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ON 2-GRAPHOIDAL COVERING NUMBER OF A GRAPH

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Abstract: A 2-graphoidal cover of a graph G is a collection ψ of paths (not necessarily open) in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most two paths in ψ and every edge of G is in exactly one path in ψ. The minimum cardinality of a 2-graphoidal cover of G is called the 2-graphoidal covering number of G and is denoted by $\eta_2(G)$ or η_2. Here, we study 2-graphoidal covering number for some classes of graphs.

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1. Introduction

A graph is a pair $G = (V, E)$, where V is the set of vertices and E is the set of edges. Here, we consider only nontrivial, finite, connected and simple graphs. The order and size of G are denoted by p and q respectively. Suppose $P = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n)$ is a path in G. Then the vertices $v_1, v_2, \ldots, v_{n-1}$ are called internal vertices of P and the vertices v_0, v_n are called external vertices of P. If $P = (v_0, v_1, v_2, \ldots, v_{n-1}, v_n = v_0)$ is a closed path, then v_0 may be taken as the external vertex. Let ψ be a collection of internally edge disjoint paths in G. A vertex of G is said to be an internal vertex of ψ if it is an internal vertex of some path(s) in ψ, otherwise it is called an external vertex of ψ. The number of internal vertices of a path P in ψ is denoted by $i_\psi(P)$; the number of internal vertices which appear exactly once in a path of ψ by $t_1(\psi)$ and $max\ t_1(\psi) = t_1$; the number of internal vertices which appear exactly twice in two paths of ψ

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by $t_2(\psi)$ and $max\ t_2(\psi) = t_2$ and the number of external vertices by t_ψ and $min\ t_\psi = t$. The concept of graphoidal cover was introduced by Acharya and Sampathkumar [1]. The reader may refer [2] and [3] for the terms not defined here.

2. Main Results

Definition 1. A 2-graphoidal cover of a graph G is a collection ψ of paths (not necessarily open) in G such that every path in ψ has at least two vertices, every vertex of G is an internal vertex of at most two paths in ψ and every edge of G is in exactly one path in ψ.

The minimum cardinality of a 2-graphoidal cover of G is called the 2-graphoidal covering number of G and is denoted by $\eta_2(G)$ or η_2.

As a particular case of this, a notation called 2-graphoidal path cover (acyclic 2-graphoidal cover) has been studied in [4].

Remark 2. We denote the 2-graphoidal covering number by η_2 and 2-graphoidal path covering number (acyclic 2-graphoidal covering number) by η_{2a}. Clearly, $\eta_2(G) \leq \eta_{2a}(G)$.

Theorem 3. (see [4]) *For any 2-graphoidal cover ψ of a (p,q)- graph G,* $\eta_2(G) = q - p - t_2 + t$.

Corollary 4. *For any graph G, $\eta_2(G) \geq q - p - t_2 + t$. Moreover, the following are equivalent:*

(i) $\eta_2(G) = q - 2p$.

(ii) There exists a 2-graphoidal cover in which every vertex is an internal vertex of exactly two paths in ψ.

(iii) There exists a set Q of edge disjoint 2-graphoidal cycle or path.(From such a set Q of paths, the required 2-graphoidal cover can be obtained by adding the edges which are not covered by the paths in Q).

Corollary 5. (see [4]) *Let G be any $(p, q)-$ graph such that $\eta_2(G) = q-2p$. Then $\delta \geq 4$ and $\Delta \geq 5$.*

Remark 6. If $\Delta \leq 3$, then $t_2 = 0$ and hence $\eta_2(G) = \eta(G)$, where η is the minimum graphoidal covering number.

