SOME PROPERTIES OF IDEAL \((\alpha)\)-CONVERGENCE
IN \((\ell)\)-GROUPS

A. Boccuto\(^1\)\(^\S\), X. Dimitriou\(^2\)

\(^1\)Dipartimento di Matematica e Informatica
Via Vanvitelli, 1, I-06123, Perugia, ITALY

\(^2\)Department of Mathematics
University of Athens
Panepistimiopolis, Athens, 15784, GREECE

Abstract: We study some relations between \((\alpha)\)-convergence and ideal \((\alpha)\)-convergence for \((\ell)\)-group-valued sequences of functions. We give some conditions for continuity of the limit function and pose some open problems.

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1. Introduction

The concept of \((\alpha)\)-convergence or continuous convergence of real-valued function sequences has been known in the literature since the beginning of last century (see [4, 6, 8]). This notion was recently further studied in [1, 3, 9]. In [1] the notion of ideal \((\alpha)\)-convergence was introduced and some of its fundamental properties were established. In [3] a similar investigation was carried out within the class of \((\ell)\)-groups paying special attention to the powerful notion of ideal exhaustiveness. In this paper we continue the investigation of this kind in the context of \((\ell)\)-groups and prove mainly some results about continuity of the limit functions and relations among \((\alpha)\)-convergence and ideal \((\alpha)\)-convergence. Finally, we pose some open problems.
2. Preliminaries

We now recall the basic concepts, which will be useful in the sequel.

**Definitions 2.1.** (a) An \((\ell)\)-group is said to be *Dedekind complete* iff every subset \(R_1 \subset R, R_1 \neq \emptyset\) bounded from above has supremum in \(R\).

(b) A bounded double sequence \((a_{i,j})_{i,j}\) in \(R\) is called \((D)\)-sequence or regulator iff for all \(i, j \in \mathbb{N}\) we have \(a_{i,j} \geq a_{i,j+1}\) and \(\wedge_j a_{i,j} = 0\) for all \(i \in \mathbb{N}\). A sequence \((x_n)_n\) in \(R\) is said to be \((D)\)-convergent to \(x \in R\) (and we write \((D)\lim_n x_n = x\)) iff there exists a \((D)\)-sequence \((a_{i,j})_{i,j}\) in \(R\), such that to every \(\varphi \in \mathbb{NN}\)

\[\text{iff there exists a regulator } (ai,j)_{i,j} \text{ such that to every } \varphi \in \mathbb{NN}, \text{ there exists a regulator } (ai,j)_{i,j} \text{ such that to every } \varphi \in \mathbb{NN}.\]

\[\varphi \in \mathbb{NN} \text{ there corresponds an } n_0 \in \mathbb{N} \text{ such that } |x_n - x| \leq \sqrt[n_0]{a_{i,\varphi(i)}} \text{ for all } n \in \mathbb{N}, n \geq n_0.\]

(c) A family of sets \(\mathcal{I} \subset \mathcal{P}(\mathbb{N})\), where \(\mathbb{N}\) is the set of the natural numbers, is called an *ideal* of \(\mathbb{N}\) iff \(A \cup B \in \mathcal{I}\) whenever \(A, B \in \mathcal{I}\) and for each \(A \in \mathcal{I}\) and \(B \subset A\) we get \(B \in \mathcal{I}\). An ideal is said to be *non-trivial* iff \(\mathcal{I} \neq \emptyset\) and \(\mathbb{N} \notin \mathcal{I}\). A non-trivial ideal \(\mathcal{I}\) is said to be *admissible* iff it contains all singletons.

(d) Let \(\mathcal{I}\) be an admissible ideal of \(\mathbb{N}\). A sequence \((x_n)_n\) in \(R\) \((D\mathcal{I})\)-converges to \(x \in R\) iff there is a \((D\mathcal{I})\)-sequence \((a_{i,j})_{i,j}\) such that

\[\{n \in \mathbb{N} : |x_n - x| \leq \sqrt[|n_0|]{a_{i,\varphi(i)}} \} \in \mathcal{I}\]

for all \(\varphi \in \mathbb{NN}\). Note that \((D)\)-convergence implies \((D\mathcal{I})\)-convergence, while the converse is in general not true (see also [3]).

