 0$.

On the other hand, since

$$\det \hat{J} = H_{x\lambda}(\rho - H_{x\lambda})(H_{k\mu}(\rho - H_{k\mu}) - H_{kk}H_{\mu\mu}),$$

then

$$H_{k\mu}(\rho - H_{k\mu}) - H_{kk}H_{\mu\mu} = \frac{\det \hat{J}}{H_{x\lambda}(\rho - H_{x\lambda})}.$$

In the fixed point P^* we have that $H_{x\lambda}^* = H_{x\lambda}(x^*, k^*, \lambda^*, \mu^*) = z_1 < 0$ and $\rho - H_{x\lambda}^* > 0$. Therefore,

$$H_{k\mu}^*(\rho - H_{k\mu}^*) - H_{kk}^*H_{\mu\mu}^* \Rightarrow \begin{cases} < 0 & \text{if } \det \hat{J}^* > 0 \\ > 0 & \text{if } \det \hat{J}^* < 0 \end{cases}$$

where * indicates that partial derivatives (as well as determinant of \hat{J}) are evaluated in the fixed point P^* .

By applying *Descartes' rule of signs* (see, e.g., [7]) to

$$Q_2(z) = z^2 - \rho z + \frac{\det \hat{J}^*}{H_{x\lambda}^* (\rho - H_{x\lambda}^*)}$$

it become clear that $Q_2(z)$ has in P^* exactly one negative root *if and only if* $\det \hat{J}^* > 0$ what completely agrees with condition (19), proved in [1].

Therefore, it is concluded that optimal choice (14) preserves indeed the saddle-point property of the bioeconomic system (4).

5.2. Policy that Favors Species Conservation

In this case, $\frac{\partial U_i(x, c)}{\partial x} = \frac{p}{x}$ in (15) and (16). Therefore, the Jacobian matrix J_2 associated with $U_2(x, c)$ is expressed as:

$$J_2 = \begin{bmatrix} \alpha \ln\left(\frac{K}{x}\right) - \alpha & -\epsilon \gamma k^{\gamma-1} & \frac{\sigma^2}{2\mu} & -\frac{\sigma^2 \lambda}{2\mu^2} \\ 0 & \gamma k^{\gamma-1} & -\frac{\sigma^2 \lambda}{2\mu^2} & \frac{q}{\mu^2} + \frac{\sigma^2 \lambda^2}{2\mu^3} \\ \frac{p}{x^2} + \frac{\alpha \lambda}{x} & 0 & \rho + \alpha - \alpha \ln\left(\frac{K}{x}\right) & 0 \\ 0 & \gamma(\gamma - 1)k^{\gamma-2}(\epsilon \lambda - \mu) & \epsilon \gamma k^{\gamma-1} & \rho - \gamma k^{\gamma-1} \end{bmatrix}.$$

Using the idea of [2] together with justification provided in Remark 2, it is observed that if the fixed-point value x^* is high enough, then we have

$$\frac{p}{x^2} + \frac{\alpha \lambda}{x} \approx 0$$

in the third row of J_2 and this term can be considered negligible. Therefore, the eigenvalues of $J_2(P^*)$ will have the same sign as the eigenvalues of $\hat{J}(P^*)$ given by (20), that is, two negative and two positive eigenvalues.

Thus, we can conclude that in case $U(x, c) = U_2(x, c)$ the optimal choice (14) also preserves the saddle-point property of the bioeconomic system (4).