Theorem 7. (see [2]) *If G is a graph with $\delta \geq 3$, then there exists a graphoidal cover ψ of G such that every vertex of G is an internal vertex of some paths in ψ.*

Theorem 8. *If G is a graph with $\delta \geq 5$, then there exists a 2-graphoidal cover ψ of G such that every vertex of G is an internal vertex of exactly two paths in ψ.*

Proof. Let $P = (v_1, v_2, v_3, \ldots, v_{n-1}, v_n)$ be a longest path in G such that all the vertices adjacent to v_1 or v_n are already in P. Now, $G - P = G'$ is a subgraph with $\delta \geq 3$. Let us choose a longest path $P' = (u_1, u_2, u_3, \ldots, u_{r-1}, u_r),\ r \leq n$ in G' such that the external vertices of P and P' are distinct and all the vertices adjacent to u_1 or u_r are already in P'. Since $\delta \geq 3$ in G', we can find vertices a, b, c and d in P' such that a, b are distinct vertices each different from u_2 and adjacent to u_1 and c, d are distinct vertices each different from u_{r-1} and adjacent to u_r. If a, b, c, d, u_1 and u_r are all distinct, then we can find $Q = (a, u_1, b)$ and $R = (c, u_r, d)$. If one of a and b coincides with u_r or one of c and d coincides with u_1, say $a = u_r$ or $d = u_1$ as the case may be, take $S = (b, u_1, u_r, c)$. Thus, $u_1, u_2, \ldots, u_r$ are all made internal vertices of one of the paths P', Q, R, S such that each of $\{P', Q, R\}$ and $\{P', S\}$ are mutually edge disjoint in G'. If these vertices exhaust all the vertices in G', then every vertex of G' is an internal vertex of exactly one path. Otherwise, let u be a vertex not in P' and Q' be a longest path of length $\leq r$ in G' containing u and internally disjoint from the paths already considered. If the ends of Q' are not in P we can make them internal vertices of some path as before. We continue this process until all the vertices of G' are exhausted. Also, we can make every vertex of P internal to exactly one path as constructed in G'. If P contains all the vertices of G, then the proof is complete. Otherwise, continue the same process in G as in G'. Hence we find a collection ψ of paths which is a 2-graphoidal cover of G. □

Corollary 9. *If G is graph with $\delta(G) \geq 5$, then $\eta_2(G) = q - 2p$.*

Proof. We know that

$$q = \sum_{P \in \psi} |E(P)| \qquad \Rightarrow \qquad |\psi| = q - p - t_2 + t$$

Since $\delta \geq 5$, every vertex is an internal vertex of exactly two paths in ψ. Hence, $t_2 = p,\ t = 0$ and so $|\psi| = \eta_2 = q - 2p$. □

Corollary 10. *For any graph G with $\delta \geq 5$, $\eta_2(G) = \eta_{2a}(G) = q - 2p$.*

Theorem 11. *Let G be a complete graph K_p. Then*

$$\eta_2(K_p) = \begin{cases} 1, & \text{if } p = 3; \\ 2, & \text{if } p = 4, 5; \\ q - 2p, & \text{if } p \geq 6. \end{cases}$$

Proof. If $p = 3, 4$ then $\Delta \leq 3$ so $t_2 = 0$. Hence, $\eta_2(K_p) = \eta(K_p)$.
If $p = 5$, then we can find two edge disjoint Hamiltonian cycles in G. Hence, $\eta_2(K_5) = q - p - t_2 + t = 10 - 5 - 4 + 1 = 2$.
If $p \geq 6$, then the result follows from Corollary 9. □

Corollary 12. $\eta_2(K_p) = \eta_{2a}(K_p)$ *if and only if* $p \geq 6$.

Definition 13. A graph is of Fan-type $2k$ if the distance $dist_G(u, v) = 2$ implies that $\max\{\deg u, \deg v\} \geq \frac{p}{2} + 2k$.