(e) Let \((X, d)\) be a metric space. A sequence \((x_n)_n\) in \(X\) is said to be \(\mathcal{I}\)-convergent to \(x \in X\) (and we write \(x = (\mathcal{I})\lim_n x_n\)) iff for every \(\varepsilon > 0\) we get \(\{n \in \mathbb{N} : d(x_n, x) > \varepsilon\} \in \mathcal{I}\).

(f) We say that \((f_n)_n\) \((\mathcal{I}\alpha)\)-converges to \(f : X \to R\) iff for every \(x \in X\) there exists a regulator \((a_{i,j})_{i,j}\) such that for each sequence \((x_n)_n\) in \(X\) with \((\mathcal{I})\lim_n x_n = x\) we get \((\mathcal{D}\mathcal{I})\lim_n f_n(x_n) = f(x)\) with respect to the regulator \((a_{i,j})_{i,j}\). We say that \((f_n)_n\) \((\alpha)\)-converges to \(f : X \to R\) iff it \((\mathcal{I}\text{fin}\alpha)\)-converges to \(f\).

(g) The sequence \((f_n)_n\) is said to be *globally \((\mathcal{I}\alpha)\)-convergent to \(f : X \to R\) iff a \((D)\)-sequence \((a_{i,j})_{i,j}\) can be found, such that for any \(x \in X\) and for each sequence \((x_n)_n\) in \(X\) with \((\mathcal{I})\lim_n x_n = x\) we have \((\mathcal{D}\mathcal{I})\lim_n f_n(x_n) = f(x)\) with respect to the \((D)\)-sequence \((a_{i,j})_{i,j}\).

(h) A function sequence \(f_n : X \to R, n \in \mathbb{N}\), is called \(\mathcal{I}\)-exhaustive at \(x \in X\) iff there exists a regulator \((a_{i,j})_{i,j}\) (depending on \(x\)) such that to every \(\varphi \in \mathbb{NN}\)
there correspond a $\delta > 0$ and an element $A \in \mathcal{I}$ (depending on $\varphi$ and $x$) such that

$$|f_n(z) - f_n(x)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

whenever $d(z, x) < \delta$ and $n \in \mathbb{N} \setminus A$. We say that $(f_n)_n$ is exhaustive iff it is $\mathcal{I}_{\text{fin}}$-exhaustive.

(i) Let $f : X \to R$ be a function and $x \in X$. Then $f$ is said to be continuous at $x$ iff there exists a regulator $(a_{i,j})_{i,j}$ (depending on $x$) such that to every $\varphi \in \mathbb{N}^\mathbb{N}$ there corresponds a $\delta > 0$ (depending on $\varphi$ and $x$) such that $|f(x) - f(z)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $d(x, z) < \delta$. A function $f : X \to R$ is continuous on $X$ iff $f$ is continuous at every point $x \in X$.

(j) A function $f : X \to R$ is called globally continuous on $X$ iff there exists a regulator $(a_{i,j})_{i,j}$ such that for every $\varphi \in \mathbb{N}^\mathbb{N}$ and $x \in X$ there is a $\delta > 0$ (depending on $\varphi$ and $x$) with $|f(x) - f(z)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$ whenever $d(x, z) < \delta$.

### 3. The Main Results

From now on let $R$ be a Dedekind complete $(\ell)$-group and $\mathcal{I}$ be an admissible ideal of $\mathbb{N}$. We prove that, for $R$-valued functions, $(\alpha)$-convergence implies $(\mathcal{I}_\alpha)$-convergence. We extend [9, Theorem 1] to the context of $(\ell)$-groups.