6. Discussion and Numerical Results

Two parameters $\alpha = 0.0353, K = 114.196$ of Gompertz model (1) were estimated using CEPES¹ observation data of *Dissostichus eleginoides* (Patagonian

¹Centro de Estudios Pesqueros, Santiago, Chile, <http://www.cepes.cl>

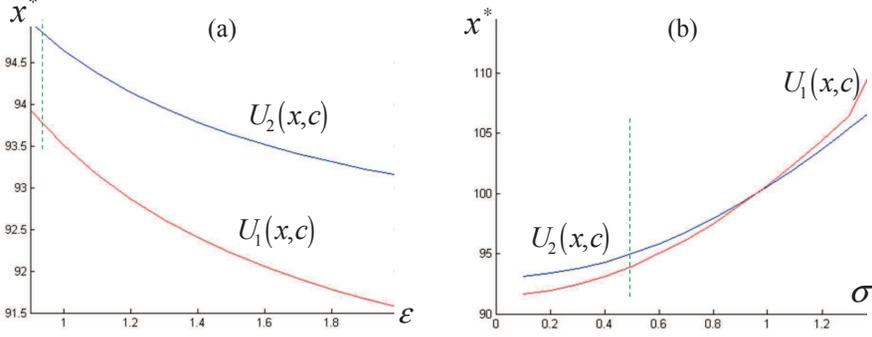


Figure 1: (a) fixed point x^* dependence on ϵ ; (b) fixed point x^* dependence on σ

toothfish or Chilean sea bass) during 1988-2006. For the dynamics (15), we use the following additional parameters specifications:

$$\gamma = \frac{1}{2}, \quad \rho = 0.05, \quad \epsilon = 0.9, \quad \sigma = \frac{1}{2}, \quad (p, q) = \begin{cases} (10, 100) & \text{for } U_1(x, c) \\ (1000, 100) & \text{for } U_2(x, c) \end{cases} \quad (21)$$

Weight parameters (p, q) are chosen differently for $U_1(x, c)$ and $U_2(x, c)$ in order to emphasize the priorities of decision-making. Namely, $U_1(x, c)$ expresses that consumption is more important than species abundance, whereas $U_2(x, c)$ implies that species conservation prevails over consumption. The coordinates of the corresponding fixed points x_1^* and x_2^* (for $U_2(x, c)$ and $U_1(x, c)$, respectively) turn out to be

$$0 < \frac{K}{e} = 42.0103 < x_1^* = 93.93439, \quad x_2^* = 94.98636 < K$$

while the eigenvalues are:

$$U_1(x, c) : \begin{cases} z_{1,2} = 0.025 \pm 1.60882 \\ z_{3,4} = 0.025 \pm 0.0525 \end{cases} \Rightarrow \hat{J}_1^* = 0.0055116 > 0$$

$$U_2(x, c) : \begin{cases} z_{1,2} = 0.025 \pm 0.60655 \\ z_{3,4} = 0.025 \pm 0.0538 \end{cases} \Rightarrow \hat{J}_2^* = 0.0008334 > 0$$

From ecological point of view, it is clear that fixed-point values x_1^* and x_2^* depend on actual level of negative effects caused by aggregated production (characterized by parameter $\epsilon > 0$), as well as on intensity of defensive expenditures (expressed by parameter $\sigma > 0$).

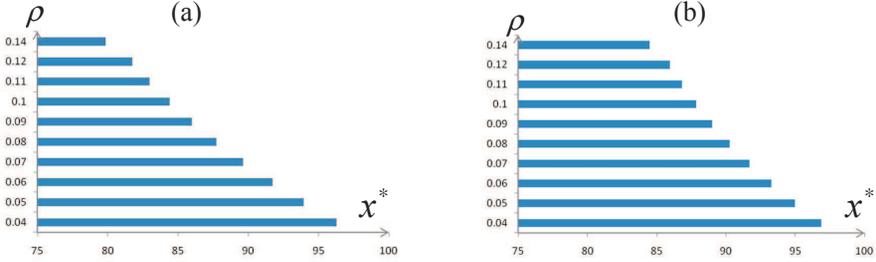


Figure 2: Impact of changes in ρ on the fixed point x^* in case of $U_1(x, c)$ and $U_2(x, c)$, respectively

In order to see what impact may have a change in one of these primary ecological parameters (ϵ, σ) upon the fixed-point values of x_1^* and x_2^* , we perform some numerical exercises by varying (ceteris paribus) each of them. The results are consistent with a priori expectations and appear in Figure 1, where vertical dashed line indicates the reference values (21).