Theorem 14. (see [8]) *Let G be a 4-connected graph of order p that is Fan-type 2. Then G contains two edge disjoint hamiltonian cycles.*

Theorem 15. *Let G be a k-regular graph. Then*

$$\eta_2(G) = \begin{cases} 1, & \text{if } k = 1, 2; \\ q - p, & \text{if } k = 3; \\ 2, & \text{if } k = 4 \text{ and Fan-type 2;} \\ q - 2p, & \text{if } k \geq 5. \end{cases}$$

Proof. If $k \leq 3$, then $\Delta \leq 3$ so $t_2 = 0$. Hence, $\eta_2(G) = \eta(G)$.
If $k = 4$, then from Theorem 14, we get $\eta_2(G) = 2$.
If $k \geq 5$, then the result follows from Corollary 9. □

Theorem 16. (see [4]) *Let G be a wheel $W_p = K_1 + C_{p-1}$. Then*

$$\eta_2(W_p) = \eta_{2a}(W_p) = \begin{cases} 2, & \text{if } p = 4, 5; \\ p - 3, & \text{if } p \geq 6. \end{cases}$$

Theorem 17. *Let G be a complete bipartite graph $K_{2,n}$. Then*

$$\eta_2(K_{2,n}) = \begin{cases} n - 1, & \text{if } n = 2, 3; \\ n - 2, & \text{if } n = 4, 5; \\ n - 3, & \text{if } n = 6, 7; \\ n - 4, & \text{if } n \geq 8. \end{cases}$$

Proof. Let $X = \{v_1, v_2\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of $K_{2,n}$.
When $2 \leq n \leq 5$, $t_2 = 0$ and so $\eta_2(K_{2,n}) = \eta(K_{2,n})$.
When $n = 6$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1), P_2 = (v_1, w_3, v_2, w_4, v_1)$, $P_3 = (v_2, w_5, v_1, w_6, v_2)$. Then $\psi = \{P_1, P_2, P_3\}$ is a 2-graphoidal cover of $K_{2,6}$ with $t_2 = 1$. Hence, $\eta_2(K_{2,6}) = q - p - 1 = 3$.

When $n = 7$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1), P_2 = (v_1, w_3, v_2, w_4, v_1)$, $P_3 = (v_2, w_5, v_1, w_6, v_2)$, $P_4 = (v_1, w_7, v_2)$. Then $\psi = \{P_1, P_2, P_3, P_4\}$ is a 2-graphoidal cover of $K_{2,7}$ with $t_2 = 1$. Hence, $\eta_2(K_{2,7}) = q - p - 1 = 4$.

When $n \geq 8$. Let $P_1 = (v_1, w_1, v_2, w_2, v_1), P_2 = (v_1, w_3, v_2, w_4, v_1)$, $P_3 = (v_2, w_5, v_1, w_6, v_2)$, $P_4 = \{v_2, w_7, v_1, w_8, v_2\}, Q_i = (v_1, w_{i+8}, v_2)$, $i = 1, 2, \ldots, n-8$. Then $\psi = \{P_1, P_2, P_3, P_4, Q_1, Q_2, \ldots, Q_{n-8}\}$ is a 2-graphoidal cover of $K_{2,n}$ with $t_2 = 2$. Hence, $\eta_2(K_{2,n}) = q - p - 2 = n - 4$. □

Corollary 18. *$\eta_2(K_{2,n}) = \eta_{2a}(K_{2,n})$ if $n = 3, 4, 5$.*

Theorem 19. (see [4]) *Let G be a complete bipartite graph $K_{3,n}$. Then*

$$\eta_2(K_{3,n}) = \eta_{2a}(K_{3,n}) = \begin{cases} 3, & \text{if } n = 3; \\ 2n - 6, & \text{if } n \geq 4. \end{cases}$$

Theorem 20. *Let G be a complete bipartite graph $K_{4,n}$. Then*

$$\eta_2(K_{4,n}) = \begin{cases} 2, & \text{if } n = 4; \\ 3, & \text{if } n = 5; \\ 2n - 8, & \text{if } n \geq 6. \end{cases}$$

Proof. Let $X = \{v_1, v_2, v_3, v_4\}$ and $Y = \{w_1, w_2, \ldots, w_n\}$ be a bipartition of $K_{4,n}$.