**Theorem 3.1.** If $f_n : X \to R$, $n \in \mathbb{N}$, $(\alpha)$-converges to $f : X \to R$, then $(f_n)_n$ $(\mathcal{I}_\alpha)$-converges to $f$ too.

**Proof.** Let $(f_n)_n$ be $(\alpha)$-convergent to $f$. By [3, Theorem 3.3], we get that $(f_n)_n$ is pointwise $(D)$-convergent to $f$ and exhaustive. Since $\mathcal{I}$ is admissible, this implies that $(f_n)_n$ is pointwise $(\mathcal{D}\mathcal{I})$-convergent (see also [2]) and $\mathcal{I}$-exhaustive. Again by [3, Theorem 3.3], we obtain that $(f_n)_n$ $(\mathcal{I}_\alpha)$-converges to $f$, that is the thesis. □

**Example 3.2.** Note that, even for real-valued functions, when $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, $(\mathcal{I}_\alpha)$-convergence does not imply $(\alpha)$-convergence, as the following example shows (see also [9]).

Let $Y$ be a set with at least two distinct elements $y_1$ and $y_2$ and $H \in \mathcal{I}$ be an infinite set. Since $\mathcal{I} \neq \mathcal{I}_{\text{fin}}$, then $H$ does exist. Set $f_n(x) = y_1$ for all $x \in X$ and $n \in \mathbb{N} \setminus H$, and $f_n(x) = y_2$ for every $x \in X$ and $n \in H$. Put $f(x) = y_1$
for each $x \in X$. For every sequence $(x_n)_n$ in $X$ with $x_0 = (I) \lim_n x_n$ we get $(I) \lim_n f_n(x_n) = y_1 = f(x_0)$, but $\lim_n f_n(x_n)$ does not exist in the usual sense. Thus the sequence $(f_n)_n$ $(I\alpha)$-converges to $f$, but not $(\alpha)$-converges to $f$.

Analogously as Theorem 3.1, using [3, Theorem 3.4], it is possible to prove the following:

**Theorem 3.3.** If $f_n : X \to R$, $n \in \mathbb{N}$, is globally $(\alpha)$-convergent to $f : X \to R$, then $(f_n)_n$ is globally $(I\alpha)$-convergent to $f$ too.

We now prove the following technical result, which will be useful in the sequel.

**Theorem 3.4.** Let $x_0 \in X$ and $(z_k)_k$ be a sequence of points of $X$ such that $\lim_k z_k = x_0$ in the ordinary sense. Let $g_n : X \to R$, $n \in \mathbb{N}$, be such that $(D\mathcal{I}) \lim_n g_n(x_n) = g(x_0)$ for every sequence $(x_n)_n$ in $X$, $\mathcal{I}$-converging to $x_0$ (with respect to a regulator, depending on $x_0$ but independent of $(x_n)_n$) and $(D\mathcal{I}) \lim_n g_n(z_k) = g(z_k)$ for all $k \in \mathbb{N}$ with respect to a common regulator. Then $(D) \lim_k g(z_k) = g(x_0)$.

**Proof.** Let $(a_{i,j})_{i,j}$ be a $(D)$-sequence associated to $(D\mathcal{I})$-convergence of $(g_n(x_n))_n$ to $g(x_0)$ and $(b_{i,j})_{i,j}$ be a regulator related with $(D\mathcal{I})$-convergence of $(g_n(z_k))_n$ to $g(z_k)$, $k \in \mathbb{N}$. Set $c_{i,j} = 2(a_{i,j} + b_{i,j})$, $i, j \in \mathbb{N}$: it is not difficult to check that $(c_{i,j})_{i,j}$ is a regulator too.