Figure 1(a) shows that an increase in the environmental impact of aggregated production (ϵk^γ), in fact reduces x in the fixed point, whereas an increase in defensive expenditures ($\sigma\sqrt{d}$) stimulates growth of x in the fixed point (Figure 1(b)). It is interesting to note that if agents do not care for species conservation (i.e., utility function $U_1(x, c)$ is used) and, nonetheless, invest considerably in the defensive expenditures ($\sigma > 1$), the effect of defensive measures will be more significant.

From economical point of view, there is another vital parameter ρ (intertemporal discount rate) that expresses how patient the decision-makers are when it comes to obtaining the maximum overall utility. Thus, higher discount rates ρ imply less patience, while lower ρ make future utility values closer to its present value ($e^{-\rho t}U(x, c) \approx U(x, c)$ when $\rho \rightarrow 0$). The impact of changes in ρ on the fixed-point value of x is presented using bar diagrams for $U_1(x, c)$ (Figure 2(a)) and $U_2(x, c)$ (Figure 2(b)). Both figures show for each value of $\rho \in [0.04, 0.14]$ the corresponding fixed-point values of x^* decrease when ρ increases. However, for the same value of ρ , the fixed-point values x_2^* (corresponding to $U_2(x, c)$) always remain higher than x_1^* . The latter is explainable due to the weight distribution within each utility function.

From biological point of view, the species abundance (and, therefore, fixed-point values of x) is related to biological rate of the species reproduction (expressed by α within the model). For species with lower reproduction rate α ,

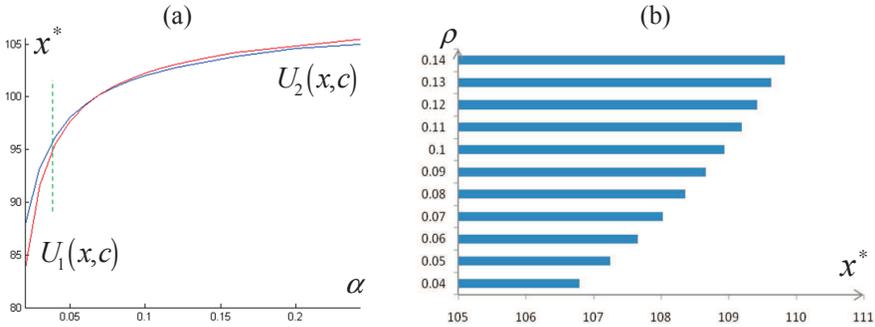


Figure 3: (a) fixed point x^* dependence on α ; (b) impact of changes in ρ on the fixed point x^* in case of $U_2(x, c)$ and $\alpha = 0.5$

the fixed point value of x will be well below the carrying capacity parameter K . An increase in α (caused by improvement of outer factors such as better climatic conditions, lower mortality rate due to natural disasters or epidemics, among others) may actually provoke an outburst of species population forcing x^* to move toward K (see Figure 3(a)). The latter is valid for both utility functions $U_1(x, c)$ and $U_2(x, c)$. On the other hand, for species with rather high reproduction rate ($\alpha = 0.5$, for example, or even more) an interesting phenomenon takes place in the case of $U_2(x, c)$. Actually, fast renovation of species population combined with increase of intertemporal discount rate ρ force x^* to move towards the carrying capacity parameter K (see Figure 3(b)). The explanation of this fact lays in the way the total utility is being composed. Since $U_2(x, c) = p \ln x + q \ln c$ and $p > q$, then the main part of overall utility will be achieved on account of x (rather than c) thanks to high reproduction rate of the species.

Acknowledgements

This research has been supported by Universidad del Valle, Cali, Colombia by way of Research Project CI-7807. First author acknowledges a three-month internship from *Centro de Modelamiento Matemático* (CMM, Universidad de Chile), supervised by Dr. Pedro Gajardo within the frameworks of REALMA Network (*Réseau Europe - Amérique Latine en Mathématiques Appliquées*, <http://www.realma.org/>).

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