When $n = 4$. Let

$$P_1 = (v_1, w_1, v_2, w_2, v_3, w_3, v_4, w_4, v_1),$$

$$P_2 = (v_1, w_2, v_4, w_1, v_3, w_4, v_2, w_3, v_1).$$

Then $\psi = \{P_1, P_2\}$ is a 2-graphoidal cover of $K_{4,4}$ with $t_2 = 7, t = 1$. Hence, $\eta_2(K_{4,4}) = q - p - 7 + 1 = 2$.

When $n = 5$. Let

$$P_1 = (v_1, w_1, v_2, w_2, v_3, w_3, v_4),$$

$$P_2 = (v_2, w_4, v_1, w_5, v_4, w_1, v_3), \quad P_3 = (v_1, w_2, v_4, w_4, v_3, w_5, v_2, w_2, v_1).$$

Then $\psi = \{P_1, P_2, P_3\}$ is a 2-graphoidal cover of $K_{4,5}$ with $t_2 = 8$. Hence, $\eta_2(K_{4,5}) = q - p - 8 = 3$.

When $n \geq 6$. Let

$$P_1 = (v_1, w_1, v_2, w_2, v_3, w_3, v_4, v_1), \quad P_2 = (v_2, w_4, v_3, w_1, v_4, w_2, v_1, w_3, v_2),$$

$$P_3 = (v_3, w_5, v_2, w_6, v_3), \quad P_4 = (v_4, w_5, v_1, w_6, v_4),$$

$$Q_i = (v_1, w_{i+6}, v_2), i = 1, 2, \ldots, n-6, \ R_i = (v_3, w_{i+6}, v_4), \quad i = 1, 2, \ldots, n-6.$$

Then $\psi = \{P_1, P_2, P_3, P_4, Q_1, Q_2, \ldots, Q_{n-7}, R_1, R_2, \ldots, R_{n-7}\}$ is a 2-graphoidal cover of $K_{4,n}$ with $t_2 = p$. Hence, $\eta_2(K_{4,n}) = q - 2p = 2n - 8$. □

Theorem 21. *Let G be a complete bipartite graph $K_{m,n}$. Then $\eta_2(K_{m,n}) = \eta_{2a}(K_{m,n}) = q - 2p$ if $m, n \geq 5$.*

Proof. Follows from Corollary 10. □

Theorem 22. (see [6]) *Let G be a unicyclic graph with n pendant vertices. Let C be the unique cycle in G and let j be the number of vertices of degree greater than 2 on C. Then*

$$\eta(G) = \begin{cases} 1 & \text{if } j = 0; \\ n+1 & \text{if } j = 1 \text{ and } deg\ v = 3, \text{ where } v \text{ is the unique} \\ & \text{vertex of degree greater than 2 on } C \\ n & \text{otherwise.} \end{cases}$$

Theorem 23. (see [6]) *Let G be a tree with n pendant vertices. Then $\eta_2(G) = \eta_{2a}(G) = n - 1 - t_2$.*

Theorem 24. *If G is a unicyclic graph with n pendant vertices and the unique cycle C, and j denote the number of vertices of degree greater than 2 on C, then*

$$\eta_2(G) = \begin{cases} 1 & \text{if } j = 0; \\ n+1-t_2 & \text{if } j = 1 \text{ and } d(v) = 3, \text{ where } v \text{ is the unique} \\ & \text{vertex of degree greater than 2 on } C; \\ n - t_2 & \text{otherwise.} \end{cases}$$

Proof. Follows from Theorem 22 and Theorem 23. □

Theorem 25. *Let G be a bicyclic graph with n pendant vertices. Also let $U(l; m)$ be the unique bicycle in G and let j be the number of vertices of* *degree greater than 2 on* $U(l;m)$. *Then*

$$\eta_2(G) = \begin{cases} 2 & \text{if } G = U(l;m); \\ n+2-t_2 & \text{if either } j = 1 \text{ and } deg\ u_0 = 5 \text{ or } j = 2,\ deg\ u_0 \\ & = 4 \text{ and exactly one vertex, say } v \text{ different} \\ & \text{from } u_0, \text{ in } U(l;m) \text{ is of degree 3;} \\ n+1-t_2 & \text{otherwise.} \end{cases}$$

Proof. *Case (i)* Let

$$C_l = (u_0, u_1, \ldots, u_{l-1}, u_0), \quad C_m = (u_0, u_l, u_{l+1}, \ldots, u_{m+l-2}, u_0)$$

then $\psi = \{C_l, C_m\}$ is a 2-graphoidal cover of G with u_0 as the external vertex for both C_l and C_m. Hence, $\eta_2(G) = q - p + 1 = 2$.