We now prove that the sequence $(g(z_k))_k$ $(D)$-converges to $g(x_0)$ with respect to the regulator $(c_{i,j})_{i,j}$. Otherwise there exist a $\varphi \in \mathbb{N}^\mathbb{N}$ and a subsequence $(z_{q_k})_k$ of $(z_k)_k$ with

$$
|g(z_{q_k}) - g(x_0)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)}.
$$

(1)

Without loss of generality, we can suppose $q_k = k$ for all $k \in \mathbb{N}$. We claim that a subsequence $(x_n)_n$ of $(z_k)_k$ can be constructed, such that $(\mathcal{I}) \lim_n x_n = x_0$, but $(g_n(x_n))_n$ does not $(D\mathcal{I})$-converge to $g(x_0)$ with respect to the regulator $(a_{i,j})_{i,j}$, getting a contradiction and so proving the theorem. We firstly consider the case

$$
\bigcup_{k \in \mathcal{I}} B_k \in \mathcal{I},
$$

where for every $k \in \mathbb{N}$, $B_k = \left\{ n \in \mathbb{N} : |g_n(z_k) - g(z_k)| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)} \right\}$, and $\varphi$ is the one of the relation (1). Note that $B_k \in \mathcal{I}$ for each $k$ by hypothesis and, if $n \notin B_k$, then $|g_n(z_k) - g(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$: otherwise we have

$$
|g(z_k) - g(x_0)| \leq |g_n(z_k) - g(x_0)| + |g_n(z_k) - g(z_k)| \leq \bigvee_{i=1}^{\infty} c_{i,\varphi(i)},
$$

(2)
which contradicts (1). Hence \(|g_n(z_n) - g(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}(n)\) for all \(n \notin \bigcup_k B_k\).

Set now \(x_n = z_n, n \in \mathbb{N}\). By hypothesis, \(\lim_n x_n = x_0\), and thus we get also (\(I\)) \(\lim_n x_n = x_0\) (see [7]). Thus the sequence \((g_n(x_n))_n\) does not (\(DI\))-converge to \(g(x_0)\) with respect to the regulator \((a_{i,j})_{i,j}\), and hence we get the claim, at least when \(\bigcup_k B_k \subseteq \mathcal{I}\).

If \(\bigcup_k B_k \not\subseteq \mathcal{I}\), then, proceeding analogously as in [9, Lemma 1] and by setting \(B_0 = \emptyset\), there is a strictly increasing sequence \((k_m)_m \in \mathbb{N}\) with \(C_m = B_{k_m} \setminus \bigcup_{i=1}^{k_m-1} B_i \neq \emptyset\). Note that \(C_m \in \mathcal{I}\) for all \(m \in \mathbb{N}\), the \(C_m\)'s are pairwise disjoint and \(\bigcup_m C_m = \bigcup_k B_k \not\subseteq \mathcal{I}\). Let us define \(x_n = z_{k_m-1}\) for all \(n \in C_m, m \in \mathbb{N}\), and \(x_n = x_0\) for \(n \notin \bigcup C_m\). Since the \(C_m\)'s are pairwise disjoint, this construction makes sense. If \(m \in \mathbb{N}\) and \(n \in C_m\), then \(n \notin B_{k_{m-1}}\), and so we get
\[
|g_n(x_n) - g(x_0)| = |g_n(z_{k_m-1}) - g(z_{k_m-1})| \leq \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}. 
\]
From this, arguing analogously as in the first case, we deduce that
\[
|g_n(x_n) - g(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}.
\]
Thus \(|g_n(x_n) - g(x_0)| \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}(n)\) for all \(n \in \bigcup C_m\), and hence the sequence \((g_n(x_n))_n\) does not \(\mathcal{I}\)-converge to \(g(x_0)\) with respect to the regulator \((a_{i,j})_{i,j}\).