Case (ii) Subcase(a). When $j = 1$ and $deg\ u_0 = 5$. Suppose C_m is the cycle in $U(l;m)$ which have no vertex of $deg > 2$ except u_0. Then $G_1 = G - C_m$ is a unicyclic graph with one vertex of deg 3 in the cycle so that $\eta_2(G_1) = n+1-t_2$. Let ψ_1 be a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is a 2-graphoidal cover of G and $|\psi| = |\psi_1|+1 = n+2-t_2$. Hence, $\eta_2(G) \leq n+2-t_2$. Again, for any 2-graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$, say u_0, are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_2(G) = q - p - t_2 + t \geq n + 1 - t_2 + 1 = n + 2 - t_2$.

Subcase(b). When $j = 2$, $deg\ u_0 = 4$ and $deg\ v = 3$ for a unique vertex v in G. Suppose v lies in C_m. Then $G_1 = G - C_l$ is a unicyclic graph with $deg\ v = 3$ in the cycle C_m so that $\eta_2(G_1) = n + 1 - t_2$. Let ψ_1 be a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_l$ is a 2-graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2 - t_2$. Hence, $\eta_2(G) \leq n + 2 - t_2$. Again, for any 2-graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $U(l;m)$, say either u_0 or v, are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_2(G) = q - p - t_2 + t \geq n + 1 + 1 - t_2 = n + 2 - t_2$.

Case (iii): Subcase(a). When $j = 1$ and $deg\ u_0 \geq 6$. Let C_l and C_m be the two cycles. Then $T = G - C_l - C_m$ is a tree with n pendant vertices so that $\eta_2(T) = n - 1 - t_2$, with ψ_1 as a minimum 2-graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a minimum 2-graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n - 1 - t_2 + 1 + 1 = n + 1 - t_2$.

Subcase(b). When $j \geq 3$. Let v, w be the vertices in $U(l;m)$ which have $deg > 2$ other than u_0. Suppose $v \in C_l$ and $w \in C_m$. Then u_0 is in $v - w$

section of $U(l;m)$ such that all internal vertices except u_0 is of *deg* 2. Let P denote this $v-w$ section. Then $\eta_2(P) = \{v-u_0, u_0-w\} = 1+1 = 2$. Also $T = G-P$ is a tree with n pendant vertices such that $\eta_2(T) = n-1-t_2$. Let ψ_1 be a minimum 2-graphoidal cover of T. Then $\psi = \psi_1 \cup P$ is a minimum 2-graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n-1-t_2+2 = n+1-t_2$.

Next, suppose $v, w \in C_l$. Then we choose $v-w$ section such that u_0 is not in $v-w$ section. Let P denote this $v-w$ section of G. Then each of its internal vertices is of *deg* 2 and $\eta_2(P) = 1$. Let ψ_1 be the minimum 2-graphoidal cover of P. Then $G_1 = G-P$ is a unicyclic graph with n pendant vertices so that $\eta_2(G_1) = n-t_2$, with ψ_2 as a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum 2-graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n+1-t_2$. □

Theorem 26. *Let G be a bicyclic graph with n pendant vertices. Also let $D(l,m;i)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on cycles in $D(l,m;i)$. Then*

$$\eta_2(G) = \begin{cases} 3 & \text{if } G = D(l,m;i); \\ n+2-t_2 & \text{if } j=2,\ deg\ u_{l-1} \geq 4 \text{ in } C_m \text{ and } deg\ u_{l+i-1} \\ & = 3 \text{ in } C_l; \text{ or } j \geq 3 \text{ and vertices of } deg \geq 3 \\ & \text{other than } u_{l-1} \text{ and } u_{l+i-1} \text{ are in one} \\ & \text{of the cycles } C_m \text{ or } C_l; \\ n+1-t_2 & \text{otherwise.} \end{cases}$$

Proof. *Case (i)* Let $C_l = (u_0, u_1, \ldots u_{l-1}, u_0)$, $P_i = (u_{l-1}, u_l, \ldots, u_{l+i-1})$ and $C_m = (u_{l+i-1}, u_{l+i}, \ldots, u_{l+m+i-2}, u_{l+i-1})$. Then $\psi = \{C_l, P_i, C_m\}$ is a 2-graphoidal cover of G with u_{l-1} and u_{l+i-1} as its external vertices. Hence, $\eta_2(G) = q-p+2 = 3$.

Case (ii) Subcase(a). When $j=2$, *deg* $u_{l-1} \geq 4$ in C_m and *deg* $u_{l+i-1} = 3$ in C_l. Here, $T = G - C_m - C_l$ is a tree with $n+1$ pendant vertices so that $\eta_2(T) = n - t_2$, with ψ_1 as a minimum 2-graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a 2-graphoidal cover of G and $|\psi| = |\psi_1| + 1 + 1 = n+2-t_2$. Hence, $\eta_2(G) \leq n+2-t_2$. Again, for any 2-graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l,m;i)$, say u_{l+i-1}, are external vertices in ψ so that $t \geq n+1$. Hence, $\eta_2(G) = q-p-t_2+t \geq n+1-t_2+1 = n+2-t_2$.

Subcase(b). When $j \geq 3$ and vertices of $deg \geq 3$ other than u_{l-1} and u_{l+i-1} are in one of the cycles C_m or C_l. Suppose v lies in C_l. Then $G_1 = G - C_m$ is a unicyclic graph with $n+1$ pendant vertices so that $\eta_2(G_1) = n+1-t_2$. Let ψ_1 be a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup C_m$ is a 2-graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2 - t_2$. Hence, $\eta_2(G) \leq n + 2 - t_2$. Again, for any 2-graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $D(l, m; i)$, say u_{l+i-1}, are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_2(G) = q - p - t_2 + t \geq n + 1 - t_2 + 1 = n + 2 - t_2$.

Case (iii) Subcase(a).When $j = 2$. Then $T = G - C_l - C_m$ is a tree with n pendant vertices so that $\eta_2(T) = n-1-t_2$, with ψ_1 as a minimum 2-graphoidal cover of T. Then $\psi = \psi_1 \cup C_l \cup C_m$ is a minimum 2-graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n - 1 - t_2 + 1 + 1 = n + 1 - t_2$.

Subcase(b). When $j \geq 3$. Let v, w be the vertices in $D(l, m; i)$ of $deg > 2$. Suppose v lies in C_m and w lies in C_l. Let P denote the $v - u_{l-1}$ section of C_m such that each of its internal vertices is of deg 2. Then $\eta_2(P) = 1$. Let ψ_1 be the minimum 2-graphoidal cover of P. Then $G_1 = G - P$ is a unicyclic graph with n pendant vertices so that $\eta_2(G_1) = n - t_2$, with ψ_2 as a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup \psi_2$ is a minimum 2-graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n + 1 - t_2$. □

Theorem 27. *Let G be a bicyclic graph with n pendant vertices. Also let $C_m(i; l)$ be the unique bicycle in G and let j be the number of vertices of degree greater than 2 on cycles in $C_m(i; l)$. Then*

$$\eta_2(G) = \begin{cases} 2 & \text{if } G = C_m(i;l); \\ n + 2 - t_2 & \text{if } j = 3 \text{ and } deg\ v = 3, \text{ where v is the unique} \\ & \text{vertex in } C_m(i;l) \text{ other than } u_0 \text{ and } u_i; \\ n + 1 - t_2 & \text{otherwise.} \end{cases}$$

Proof. Case (i) Let $C_m = (u_0, u_1, \ldots, u_{m-1}, u_0)$ with $m \geq 4$ be the cycle and $P_l = (u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i)$, $2 \leq i \leq m-2$, be the chord in $C_m(i; l)$. Then $\psi = \{C_m, P_l\}$ is a minimum 2-graphoidal cover of G such that any one vertex in C_m can be taken as an external vertex. Hence, $\eta_2(G) = 2$.