If we prove that (\(I\)) \(\lim_n x_n = x_0\), then we get the requested. Fix arbitrarily \(\varepsilon > 0\). If \(n \in \mathbb{N}\) is such that \(d(x_0, x_n) > \varepsilon\), then obviously \(x_n \neq x_0\), and by construction there exists a unique \(m = m(n) \in \mathbb{N}\) such that \(n \in C_m(n)\) and hence \(x_n = z_{k_m(n)-1}\). This may happen only when \(z_{k_m(n)-1} \neq x_0\). In any case, we get
\[
\{n \in \mathbb{N} : d(x_0, x_n) > \varepsilon\} \subset \{n \in \mathbb{N} : d(x_0, z_{k_m(n)-1}) > \varepsilon\}. \tag{3}
\]
Note that, since \(\lim_n z_n = x_0\) in the usual sense, then \(\lim_m z_{k_m} = x_0\) in the ordinary sense too, and hence \(d(x_0, z_{k_m-1}) > \varepsilon\) only for finitely many \(m\)'s. From this and (3) it follows that the set \(\{n \in \mathbb{N} : d(x_0, x_n) > \varepsilon\}\) is contained in a finite union of \(C_m\)'s and so belongs to \(\mathcal{I}\). Hence, (\(I\)) \(\lim_n x_n = x_0\). This ends the proof of the theorem. \(\square\)

We now give some conditions for continuity of the limit function.
Corollary 3.5. If \( f_n : X \to R, n \in \mathbb{N} \) is pointwise \((DI)\)-convergent with respect to a common regulator and \((I\alpha)\)-convergent to \( f : X \to R \), then \( f \) is continuous.

Proof. Choose arbitrarily \( x_0 \in X \). We first claim that, if \((z_k)_k\) is a sequence in \( X \), convergent to \( x_0 \) in the usual sense, then we have \((D)\lim_k f(z_k) = f(x_0)\) with respect to a regulator \((c_{i,j})_{i,j}\), independent of the choice of \((z_k)_k\).

Let \((z_k)_k\) be as above. By \((I\alpha)\)-convergence of \((f_n)_n\) to \( f \), we have

\[
(DI)\lim_n f_n(z_k) = f(x_0)
\]

with respect to a regulator \((a_{i,j})_{i,j}\), depending on \( x_0 \) and independent of the choice of \((z_k)_k\). Let \((b_{i,j})_{i,j}\) be a regulator associated to \((DI)\)-pointwise convergence of \((f_n)_n\) to \( f \): by hypothesis, \((b_{i,j})_{i,j}\) does not depend on the chosen sequence \((z_k)_k\). Thus, the hypotheses of Theorem 3.4 are fulfilled, and so, if \( c_{i,j} = 2(a_{i,j} + b_{i,j}), i, j \in \mathbb{N} \), then \((c_{i,j})_{i,j}\) is the requested regulator.

We now prove that \( f \) is continuous at \( x_0 \) with respect to the \((D)\)-sequence \((c_{i,j})_{i,j}\). Otherwise there exists \( \varphi \in \mathbb{N}^\mathbb{N} \) such that to every \( k \in \mathbb{N} \) there is \( z_k \in X \) with \( d(z_k, x_0) \leq 1/k \) but \( |f(z_k) - f(x_0)| \leq \bigvee_{i=1}^\infty c_{i,\varphi(i)} \) for all \( k \in \mathbb{N} \). So \((z_k)_k\) converges to \( x_0 \) in the usual sense, but \((f(z_k))_k\) does not converge to \( f(x_0) \) with respect to \((c_{i,j})_{i,j}\), a contradiction. Thus we get continuity of \( f \), by arbitrariness of \( x_0 \in X \). \( \square \)

Similarly as Corollary 3.5, it is possible to prove the following

Corollary 3.6. If \( f_n : X \to R, n \in \mathbb{N} \) is globally \((I\alpha)\)-convergent to \( f : X \to R \), then \( f \) is globally continuous.

Open problems: (a) Find conditions under which \((I\alpha)\)-convergence implies \((\alpha)\)-convergence, when \( I \neq I_{\text{fin}} \).

(b) Find more general sufficient conditions to obtain continuity of the limit function.

References