Case (ii) Let $P_l = (u_0, u_m, u_{m+1}, \ldots, u_{l+m-2}, u_i)$, $2 \leq i \leq m - 2$, be the chord in $C_m(i; l)$ such that $\eta_2(P_l) = 1$. Then $G_1 = G - P_l$ is a unicyclic graph with n pendant vertices so that $\eta_2(G_1) = n + 1 - t_2$. Let ψ_1 be a minimum

2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_l$ is a 2-graphoidal cover of G and $|\psi| = |\psi_1| + 1 = n + 2 - t_2$. Hence, $\eta_2(G) \leq n + 2 - t_2$. Again, for any 2-graphoidal cover ψ of G, the n pendant vertices of G and at least one vertex in $C_m(i; l)$, say v, are external vertices in ψ so that $t \geq n + 1$. Hence, $\eta_2(G) = q - p - t_2 + t \geq n + 1 - t_2 + 1 = n + 2 - t_2$.

Case (iii) Subcase(a). When $j = 2$, $deg\ u_0 = 3$ and $deg\ u_i \geq 4$. Let C_m be the cycle with u_i as an external vertex and P_l be the chord in G along with its n pendant vertices. Then $T = G - C_m$ is a tree with $n + 1$ pendant vertices so that $\eta_2(T) = n - t_2$, with ψ_1 as a minimum 2-graphoidal cover of T. Now, $\psi = \psi_1 \cup C_m$ is a minimum 2-graphoidal cover of G such that every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n + 1 - t_2$.

Subcase(b). When $j \geq 3$. Let $P_l = (u_0 u_m u_{m+1} \ldots u_{l+m-2} u_i), 2 \leq i \leq m-2$, be the chord in G such that $\eta_2(P_l) = 1$. Then $G_1 = G - P_l$ is a unicyclic graph with n pendant vertices so that $\eta_2(G_1) = n - t_2$. Let ψ_1 be a minimum 2-graphoidal cover of G_1. Then $\psi = \psi_1 \cup P_l$ is a minimum 2-graphoidal cover of G and every vertex of degree greater than 1 is an internal vertex of some path(s) in ψ. Hence, $\eta_2(G) = n + 1 - t_2$. □

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References

[1] B.D. Acharya, E. Sampathkumar, Graphoidal covers and graphoidal covering number of a graph, *Indian J. Pure Appl. Math.*, **18**, No. 10 (1987), 882-890.

[2] S. Arumugam, B.D. Acharya, E. Sampathkumar, Graphoidal covers of a graph: a creative review, In: *Proc. National Workshop on Graph Theory and its Applications*, Manonmaniam Sundaranar University, Tirunelveli, Tata McGraw-Hill, New Delhi (1997), 1-28.

[3] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA (1969).

[4] K. Nagaranjan, A. Nagaranjan, S. Somasundaram, 2-graphoidal path covers, *International Journal of Applied Mathematics*, **21**, No. 4 (2008), 615-628.

[5] C. Pakkiam, S. Arumugam, On the graphoidal covering number of a graph, *Indian J. Pure Appl. Math.*, **20**, No. 4 (1989), 330-333.

[6] C. Pakkiam, S. Arumugam, On the graphoidal covering number of unicyclic graphs, *Indian J. Pure Appl. Math.*, **23** (1992), 141-143.

[7] K. Ratan Singh, P.K. Das, On graphoidal covers of bicyclic graphs, *Int. Math. Forum*, **5**, No. 42 (2010), 2093-2101.

[8] M. Zhou, Decomposition of some product graphs into 1-factors and hamiltonian cycles, *Ars Comb.*, **28** (1989), 258-268.